STRUCTURES
CLUSTER OF
EXCELLENCE

UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386

A zero-dimensional Field Theory of Noncommutative Geometry

QFT-Seminar, Theoretisch-Physikalisches Institut der FSU, Jena

Carlos I. Pérez-Sánchez
ITP, University of Heidelberg
`www.thphys.uni-heidelberg.de/~perez/carlos.html`
`perez@thphys.uni-heidelberg.de`

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1912.13288; 2007.10914; 2102.06999; 2105.01025; 2111.02858

ERC, indirectly & DFG-Structures Excellence Cluster

Made in Jena: *Jenaer Erklärung* «Thm: $\nexists$ human subspecies » [Fischer-Hoßfeld-Krause-Richter, 2019]

$-\infty$ in human story

humans appear in Africa, $\sim$ 250 000 years ago

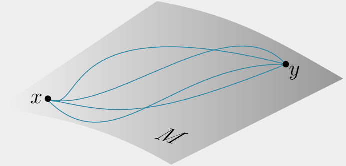
East-Africans populate the globe, $\sim$60 000 years ago

Zlatý Kůň, Bichon, . . . $\in$ Center Europe $\sim$ 40 000 to 9 000 years ago

$\sim$ 5 000 years ago, plant-based diet in Europe favours light skin (which in winter produces enough vitamin D, unlike dark skin)

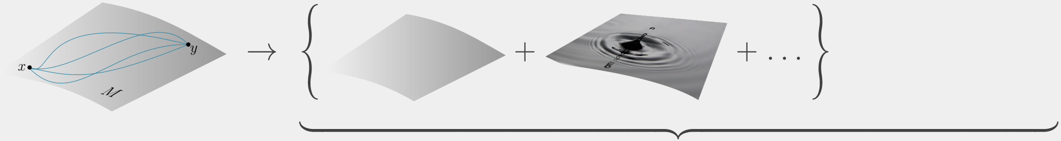
Credits on individual pictures: T. Björklund (Zlatý Kůň woman), else Wikicommons-Authors: Kious *et. al*, Martin23230, Joe Roe

Path integrals and (Euclidean) quantum gravity



- 1st. challenge:
  - Whereas in quantum mechanics: path integrals <u>on</u> a fixed spacetime M

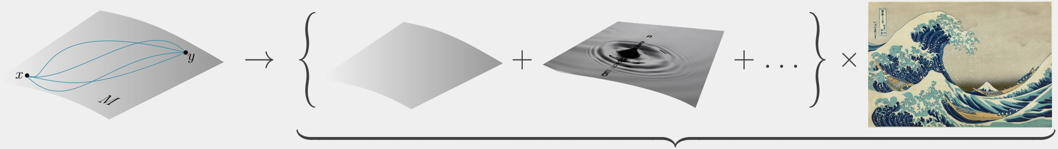
Path integrals and (Euclidean) quantum gravity



Quantum superposition of geometries

- 1st. challenge:
  - Whereas in quantum mechanics: path integrals on a fixed spacetime M
  - In quantum gravity: path integrals of space~~time~~, $Z = \int_{\text{METRIC}} \mathrm{e}^{-\frac{1}{\hbar} S_{\text{EH}}[g]} \mathrm{d}g$

Path integrals and (Euclidean) quantum gravity



Quantum superposition of geometries (small perturbations+ instantons)

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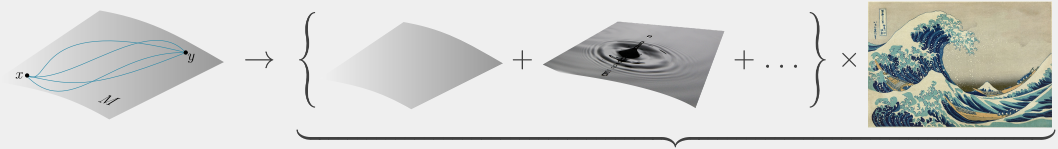
[Hokusai]

Path integrals and (Euclidean) quantum gravity



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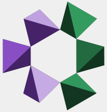
- 2nd. challenge: replace C^{∞}-category

discrete

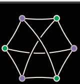


single geometry
← paradigms →

algebraic

[Hokusai][Tetrahedra from Wikipedia]

↑ from 'traces of tensors'



History of the spectral formalism

- *Weyl's law* (1911) on the Laplace spectrum of $\Omega \subset \mathbb{R}^d$ ($\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \ldots$):

$$\#\{i : \lambda_i \leqslant \Lambda\} \sim \mathrm{vol}\,\Omega \cdot \Lambda^{d/2} + \mathrm{o}(\Lambda^{d/2})$$

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[Gordon, Webb, Wolpert, *Invent. Math.* '92 after Milnor, Sunada, Bérard, …] [Connes, *JNCG* 2013]

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- differential *noncommutative (nc) geometry* = nc topology + metric data

$$\begin{tikzcd}
\{\text{nice topological spaces}\} \arrow[r, phantom, "\simeq"] \arrow[d, hook] & \{\text{unital } \textit{commutative } C^*\text{-algebras}\} \arrow[d, hook] \\
\{\text{nice `nc topological spaces'}\} \arrow[r, phantom, "\simeq"] & \{\text{unital } \cancel{\textit{commutative}}\ C^*\text{-algebras}\}
\end{tikzcd}$$

[Gordon, Webb, Wolpert, *Invent. Math.* '92 after Milnor, Sunada, Bérard, …] [Connes, *JNCG* 2013]
[Gelfand, Najmark *Mat. Sbornik* '43]

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\end{tikzcd}$$

- spectral triples $(\boxed{A}, H, \boxed{D})$, cf. spin geometry $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

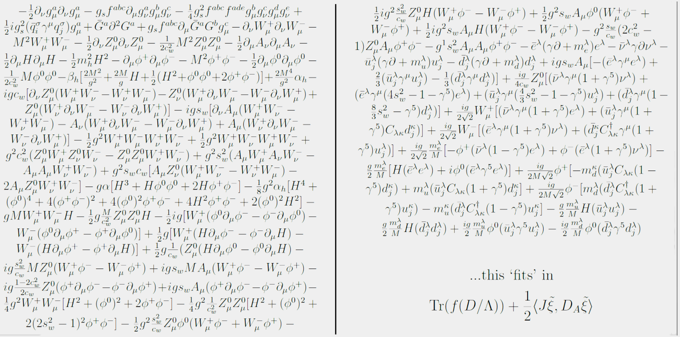
topology

geometry

[Gordon, Webb, Wolpert, *Invent. Math.* '92 after Milnor, Sunada, Bérard, ...] [Connes, *JNCG* 2013]
[Gelfand, Najmark *Mat. Sbornik* '43] [Connes, *NCG* '94]

Physical motivation of spectral triples (deterministic)

- Physics ∩ Noncommutative Geometry ∋ The Standard Model from the Spectral Action



$$
\begin{gathered}
-\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4}g^2_s f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu + \\
\tfrac{1}{2}ig^2_s(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu + \bar{G}^a\partial^2G^a + g_s f^{abc}\partial_\mu\bar{G}^aG^bg^c_\mu - \partial_\nu W^+_\mu\partial_\nu W^-_\mu - \\
M^2W^+_\mu W^-_\mu - \tfrac{1}{2}\partial_\nu Z^0_\mu\partial_\nu Z^0_\mu - \tfrac{1}{2c^2_w}M^2Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu\partial_\mu A_\nu - \\
\tfrac{1}{2}\partial_\mu H\partial_\mu H - \tfrac{1}{2}m^2_hH^2 - \partial_\mu\phi^+\partial_\mu\phi^- - M^2\phi^+\phi^- - \tfrac{1}{2}\partial_\mu\phi^0\partial_\mu\phi^0 - \\
\tfrac{1}{2c^2_w}M\phi^0\phi^0 - \beta_h[\tfrac{2M^2}{g^2} + \tfrac{2M}{g}H + \tfrac{1}{2}(H^2+\phi^0\phi^0+2\phi^+\phi^-)] + \tfrac{2M^4}{g^2}\alpha_h - \\
igc_w[\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + \\
Z^0_\mu(W^+_\nu\partial_\nu W^-_\mu - W^-_\nu\partial_\nu W^+_\mu)] - igs_w[\partial_\nu A_\mu(W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - A_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + A_\mu(W^+_\nu\partial_\nu W^-_\mu - \\
W^-_\nu\partial_\nu W^+_\mu)] - \tfrac{1}{2}g^2W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \tfrac{1}{2}g^2W^+_\mu W^-_\nu W^+_\mu W^-_\nu + \\
g^2c^2_w(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2s^2_w(A_\mu W^+_\mu A_\nu W^-_\nu - \\
A_\mu A_\mu W^+_\nu W^-_\nu) + g^2s_wc_w[A_\mu Z^0_\nu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - \\
2A_\mu Z^0_\mu W^+_\nu W^-_\nu] - g\alpha[H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-] - \tfrac{1}{8}g^2\alpha_h[H^4 + \\
(\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2] - \\
gMW^+_\mu W^-_\mu H - \tfrac{1}{2}g\tfrac{M}{c^2_w}Z^0_\mu Z^0_\mu H - \tfrac{1}{2}ig[W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - \\
W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)] + \tfrac{1}{2}g[W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) - \\
W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)] + \tfrac{1}{2}g\tfrac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) - \\
ig\tfrac{s^2_w}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_wMA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) - \\
ig\tfrac{1-2c^2_w}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + igs_wA_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \\
\tfrac{1}{4}g^2W^+_\mu W^-_\mu[H^2 + (\phi^0)^2 + 2\phi^+\phi^-] - \tfrac{1}{4}g^2\tfrac{1}{c^2_w}Z^0_\mu Z^0_\mu[H^2 + (\phi^0)^2 + \\
2(2s^2_w - 1)^2\phi^+\phi^-] - \tfrac{1}{2}g^2\tfrac{s^2_w}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) -
\end{gathered}
$$

$$
\begin{gathered}
\tfrac{1}{2}ig^2\tfrac{s^2_w}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \tfrac{1}{2}g^2s_wA_\mu\phi^0(W^+_\mu\phi^- + \\
W^-_\mu\phi^+) + \tfrac{1}{2}ig^2s_wA_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\tfrac{s_w}{c_w}(2c^2_w - \\
1)Z^0_\mu A_\mu\phi^+\phi^- - g^1s^2_wA_\mu A_\mu\phi^+\phi^- - \bar{e}^\lambda(\gamma\partial + m^\lambda_e)e^\lambda - \bar{\nu}^\lambda\gamma\partial\nu^\lambda - \\
\bar{u}^\lambda_j(\gamma\partial + m^\lambda_u)u^\lambda_j - \bar{d}^\lambda_j(\gamma\partial + m^\lambda_d)d^\lambda_j + igs_wA_\mu[-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \\
\tfrac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \tfrac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)] + \tfrac{ig}{4c_w}Z^0_\mu[(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + \\
(\bar{e}^\lambda\gamma^\mu(4s^2_w - 1 - \gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(\tfrac{4}{3}s^2_w - 1 - \gamma^5)u^\lambda_j) + (\bar{d}^\lambda_j\gamma^\mu(1 - \\
\tfrac{8}{3}s^2_w - \gamma^5)d^\lambda_j)] + \tfrac{ig}{2\sqrt{2}}W^+_\mu[(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(1 + \\
\gamma^5)C_{\lambda\kappa}d^\kappa_j)] + \tfrac{ig}{2\sqrt{2}}W^-_\mu[(\bar{e}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{d}^\kappa_jC^\dagger_{\lambda\kappa}\gamma^\mu(1 + \\
\gamma^5)u^\lambda_j)] + \tfrac{ig}{2\sqrt{2}}\tfrac{m^\lambda_e}{M}[-\phi^+(\bar{\nu}^\lambda(1-\gamma^5)e^\lambda) + \phi^-(\bar{e}^\lambda(1+\gamma^5)\nu^\lambda)] - \\
\tfrac{g}{2}\tfrac{m^\lambda_e}{M}[H(\bar{e}^\lambda e^\lambda) + i\phi^0(\bar{e}^\lambda\gamma^5e^\lambda)] + \tfrac{ig}{2M\sqrt{2}}\phi^+[-m^\kappa_d(\bar{u}^\lambda_jC_{\lambda\kappa}(1 - \\
\gamma^5)d^\kappa_j) + m^\lambda_u(\bar{u}^\lambda_jC_{\lambda\kappa}(1+\gamma^5)d^\kappa_j] + \tfrac{ig}{2M\sqrt{2}}\phi^-[m^\lambda_d(\bar{d}^\lambda_jC^\dagger_{\lambda\kappa}(1 + \\
\gamma^5)u^\kappa_j) - m^\kappa_u(\bar{d}^\lambda_jC^\dagger_{\lambda\kappa}(1-\gamma^5)u^\kappa_j] - \tfrac{g}{2}\tfrac{m^\lambda_u}{M}H(\bar{u}^\lambda_ju^\lambda_j) - \\
\tfrac{g}{2}\tfrac{m^\lambda_d}{M}H(\bar{d}^\lambda_jd^\lambda_j) + \tfrac{ig}{2}\tfrac{m^\lambda_u}{M}\phi^0(\bar{u}^\lambda_j\gamma^5u^\lambda_j) - \tfrac{ig}{2}\tfrac{m^\lambda_d}{M}\phi^0(\bar{d}^\lambda_j\gamma^5d^\lambda_j)
\end{gathered}
$$

...this 'fits' in

$$\mathrm{Tr}(f(D/\Lambda)) + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle$$

\# of generations and $\mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C}) \rightarrowtail$ NCG $\rightarrowtail$ *Classical* Standard Model

[Connes, Lott, *Nucl. Phys. B* '91; . . . Chamseddine, Connes, Marcolli *ATMP* '07 (Euclidean)]
[Barrett *J. Math. Phys.* '07 (Lorenzian); Connes-Chamseddine *JHEP* '12; van Suijlekom's textbook NCG∩HEP '15]

PHYSICAL MOTIVATION OF SPECTRAL TRIPLES (DETERMINISTIC)

- Physics $\cap$ Noncommutative Geometry $\ni$ The Standard Model from the Spectral Action

$$D_F = \begin{pmatrix}
0 & 0 & \Upsilon_\nu^* & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_R^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Upsilon_e^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Upsilon_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \Upsilon_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^* & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^* & \otimes 1_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Upsilon_u & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Upsilon_d & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Upsilon_R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_\nu^T & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_e^T & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^T & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^T & \otimes 1_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_u & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_d & 0 & 0 &
\end{pmatrix} \in M_{96}(\mathbb{C})_{\text{s.a.}}$$

# of generations and $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \rightarrowtail$ NCG $\rightarrowtail$ *Classical* Standard Model

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Towards a quantum theory of noncommutative spaces

« *The far distant goal is to set up a functional integral evaluating (...)*

observables $\mathscr{S}$ $\quad \langle \mathscr{S} \rangle = \int \mathscr{S}\, \mathrm{e}^{-\operatorname{Tr} f(D/\Lambda) - \frac{1}{2}\langle J\psi, D\psi\rangle + \rho(e,D)}\, \mathrm{d}e\, \mathrm{d}\psi\, \mathrm{d}D \quad (*)$ »

[∗ Eq. 1.892, Connes, Marcolli, *NCG, QFT and motives*, 2007] [Connes; Monge-Kantorovich]

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functional integral $\xrightarrow[\text{paradigm shift}]{}$ operator integral

$$\int_{\text{METRIC}} \mathrm{e}^{-\frac{1}{\hbar} S_{\text{EH}}[g]}\, \mathrm{d}g \xrightarrow{\text{Einstein-Hilbert} \longrightarrow \text{spectral}} \int_{\text{DIRAC}} \mathrm{e}^{-\frac{1}{\hbar} \operatorname{Tr} f(D)}\, \mathrm{d}D$$

(hard to define for manifolds)

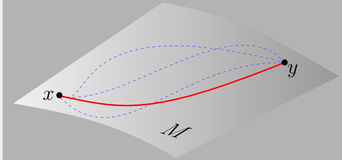
$f : \mathbb{R} \longrightarrow \mathbb{R}$ with $f(D) \longrightarrow \infty$ at large argument

[$*$ Eq. 1.892, Connes, Marcolli, *NCG, QFT and motives*, 2007] [Connes; Monge-Kantorovich]

$\inf_{\gamma: x \to y} \{\int_\gamma \mathrm{d}s\} =: d(x,y) = \sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| : \ ||D_M f - f D_M|| \leqslant 1 \}$ ⊳⊳ `comm. sp. str.`

Replace spin manifold (M, g) by $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

Connes' geodesic distance



$\gamma : \mathbb{R} \to M$

$\inf_{\gamma \text{ as above}} \{\int_\gamma \mathrm{d}s\} = d(x, y)$

Replace spin manifold (M, g) by $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

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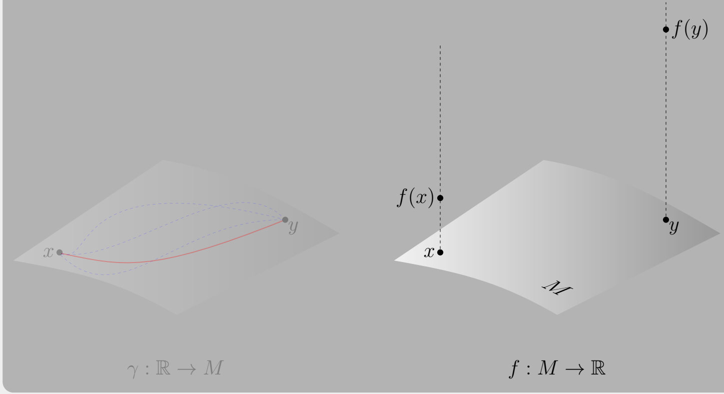

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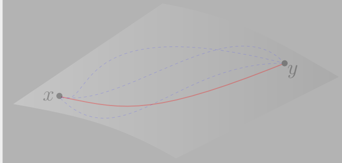

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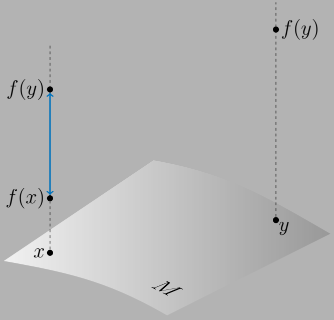
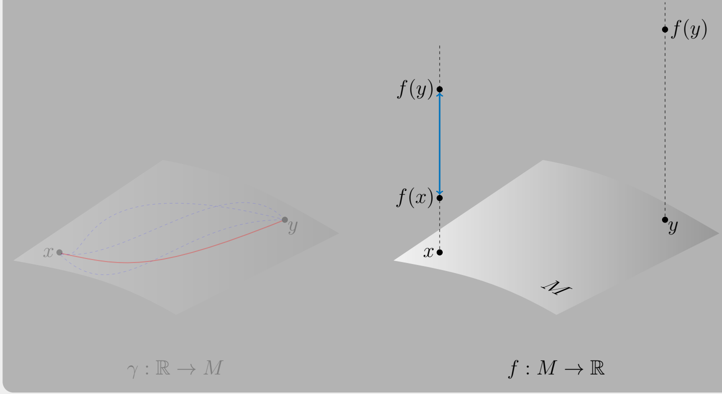


$f : M \to \mathbb{R}$

$$|\mathrm{ev}_x(f) - \mathrm{ev}_y(f)|$$

Replace spin manifold (M, g) by $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

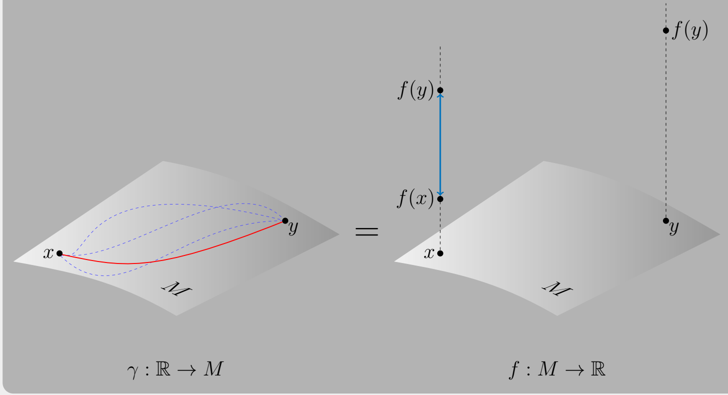
Connes' geodesic distance



$$\sup_{f \in C^\infty(M)} \left\{ |\mathrm{ev}_x(f) - \mathrm{ev}_y(f)| \ : \ ||D_M f - f D_M|| \leqslant 1 \right\}$$

Replace spin manifold (M, g) by $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

Connes' geodesic distance

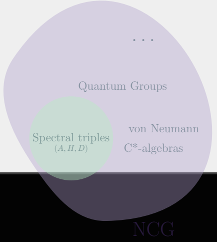


$$\inf_{\gamma \text{ as above}} \{\textstyle\int_\gamma \mathrm{d}s\} = d(x, y) \;=\; \sup_{f \in C^\infty(M)} \left\{ |\mathrm{ev}_x(f) - \mathrm{ev}_y(f)| \;:\; ||D_M f - f D_M|| \leqslant 1 \right\}$$

Commutative spectral triples

A spin manifold M yields (A_M, H_M, D_M)

- $A_M = C^\infty(M)$ is a comm. $*$-algebra
- $H_M := L^2(M, \mathbb{S})$ a repr. of A_M
- $D_M = -\mathrm{i}\gamma^\mu(\partial_\mu + \omega_\mu)$ is s.a.
- for each $a \in A_M$, $[D_M, a]$ is bounded, and in fact $[D_M, x^\mu] = -\mathrm{i}\gamma^\mu$
- D_M has compact resolvent ...



. . .

Quantum Groups

Spectral triples
(A, H, D)

von Neumann

C*-algebras

NCG

A *spectral triple* (A, H, D) consists of

- a ~~commutative~~ $*$-algebra A
- a representation H of A
- a self-adjoint operator D on H with compact resolvent and such that $[D, a]$ is bounded for each $a \in A$
- $\Omega^1_D(A) := \left\{ \sum_{\text{(finite)}} b[D, a] \mid a, b \in A \right\}$

Commutative spectral triples

A spin manifold M yields (A_M, H_M, D_M)

- $A_M = C^\infty(M)$ is a comm. $*$-algebra
- $H_M := L^2(M, \mathbb{S})$ a repr. of A_M
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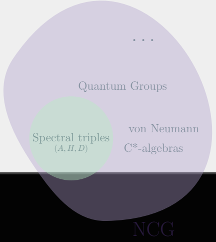

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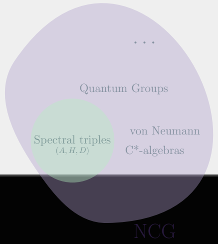

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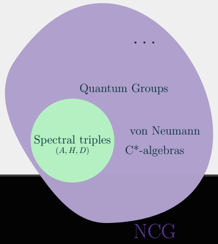

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NCG toolkit in high energy physics

- On a spectral triple (A, H, D) the (bosonic) classical action reads

$$S(D) = \mathrm{Tr}_H\left[\underbrace{f}_{\text{bump function}}\left(\underbrace{D}_{\text{abstract Dirac operator}}/\underbrace{\Lambda}_{\text{energy scale}}\right)\right]$$

[Chamseddine-Connes *CMP* ‘97]

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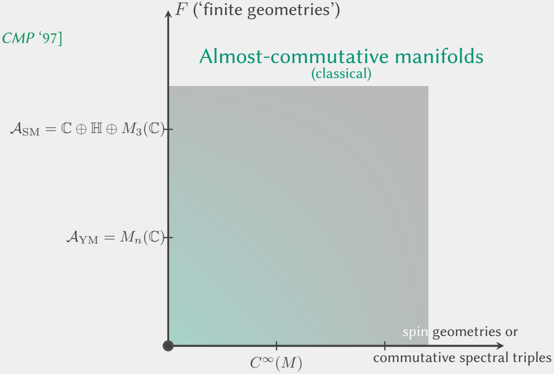
- Realistic, classical models come from *almost-commutative manifolds* $M \times F$, where F is a finite-dim. spectral triple

$$\left(C^\infty(M, A_F), H_M \otimes H_F, D_M \otimes 1_F + \gamma_5 \otimes D_F \right)$$



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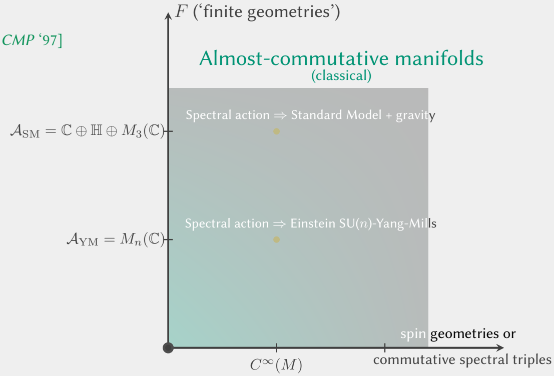

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bump function

energy scale

abstract Dirac operator

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- *connections*: if S^G is a G-invariant functional on M

$$S^G \rightsquigarrow S^{\mathrm{Maps}(M,G)}$$

$$\mathrm{d} \rightsquigarrow \mathrm{d} + \mathbb{A} \qquad \mathbb{A} \in \Omega^1(M) \otimes \mathfrak{g}$$

$$\mathbb{A}' = u \mathbb{A} u^{-1} + u \mathrm{d} u^{-1} \qquad u \in \mathrm{Maps}(M, G)$$

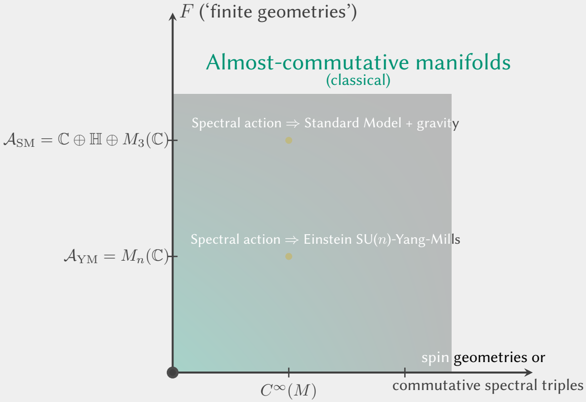
- given (A, H, D) and $A \simeq_{\mathrm{m}} B$ (i.e. $\mathrm{End}_A(E) \cong B$) yields $(B, E \otimes_A H, D\text{'s})$. For $A = B$, a tower

$$\left\{ (A, H, D + \omega \pm J \omega J^{-1}) \right\}_{\omega \in \Omega^1_D(A)}$$

$$D_\omega \mapsto \mathrm{Ad}(u) D_\omega \mathrm{Ad}(u)^* = D_{\omega_u}$$

$$\omega \mapsto \omega_u = u \omega u^* + u[D, u^*] \qquad u \in \mathcal{U}(A)$$

First result (almost there)



F ('finite geometries')

Almost-commutative manifolds
(classical)

Spectral action ⇒ Standard Model + gravity

$\mathcal{A}_{\mathrm{SM}} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$

Spectral action ⇒ Einstein SU(n)-Yang-Mills

$\mathcal{A}_{\mathrm{YM}} = M_n(\mathbb{C})$

spin geometries or
commutative spectral triples

$C^\infty(M)$

SPECTRAL ACTION

Classical

$$S(D) = \operatorname{Tr} f(D/N) \qquad \text{(bosons)}$$

$$\sim \sum_{s \in \mathrm{Sp\,Dim} \cap \mathbb{R}_+} f_s N^s ⨍ |D|^{-s} + f(0)\zeta(0) \dots$$

Quantum

$$\mathcal{Z}_{\mathrm{AC}} \overset{?}{=} \int_{\text{DIRAC}} \mathrm{e}^{-\frac{1}{\hbar} \operatorname{Tr} f(D/N)} \mathrm{d}D$$

(hard to define for almost-comm. manifolds)

[Chamseddine, Connes, Marcolli *ATMP* '07] using heat kernel expansion, for 4-manifolds:

$$N^4 ⨍ |D|^{-4} = c_4(N) \mathrm{vol}(M)$$

[cosmological constant]

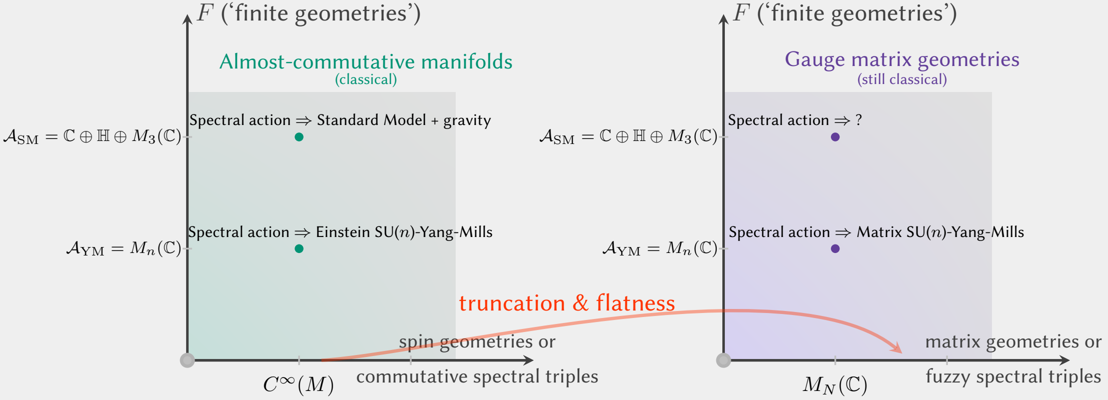
$$N^2 ⨍ |D|^{-2} = c_2(N) \int R$$

[Einstein-Hilbert]

$$\zeta_D(0) = c_0 \int (R^* R^*) + c_0' \int C^2$$

[Gauß-Bonnet + conformal gravity]

First result



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spin geometries or
commutative spectral triples

truncation & flatness

F ('finite geometries')

Gauge matrix geometries
(still classical)

$\mathcal{A}_{\mathrm{SM}} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$

Spectral action ⇒ ?

$\mathcal{A}_{\mathrm{YM}} = M_n(\mathbb{C})$

Spectral action ⇒ Matrix SU(n)-Yang-Mills

$M_N(\mathbb{C})$

matrix geometries or
fuzzy spectral triples

Matrix Yang-Mills(-Higgs) functional obeying spectral triple axioms; its partition function is a multi-matrix model. Red arrow is a more general $C(\mathbb{S}^2) \twoheadrightarrow \langle$  $\rangle$

[CP *Ann. Henri Poincaré* **23**, 2022 `arXiv:2105.01025`]

Second result

- which is the low-energy limit?

$$``\langle \ , \ , \ldots , \ \rangle" = M_N(\mathbb{C}) \xrightarrow{?} C^\infty(M)$$

- adopting the functional renormalisation group (FRG) viewpoint I failed
- but still, as a corollary: important step in the computation of the FRG for general multimatrix models

Multimatrix models with multi-traces

- A chord-diagram formula computes the spectral action (in any signature)

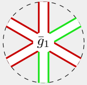
$$\mathcal{Z} = \int_{\textsc{Dirac}} \mathrm{e}^{-\operatorname{Tr}_H f(D)} \mathrm{d}D \quad (\hbar = 1)$$

$$= \int_{M_{p,q}} \mathrm{e}^{-N \operatorname{Tr}_N P - \operatorname{Tr}_N^{\otimes 2}(Q_{(1)} \otimes Q_{(2)})} \mathrm{d}\mathbb{X}_{\textsc{Leb}}$$

- $\mathbb{X} \in M_{p,q}$ = products of $\mathfrak{su}(N)$ and $\mathcal{H}_N$
- $\mathrm{d}\mathbb{X}_{\textsc{Leb}}$ is the Lebesgue measure on $M_{p,q}$
- P, $Q_{(i)}$ in $\mathbb{C}_{\langle k \rangle} = \mathbb{C}\langle \mathbb{X} \rangle$ nc-polynomials

- $\mathcal{Z}_{\textsc{formal}}$ leads to colored ribbon graphs

$$g_1 \operatorname{Tr}_N(ABBBAB) \quad \leftrightarrow$$



Chord-diagram is what it sounds like: 

[CP '19, CP '21, CP '22a, CP '22b]

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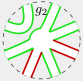
$g_2 \operatorname{Tr}_N^{\otimes 2}(AABABA \otimes AA)$ ↔



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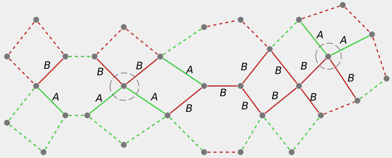
- Ribbon graphs: Enumeration of maps, here 'face-worded'



- Multitrace: 'touching interactions' [Klebanov, *Phys. Rev. D.* '95], wormholes [Ambjørn-Jurkiewicz-Loll-Vernizzi, *JHEP* '01], 'stuffed maps' [Borot *Ann. Inst. Henri Poincaré Comb. Phys. Interact.* '14], AdS/CFT [Witten, `hep-th/0112258`]

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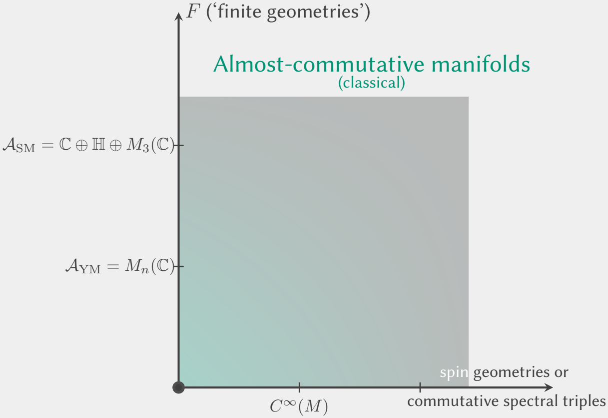

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spin geometries or

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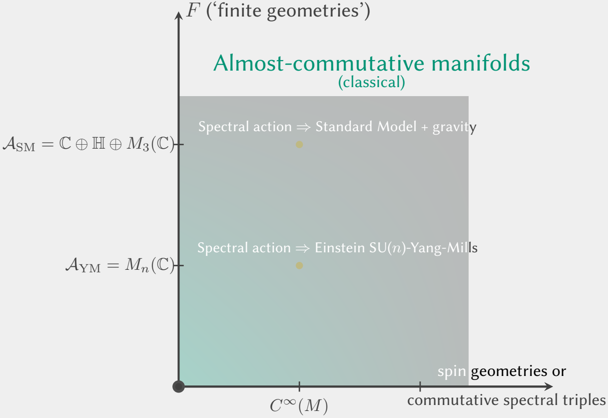

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(hard for almost-commutative manifolds)

Barrett's matrix geometries ⊂ spectral triples

$A = M_N(\mathbb{C})$ ⻎ a bit more precise?

$H = \mathbb{S} \otimes M_N(\mathbb{C})$

$D = \gamma^\mu \otimes \mathbb{X}_\mu + \gamma^{\alpha_1\alpha_2\alpha_3} \otimes \mathbb{X}_{\alpha_1\alpha_2\alpha_3} + \dots$

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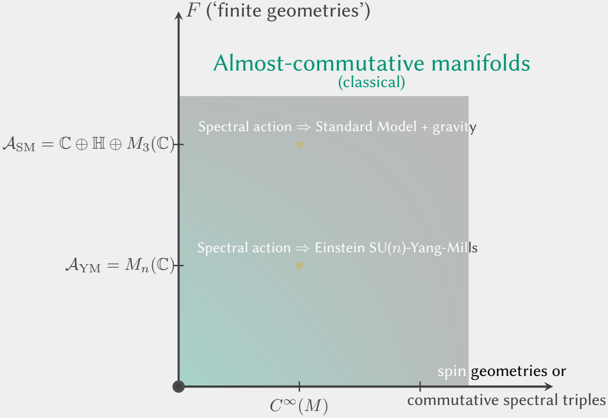

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DEFINITION [CP' 21]. A *gauge matrix spectral triple* $G_\ell \times F$ is the spectral triple product of a matrix geometry G_ℓ with a finite geometry $F = (A_F, H_F, D_F)$, $\dim A_F < \infty$.

(Lemma-)Definition [CP' 21]. Consider a gauge matrix spectral triple $G_\ell \times F$ with

$$F = (M_n(\mathbb{C}), M_n(\mathbb{C}), D_F)$$

and G_ℓ Riemannian ($d = 4$) fuzzy geometry on $M_N(\mathbb{C})$, whose fluctuated Dirac op. is

$$D_\omega = \sum_{\mu=0}^{3} \overbrace{\gamma^\mu \otimes (\ell_\mu + \mathscr{a}_\mu) + \gamma^{\hat{\mu}} \otimes (\mathscr{x}_\mu + \mathscr{s}_\mu)}^{D_{\text{gauge}}} + \overbrace{\gamma \otimes \Phi}^{D_{\text{Higgs}}}\,, \qquad \mathscr{a}_\mu = \text{‘gauge potential’},\ \mathscr{x}_\mu = \text{spin connection?}$$

The *field strength* is given by $\mathscr{F}_{\mu\nu} := [\overbrace{\ell_\mu + \mathscr{a}_\mu}^{\mathscr{d}_\mu}, \ell_\nu + \mathscr{a}_\nu] =: [\mathsf{F}_{\mu\nu}, \ \cdot\ ]$

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Lemma. The gauge group $\mathrm{G}(\mathcal{A}) \cong \mathcal{U}(\mathcal{A})/\mathcal{U}(Z(\mathcal{A})) \cong \mathrm{PU}(N) \times \mathrm{PU}(n)$ acts as follows

$$\mathsf{F}_{\mu\nu} \mapsto \mathsf{F}^{u}_{\mu\nu} = u\mathsf{F}_{\mu\nu}u^* \qquad \text{for all } u \in \mathrm{G}(\mathcal{A})$$

Meaning	Random matrix case, flat $d=4$ Riem. Tr = trace of ops. $M_N \otimes M_n \to M_N \otimes M_n$	Smooth operator
Derivation	$\ell_\mu = [L_\mu \otimes 1_n,\ \cdot\ ]$	∂_i
Gauge potential	$\mathscr{a}_\mu = [A_\mu,\ \cdot\ ]$	$\mathbb{A}_i$
Covariant derivative	$\mathscr{d}_\mu = \ell_\mu + \mathscr{a}_\mu$	$\mathbb{D}_i = \partial_i + \mathbb{A}_i$

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Field strength	$[\mathscr{d}_\mu, \mathscr{d}_\nu] = \overbrace{[\ell_\mu, \ell_\nu]}^{\not\equiv 0} + [\ell_\mu, \mathscr{a}_\nu] - [\ell_\nu, \mathscr{a}_\mu] + [\mathscr{a}_\mu, \mathscr{a}_\nu]$	$[\mathbb{D}_i, \mathbb{D}_j] = \overbrace{[\partial_i, \partial_j]}^{\equiv 0} + \partial_i \mathbb{A}_j - \partial_j \mathbb{A}_i + [\mathbb{A}_i, \mathbb{A}_j]$
Yang-Mills action	$-\frac{1}{4} \mathrm{Tr}(\mathscr{F}_{\mu\nu} \mathscr{F}^{\mu\nu})$	$-\frac{1}{4} \int_M \mathrm{Tr}_{\mathfrak{su}(n)} (\mathbb{F}_{ij} \mathbb{F}^{ij}) \mathrm{vol}$
Higgs field	Φ	h
Higgs potential	$\mathrm{Tr}(f_2 \Phi^2 + \Phi^4)$	$\int_M \big(- \mu^2 \lvert h \rvert^2 + \lambda \lvert h \rvert^4 \big) \mathrm{vol}$
Gauge-Higgs coupling	$-\mathrm{Tr}(\mathscr{d}_\mu \Phi \mathscr{d}^\mu \Phi)$	$-\int_M \lvert \mathbb{D}_i h \rvert^2 \mathrm{vol}$

III. Functional renormalisation in random matrices

Motivation from ‘2D-Quantum Gravity’

discrete surfaces ↔ matrix integrals $\mathcal{Z}(\lambda)$

[B. Eynard, *Counting Surfaces* ’16]

smooth surface ↔ $\langle \text{area} \rangle$ finite & mesh $\mathfrak{a} \to 0$

all topologies ↔ $\mathcal{Z}(\lambda) = \sum_g N^{2-2g} \underset{\underset{(\lambda_c - \lambda)^{(2-2g)/\theta}}{\wr}}{\mathcal{Z}_g(\lambda)}$

$\Uparrow$

double-scaling limit $N(\lambda_c - \lambda)^{1/\theta} = C$

lin. RG-flow near a fixed point ↔ $\lambda(N) = \lambda_c + (N/C)^{-\theta}$

$\theta = -(\partial\beta/\partial\lambda)|_{\lambda_c}$

[Eichhorn-Koslowski, *Phys. Rev. D.* ’13]

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[B. Eynard, *Counting Surfaces* ’16]

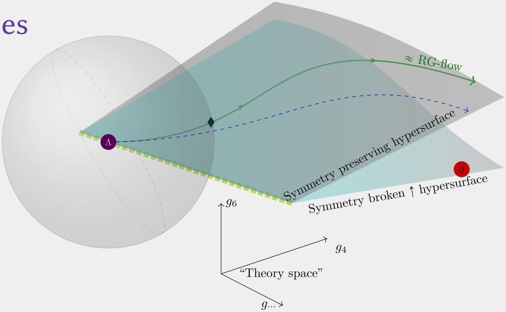
smooth surface ↔ $\langle\text{area}\rangle$ finite & mesh $\mathfrak{a} \to 0$

all topologies ↔ $\mathcal{Z}(\lambda) = \sum_g N^{2-2g} \underset{\overset{\wr}{(\lambda_c - \lambda)^{(2-2g)/\theta}}}{\mathcal{Z}_g(\lambda)}$

$\Uparrow$

double-scaling limit $N(\lambda_c - \lambda)^{1/\theta} = C$

lin. RG-flow near a fixed point ↔ $\lambda(N) = \lambda_c + (N/C)^{-\theta}$, $\theta = -(\partial\beta/\partial\lambda)|_{\lambda_c}$



Λ Chosen bare action $S = \Gamma_{N=\Lambda}$

0 Full effective action $\Gamma = \Gamma_{N=0}$

◆ Interpolating action $\Gamma_{N=\Lambda-\rho}$ (projected & truncated)

⟶ RG-flow with truncation and projection

┈┈ Moduli of Dirac operators ↪ theory space

⇢ RG-flow without truncation nor projection

$g_{\dots}$ Rest of coupling constants

[Eichhorn-Koslowski, *Phys. Rev. D.* ’13]

[CP ’21]

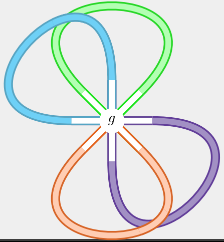
Notation

- Wick's theorem [L. Isserlis *Biometrika* 1918]: for zero-mean x_i's, ...

$$\mathbb{E}[x_{j_1}\cdots x_{j_{2n}}] := \langle x_{j_1}\cdots x_{j_{2n}}\rangle = \sum_{\substack{\pi\in P_2(2n)\\ \text{(pairings)}}} \prod_{(p,q)\in\pi} \langle x_{j_p} x_{j_q}\rangle$$

- k = number of Hermitian matrices of size N, $X_1^{(N)}, \dots, X_k^{(N)}$
- **Ribbon graphs:** For $\langle (X_\mu^{(N)})_{i,j}(X_\rho^{(N)})_{l,m}\rangle = N^{-1}\delta_{\mu\rho}\delta_{im}\delta_{jl}$ $\quad \mu,\rho = 1,\dots,k;\ i,\dots,m = 1,\dots,N$

$$(gN)\cdot\left\langle \mathrm{Tr}_N\left(X_1^{(N)}X_2^{(N)}X_1^{(N)}X_2^{(N)}X_3^{(N)}X_4^{(N)}X_3^{(N)}X_4^{(N)}\right)\right\rangle =$$



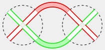
$$\sim N^{\chi(\Sigma_g)|_{g=2}} = N^{-2}$$

Functional Renormalisation for k-matrix models (w/multitrace-measures)

Quantum theories 'flow' with energy, here in RG-time $t = \log N$, $1 \ll N < \mathscr{N}$. E.g. for $k = 2$ and with bare action

$$S[A, B] = \mathscr{N}\,\mathrm{Tr}_{\mathscr{N}}\left\{\frac{1}{2}A^2 + \frac{1}{2}B^2 + g_{A^4}\frac{1}{4}A^4 + g_{B^4}\frac{1}{4}B^4 + \frac{1}{2}g_{ABAB}ABAB\right\}$$

A, B with $A^\dagger = A$, $B^\dagger = B$

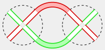
radiative corrections 'generate' *effective vertices*, e.g.  generates $\mathrm{Tr}\otimes\mathrm{Tr}(ABBA\otimes 1)$.

$$\Gamma_N[A, B] = \mathrm{Tr}_N\left\{\overbrace{\frac{Z_A}{2}A^2 + \frac{Z_B}{2}B^2 + \bar{g}_{A^4}\frac{1}{4}A^4 + \bar{g}_{B^4}\frac{1}{4}B^4 + \frac{1}{2}\bar{g}_{ABAB}ABAB}^{\text{operators from the bare action (but with 'running couplings')}} + \overbrace{\frac{1}{2}\bar{g}_{ABBA}ABBA + \frac{1}{2}\bar{g}_{A|A}\,\mathrm{Tr}_N(A)\times A + \dots}^{\text{radiative corrections}}\right\}$$

Functional Renormalisation for *k*-matrix models (w/multitrace-measures)

Quantum theories 'flow' with energy, here in RG-time $t = \log N$, $1 \ll N < \mathcal{N}$. E.g. for $k = 2$ and with bare action

$$S[A,B] = \mathcal{N}\,\mathrm{Tr}_{\mathcal{N}}\left\{\frac{1}{2}A^2 + \frac{1}{2}B^2 + g_{A^4}\frac{1}{4}A^4 + g_{B^4}\frac{1}{4}B^4 + \frac{1}{2}g_{ABAB}ABAB\right\} \qquad A,\ B \text{ with } A^\dagger = A,\ B^\dagger = B$$

radiative corrections 'generate' *effective vertices*, e.g.  generates $\mathrm{Tr}\otimes\mathrm{Tr}(ABBA\otimes 1)$.

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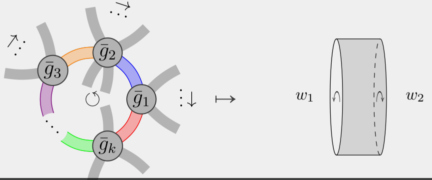
Effective vertex O_G^{eff}:

$$O_G^{\text{eff}} = \overbrace{\mathrm{Tr}_N(w_1)\times\mathrm{Tr}_N(w_2)\times\cdots\times\mathrm{Tr}_N(w_s)}^{\text{from pieces contracted with propagators}}$$
$$\times\underbrace{\mathrm{Tr}_N(U_1)\times\mathrm{Tr}_N(U_2)\dots\times\mathrm{Tr}_N(U_r)}_{\text{from pieces uncontracted with propagators}}$$

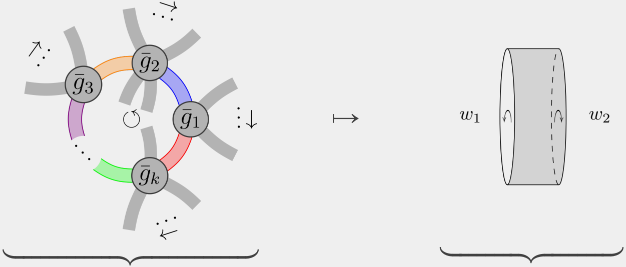


Comment on the FRGE on the *ABAB*-model:
[CP *J. High Energ. Phys* 2021]

gral. situation (yet, without multitraces)

inner word $w_1 \otimes$ outer word w_2

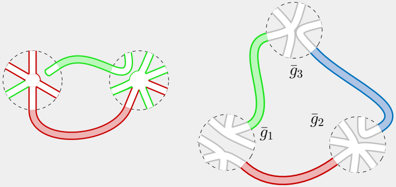
So the actual question is: find the pre-image of the map



gral. situation (yet, without multitraces)

inner word $w_1 \otimes$ outer word w_2

For multitrace operators, pre-image of $O = \prod_\alpha \mathrm{Tr}_N \, w_\alpha$:



One then sums over the product of all g_j's appearing in such 1-loops.
These polynomials span the β-function for O.

Two steps

1. Understanding the FRGE

[CP 2007.10914, *Ann. Henri Poincaré* 2021]

- prove Wetterich Equation, or FRGE; its proof determines the algebra that governs geometric series in the Hessian of Γ
- [A. Eichhorn, T. Koslowski, *Phys. Rev. D* '13] oriented us, but the proof of the FRGE dictates an **algebra not reported there**
- β-equations found for a sextic truncation (48 running operators). For the unique real solution g^* leading to a single relevant direction (positive e.v. of $-(\partial\beta_i/\partial g_j)_{i,j}|_{g^*}$) yields ⊳⊳go to β-functions $g^*_{A^4} = 1.002\times\left(g^*_{A^4}\big|_{\text{[Kazakov-Zinn-Justin, } \textit{Nucl. Phys. B} \text{ '99]}}\right)$

2. Unicity (using a ribbon graph argument)

[CP 2111.02858 *Lett. Math. Phys.* 2022]

- write down Wetterich Equation "$\dot\Gamma = \frac{1}{2}\operatorname{Tr}_{M_k(\mathcal{A})}\left\{\dot R_N/[\Gamma^{(2)}+R_N]\right\}$"
- assume an expansion of its rhs in unitary-invariant operators ($\neq$ exact RG)
- impose the one-loop structure and solve for the algebra $\mathcal{A}=\mathcal{A}_{k,N}$
- determine from it the algebra that computes Wetterich equation; it is unique and the one reported before in [CP' 21] (cf. left column)

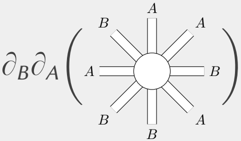
- *nc-derivative* $\partial_A : \mathbb{C}_{\langle k \rangle} \to \mathbb{C}_{\langle k \rangle}^{\otimes 2}$ sums over ‘replacements of A by $\otimes$’ [Turnbull+Rota-Sagan-Stein+Voiculescu]:

$$\begin{aligned} \partial_A(PAAR) &= P \otimes AR + PA \otimes R\,, \text{ but} \\ \partial_A(ALGEBRA) &= 1 \otimes LGEBRA + ALGEBR \otimes 1 \end{aligned}$$

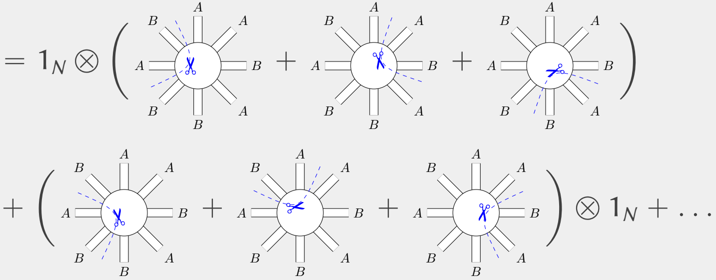
- this is not enterely abstract, just ‘take entries’ of the matrices:

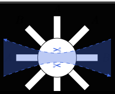
$$\frac{\partial}{\partial A_{b,c}}(W)_{a,d} = (\partial_A W)_{ab;cd}$$

- $W \in \mathbb{C}_{\langle k \rangle}$, the *nc-Hessian* [CP '21] Hess $\mathrm{Tr}_N\ W \in M_k(\mathbb{C}_{\langle k \rangle} \otimes \mathbb{C}_{\langle k \rangle})$ has entries are $\mathrm{Hess}_{b,a}\,\mathrm{Tr}\ W = (\partial_{X_b} \circ \partial_{X_a})\,\mathrm{Tr}_N\ W$. Are computed by 'cuts': e.g. $W = ABAABABB$



go to examples of nc-Hessians ∇

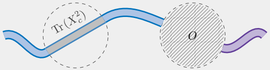


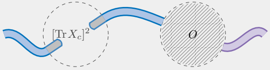
in ellipsis $\sum_{cuts}$ like  $\rightarrow BAA \otimes ABB$

- products of traces ⇒ extend by $\boxtimes$, $\mathcal{A}_k = (\mathbb{C}_{\langle k\rangle} \otimes \mathbb{C}_{\langle k\rangle}) \oplus (\mathbb{C}_{\langle k\rangle} \boxtimes \mathbb{C}_{\langle k\rangle})$

$$\mathrm{Hess}_{a,b}(\mathrm{Tr}\, P \cdot \mathrm{Tr}\, Q) = \mathrm{Tr}\, P \cdot \mathrm{Hess}_{a,b}[\mathrm{Tr}\, Q] + (\partial_{X_a} \mathrm{Tr}\, P) \boxtimes (\partial_{X_b} \mathrm{Tr}\, Q) + (P \leftrightarrow Q)$$

- products of traces ⇒ extend by $\boxtimes$, $\mathcal{A}_k = (\mathbb{C}_{\langle k\rangle} \otimes \mathbb{C}_{\langle k\rangle}) \oplus (\mathbb{C}_{\langle k\rangle} \boxtimes \mathbb{C}_{\langle k\rangle})$

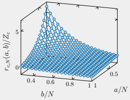




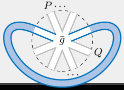
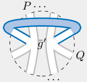
$$\mathrm{Hess}_{a,b}(\mathrm{Tr}\,P \cdot \mathrm{Tr}\,Q) = \mathrm{Tr}\,P \cdot \mathrm{Hess}_{a,b}[\mathrm{Tr}\,Q] + (\partial_{X_a} \mathrm{Tr}\,P) \boxtimes (\partial_{X_b} \mathrm{Tr}\,Q) + (P \leftrightarrow Q)$$

- Wetterich Eq. governs the functional RG with time $t = \log N$

$$\partial_t \Gamma_N[\mathbb{X}] = \frac{1}{2} \mathrm{Tr}_k \otimes \mathrm{Tr}_{\mathcal{A}_n} \left\{ \frac{\partial_t R_N}{\mathrm{Hess}\,\Gamma_N[\mathbb{X}] + R_N} \right\}$$



$$\overset{\text{piecewise cte. } R_N}{=} \sum_{k=0}^{\infty} \bar{h}_k(N, \eta_1, \dots, \eta_n) \times \underbrace{\frac{1}{2}(-1)^k \mathrm{Tr}_{M_k(\mathcal{A})} \left\{ (\mathrm{Hess}\,\Gamma_N^{\textsc{Int}}[\mathbb{X}])^{\star k} \right\}}_{\text{regulator-independent part}}$$

- From   $\Rightarrow \mathrm{Tr}_{\mathcal{A}_n}(P \otimes Q) = \mathrm{Tr}_N\,P \cdot \mathrm{Tr}_N\,Q$, $\mathrm{Tr}_{\mathcal{A}_n}(P \boxtimes Q) = \mathrm{Tr}_N(PQ)$

Thm. [CP '22] If the RG-flow is computable in terms of $\mathrm{U}(N)$-invariants, the algebra of Functional Renormalisation is $M_k(\mathcal{A}_{k,N},\star)$ where

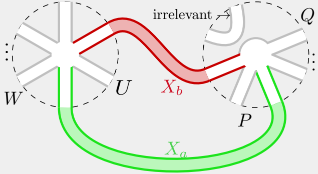
$$\mathcal{A}_{k,N} = (\mathbb{C}^{(N)}_{\langle k\rangle} \otimes \mathbb{C}^{(N)}_{\langle k\rangle}) \oplus (\mathbb{C}^{(N)}_{\langle k\rangle} \boxtimes \mathbb{C}^{(N)}_{\langle k\rangle})$$

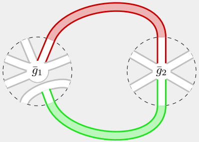
whose product in hom. elements reads:

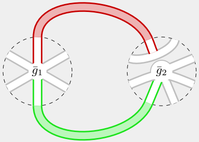
$$\begin{aligned}
(U \otimes W) \star (P \otimes Q) &= PU \otimes WQ\,,\\
(U \boxtimes W) \star (P \otimes Q) &= U \boxtimes PWQ\,,\\
(U \otimes W) \star (P \boxtimes Q) &= WPU \boxtimes Q\,,\\
(U \boxtimes W) \star (P \boxtimes Q) &= \mathrm{Tr}_N(WP)\, U \boxtimes Q\,,
\end{aligned}$$

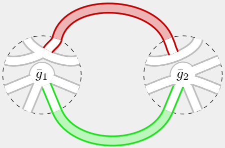
and traces $\mathrm{Tr}_k \otimes \mathrm{Tr}_{\mathcal{A}_k}$

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{A}_n}(P \otimes Q) &= \mathrm{Tr}_N\, P \cdot \mathrm{Tr}_N\, Q\,,\\
\mathrm{Tr}_{\mathcal{A}_n}(P \boxtimes Q) &= \mathrm{Tr}_N(PQ)\,.
\end{aligned}$$







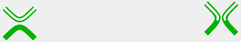


Remark: To be more precise, any occurrence of the free algebra in $\mathcal{A}_{k,N}$ should be replaced by the algebra of 'trace polynomials' (e.g. $\mathrm{Tr}_N(X_1X_3)X_2 + N\,\mathrm{Tr}_N(X_2^2)$) [D. Jekel-W. Li-D. Shlyakhtenko, '21]

Example: a Hermitian 3-matrix model

Consider two operators $O_1 = \frac{\bar g_1}{2}[\mathrm{Tr}_N(\frac{A^2}{2})]^2$ and $O_2 = \bar g_2\,\mathrm{Tr}_N(ABC)$. We compute $g_1 g_2^2$-coefficients:

$$\mathrm{Hess}_{I,J}\, O_1 = \delta^J_I \delta^A_I \bar g_1 \{\underbrace{\mathrm{Tr}_N(A^2/2)\cdot[1_N\otimes 1_N]}_{} + \underbrace{A\boxtimes A}_{}\}\,,$$



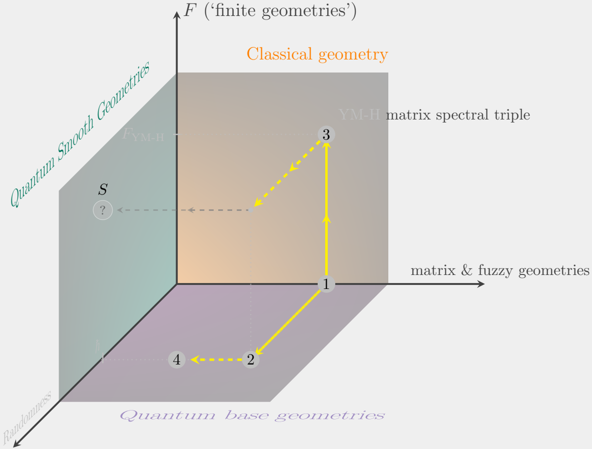
where a ‘filled ribbon’ means contracted in the one-loop graph, and ‘empty ribbon’ uncontracted.

$$\mathrm{Hess}\, O_2 = \bar g_2 \begin{bmatrix} 0 & C\otimes 1_N & B\otimes 1_N \\ 1_N\otimes C & 0 & A\otimes 1_N \\ 1_N\otimes B & 1_N\otimes A & 0 \end{bmatrix} \Rightarrow [\mathrm{Hess}\, O_2]^{\star 2} = \bar g_2^2 \begin{bmatrix} \overbrace{C\otimes C}^{} + \overbrace{B\otimes B}^{} & B\otimes A & C\otimes A \\ A\otimes B & A\otimes A + C\otimes C & C\otimes B \\ A\otimes C & B\otimes C & B\otimes B + A\otimes A \end{bmatrix}.$$

$$[\bar g_1 \bar g_2^2]\,\mathrm{STr}\{\mathrm{Hess}\, O_1 \star [\mathrm{Hess}\, O_2]^{\star 2}\} = \mathrm{Tr}_N(A^2/2)\times\left[(\mathrm{Tr}_N\, C)^2 + (\mathrm{Tr}_N\, B)^2\right] + \mathrm{Tr}_N(ACAC + ABAB)\,.$$

These are effective vertices of the four one-loop graphs that can be formed with the contractions of (the filled ribbon half-edges of) any of $\Big\{$  $\Big\}$ with any of $\Big\{$  $\Big\}$

CONCLUSION: SOME PROGRESS



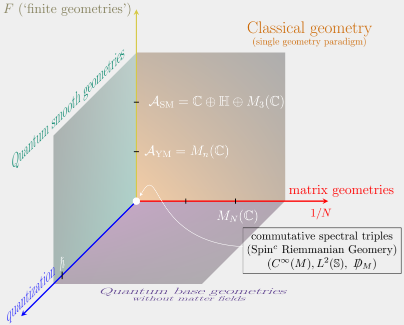
1. Matrix Geometries [Barrett, *J. Math. Phys.* '15]
2. Dirac Ensembles [Barrett-Glaser, *J. Phys A*, '16] and how to compute the spectral action [CP '19]
3. Gauge matrix spectral triples [CP '22a]
4. Functional Renormalisation (Wetterich equation) in formal random matrix theory [CP '21a, CP '21b] and [CP '22b]

References: [CP *J. Noncommut. Geom.* 2023] on the spectral action, [CP *Ann. Henri Poincaré 2022*] on Yang-Mills-Higgs.
Related: [CP *Ann. Henri Poincaré 2021*] on Wetterich Eq., [CP *J. High Energ. Phys* 2021] [CP *Lett. Math. Phys.* 2022] on algebra and FRG

CONCLUSION

- small step towards [Eq. 1.892, Connes, Marcolli, *NCG, QFT and motives*, 2007]

> « *The far distant goal is to set up a functional integral evaluating spectral observables* $\mathscr{S}$ $\quad \langle \mathscr{S} \rangle = \int \mathscr{S}\, \mathrm{e}^{-\operatorname{Tr} f(D/\Lambda) - \frac{1}{2}\langle J\psi, D\psi\rangle + \rho(e,D)}\, \mathrm{d}e\, \mathrm{d}\psi\, \mathrm{d}D$ »



OUTLOOK (PHYSICS)

- Random/Quantum YM
- Missing is a fully mathematicallly correct FRG for arbitrary regulator
- Tensor Models & FRG (MSc. thesis of Leena Tharwat, on the job market)
- Software for Graph Theory & FRG (MSc. thesis of. Niels Gehring)

Thanks for listening!

References: [CP *J. Noncommut. Geom.* 2023] on the spectral action, [CP *Ann. Henri Poincaré 2022*] on Yang-Mills-Higgs.
Related: [CP *Ann. Henri Poincaré 2021*] on Wetterich Eq., [CP *J. High Energ. Phys* 2021] [CP *Lett. Math. Phys.* 2022] on algebra and FRG

Classical Dirac operators (assume d even)

- M (space~~time~~) will be a closed, Riemannian manifold
- if M is spin, there is a vector bundle $\mathbb{S}$ with fibers satisfying $\mathrm{End}(\mathbb{S}_x) \cong \mathbb{C}\ell(d)$ ($x \in M$). The sections $\Gamma(\mathbb{S})$ are spinors
- the Levi-Civita connection ∇^{LC} can be also lifted to the *spin connection* $\nabla^s : \Gamma(\mathbb{S}) \to \Omega^1(M) \otimes \Gamma(\mathbb{S})$

$$\nabla^s c(\omega)\psi = c(\nabla^{\mathrm{LC}}\omega)\psi + c(\omega)\nabla^s\psi$$

$$\psi \in \Gamma(\mathbb{S}), \omega \in \Omega^1(M)$$

being c Clifford multiplication, basically $c(\mathrm{d}x^\mu) = \gamma^\mu$

- on the space of square integrable spinors $L^2(M, \mathbb{S})$ there is an (ess.) self-adjoint operator, the *Dirac operator*,

$$D_M = -\mathrm{i}c \circ \nabla^s \overset{\text{loc.}}{=} -\mathrm{i} \sum_{\mu=1}^{d} \gamma^\mu(\partial_\mu + \omega_\mu)$$

and by Leibniz rule

$$[D_M, a] = -\mathrm{i}c(\mathrm{d}a) \quad a \in C^\infty(M)$$

which is bounded

`back to ‘spectral triples’` ⋘

Sketch of the Standard Model derivation from NCG [Chamseddine, Connes, Marcolli *ATMP* ’07]

One starts with the $M \times_{\text{s.t.}} F$ and $\mathcal{A}_{LR} = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C})$

- $F = (\mathcal{A}_{LR}, \mathcal{M}_F^{\#\text{generations}}, D_F)$, $\mathcal{M}_F$ an $\mathcal{A}_{LR}$-module
- $\mathcal{M}_F$ has to be of the form $\mathcal{M}_F = \mathcal{E} \otimes \mathcal{E}^o$, with

$$\mathcal{E} = (2_L \otimes 1^o) \oplus (2_R \otimes 1^o) \oplus (2_L \otimes 3^o) \oplus (2_L \otimes 3^o)\,, \quad \dim_{\mathbb{C}} \mathcal{E} = 16$$

- Thus the $\mathcal{H}_F \cong \mathbb{C}^{32\times 3}$. The 96×96 matrix D_F can have off-diagonal elements only for the maximal subalgebra

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

- Lie group part of $\mathrm{SU}(\mathcal{A}_F) = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$

Sketch of the Standard Model derivation from NCG

With $Q : \mathbb{C} \hookrightarrow \mathbb{H}$, $Q_\lambda = \mathrm{diag}(\lambda, \bar{\lambda})$ and $Q_\lambda \, |\pm\rangle = \pm\lambda \, |\pm\rangle$,

- Weak hypercharge:

	ν	e	u	d
Y	$\|+\rangle \otimes 1^o$	$\|-\rangle \otimes 1^o$	$\|+\rangle \otimes 3^o$	$\|-\rangle \otimes 3^o$
L	-1	-1	$+1/3$	$+1/3$
R	0	-2	$+4/3$	$-2/3$

- $\mathrm{SU}(2)$-adjoint action is 2 on $\mathcal{H}_L$ or trivial in the $\mathcal{H}_R$ sector
- $\mathrm{SU}(3)$-adjoint action is the color action on $\mathcal{H}_q$ and trivial on $\mathcal{H}_\ell$

$$\mathrm{Lie}(\mathrm{SU}(\mathcal{A}_F)) = \mathrm{U}(1)_Y \times \mathrm{SU}(2)_L \times \mathrm{SU}(3)_{\mathrm{color}}$$

- All D_F such that $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ is a spectral triple are

$$D_F(\Upsilon_R, \Upsilon_\nu, \Upsilon_e, \Upsilon_u, \Upsilon_d) \qquad \dim\{\text{ Dirac operators}\} = 31 = \#\text{ Yukawa coupl. in } \nu\text{MSM}$$

Fermionic Spectral Action

- The fermionic part is not treated here but is essentially given by [not needed here]

$$S_f(D) = \frac{1}{2}\langle J\psi \mid D\psi\rangle$$

where ψ are classical fermions, J implements charge-conjugation (J fixes the spin structure)

Dirac F_{SM} operator

$$D_F = \left(\begin{array}{cccccccccccccccccc}
0&0&\Upsilon_\nu^*&0&0&0&0&0&0&\Upsilon_R^*&0&0&0&0&0&0&0&0\\
0&0&0&\Upsilon_e^*&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\Upsilon_\nu&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&\Upsilon_e&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&\Upsilon_u^*&0&&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\Upsilon_d^*&\otimes 1_3&0&0&0&0&0&0&0&0&0\\
0&0&0&0&\Upsilon_u&0&0&0&&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\Upsilon_d&0&0&&0&0&0&0&0&0&0&0&0\\
\Upsilon_R&0&0&0&0&0&0&0&0&0&0&\Upsilon_\nu^T&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&\Upsilon_e^T&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&\bar{\Upsilon}_\nu&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&\bar{\Upsilon}_e&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\Upsilon_u^T&0&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\Upsilon_d^T&\otimes 1_3\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\bar{\Upsilon}_u&0&0&0&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\bar{\Upsilon}_d&0&0&
\end{array}\right) \in M_{96}(\mathbb{C})_{\mathtt{s.a.}}$$

$\sim 10^4$ zeroes from geometry.

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Operator	Its noncommutative Hessian
$\operatorname{Tr}(A)\operatorname{Tr}(A^3)$	$3\cdot\begin{pmatrix} \operatorname{Tr} A\cdot(A\otimes 1+1\otimes A) +1\boxtimes A^2+A^2\boxtimes 1 & 0 \\ 0 & 0 \end{pmatrix}$
$\operatorname{Tr} A\operatorname{Tr}(AAABB)$	$\begin{pmatrix} (\operatorname{Tr}(A)1\otimes(ABB)+\operatorname{Tr}(A)1\otimes(BBA)+\operatorname{Tr}(A)(ABB)\otimes 1+\operatorname{Tr}(A)(BBA)\otimes 1+1\boxtimes(AABB)+1\boxtimes(ABBA)+1\boxtimes(BBAA)+(AABB)\boxtimes 1+(ABBA)\boxtimes 1+(BBAA)\boxtimes 1+\operatorname{Tr}(A)A\otimes B^2+\operatorname{Tr}(A)B^2\otimes A) & \operatorname{Tr}(A)(1\otimes(AAB)+(BAA)\otimes 1+A\otimes AB+B\otimes A^2+A^2\otimes B+BA\otimes A)+1\boxtimes(A^3B)+1\boxtimes(BA^3) \\ \operatorname{Tr}(A)(1\otimes(BAA)+(AAB)\otimes 1+A\otimes BA+B\otimes A^2+A^2\otimes B+AB\otimes A)+(A^3B)\boxtimes 1+(BA^3)\boxtimes 1 & \operatorname{Tr}(A)(1\otimes(A^3)+(A^3)\otimes 1) \end{pmatrix}$

Table: Some Hessians of operators. Here $\operatorname{Tr}=\operatorname{Tr}_N$.

β-functions of NCG two-matrix models, signature $\eta = \mathrm{diag}(e_1, e_2)$

$$2h_1(\mathrm{a}_4 + \mathrm{c}_{22} + 2\mathrm{d}_{2|02} + 6\mathrm{d}_{2|2}) = \eta_\mathrm{a}$$

$$2h_1(\mathrm{b}_4 + \mathrm{c}_{22} + 6\mathrm{d}_{02|02} + 2\mathrm{d}_{2|02}) = \eta_\mathrm{b}$$

$$-h_1[e_\mathrm{a}(\mathrm{a}_4 - \mathrm{c}_{1111}) + 2\mathrm{d}_{1|12} + 6\mathrm{d}_{1|3}] + \mathrm{d}_{1|1}(\eta + 1) = \beta(\mathrm{d}_{1|1})$$

$$-h_1[e_\mathrm{b}(\mathrm{b}_4 - \mathrm{c}_{1111}) + 6\mathrm{d}_{01|03} + 2\mathrm{d}_{01|21}] + \mathrm{d}_{01|01}(\eta + 1) = \beta(\mathrm{d}_{01|01})$$

The next block encompasses the connected quartic couplings:

$$\begin{aligned} & h_2\left(4\mathrm{a}_4^2 + 4\mathrm{c}_{22}^2\right) + \mathrm{a}_4(2\eta + 1) \\ & -h_1(24\mathrm{a}_6 e_\mathrm{a} + 4\mathrm{c}_{42} e_\mathrm{b} + 4\mathrm{d}_{02|4} e_\mathrm{b} + 4\mathrm{d}_{2|4} e_\mathrm{a}) = \beta(\mathrm{a}_4) \end{aligned}$$

$$\begin{aligned} & h_2\left(4\mathrm{b}_4^2 + 4\mathrm{c}_{22}^2\right) + \mathrm{b}_4(2\eta + 1) \\ & -h_1(24\mathrm{b}_6 e_\mathrm{b} + 4\mathrm{c}_{24} e_\mathrm{a} + 4\mathrm{d}_{02|04} e_\mathrm{b} + 4\mathrm{d}_{2|04} e_\mathrm{a}) = \beta(\mathrm{b}_4) \end{aligned}$$

$$\begin{aligned} & -h_1\left(2e_\mathrm{a}\mathrm{c}_{1212} + e_\mathrm{b}2\mathrm{c}_{2121} + 3e_\mathrm{a}\mathrm{c}_{24} + 3e_\mathrm{b}\mathrm{c}_{42} + e_\mathrm{a}\mathrm{d}_{02|22} + e_\mathrm{b}\mathrm{d}_{2|22}\right) \\ & +h_2\left(2\mathrm{a}_4\mathrm{c}_{22} + 2\mathrm{b}_4\mathrm{c}_{22} + 2e_\mathrm{a}e_\mathrm{b}\mathrm{c}_{1111}^2 + 2e_\mathrm{a}e_\mathrm{b}\mathrm{c}_{22}^2\right) + \mathrm{c}_{22}(2\eta + 1) = \beta(\mathrm{c}_{22}) \end{aligned}$$

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$$2h_2(6\mathrm{a}_4\mathrm{a}_6 + e_\mathrm{a}e_\mathrm{b}\mathrm{c}_{22}\mathrm{c}_{42}) + \mathrm{a}_6(3\eta + 2) = \beta(\mathrm{a}_6)$$

$$2h_2(6\mathrm{b}_4\mathrm{b}_6 + e_\mathrm{a}e_\mathrm{b}\mathrm{c}_{22}\mathrm{c}_{24}) + \mathrm{b}_6(3\eta + 2) = \beta(\mathrm{b}_6)$$

$$\begin{aligned}4h_2\{\mathrm{a}_4\mathrm{c}_{3111} + e_\mathrm{a}e_\mathrm{b}[\mathrm{c}_{22}(\mathrm{c}_{1311} + 2\mathrm{c}_{3111})\\ -\mathrm{c}_{1111}(2\mathrm{c}_{2121} + \mathrm{c}_{42})]\} + \mathrm{c}_{3111}(3\eta + 2) &= \beta(\mathrm{c}_{3111})\end{aligned}$$

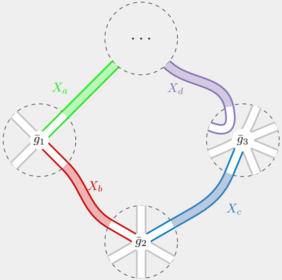
$$\begin{aligned}2h_2[2\mathrm{a}_4\mathrm{c}_{2121} + e_\mathrm{a}e_\mathrm{b}(-2\mathrm{c}_{1111}\mathrm{c}_{3111}\\ +4\mathrm{c}_{2121}\mathrm{c}_{22} + \mathrm{c}_{22}\mathrm{c}_{24})] + \mathrm{c}_{2121}(3\eta + 2) &= \beta(\mathrm{c}_{2121})\end{aligned}$$

$$\begin{aligned}2h_2[\mathrm{a}_4\mathrm{c}_{24} + 3\mathrm{b}_4\mathrm{c}_{24} + 2e_\mathrm{a}e_\mathrm{b}(\mathrm{c}_{22}(3\mathrm{b}_6 + \mathrm{c}_{2121} + \mathrm{c}_{24} + \mathrm{c}_{42})\\ -\mathrm{c}_{1111}\mathrm{c}_{1311})] + \mathrm{c}_{24}(3\eta + 2) &= \beta(\mathrm{c}_{24})\end{aligned}$$

$$\begin{aligned}4h_2\{\mathrm{b}_4\mathrm{c}_{1311} + e_\mathrm{a}e_\mathrm{b}[\mathrm{c}_{22}(2\mathrm{c}_{1311} + \mathrm{c}_{3111})\\ -\mathrm{c}_{1111}(2\mathrm{c}_{1212} + \mathrm{c}_{24})]\} + \mathrm{c}_{1311}(3\eta + 2) &= \beta(\mathrm{c}_{1311})\end{aligned}$$

$$\begin{aligned}2h_2[2\mathrm{b}_4\mathrm{c}_{1212} + e_\mathrm{a}e_\mathrm{b}(\mathrm{c}_{22}(4\mathrm{c}_{1212} + \mathrm{c}_{42})\\ -2\mathrm{c}_{1111}\mathrm{c}_{1311})] + \mathrm{c}_{1212}(3\eta + 2) &= \beta(\mathrm{c}_{1212})\end{aligned}$$

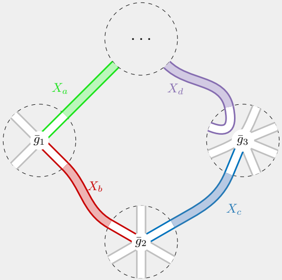
Finding $\star$

Want: 

$\subset \mathrm{Hess}_{a,b}\, O_1 \star \mathrm{Hess}_{b,c}\, O_2 \star \mathrm{Hess}_{c,d}\, O_3 \star \dots \star \mathrm{Hess}_{*,a}\, O_\ell$

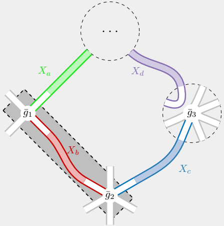
Finding ⋆

Want:

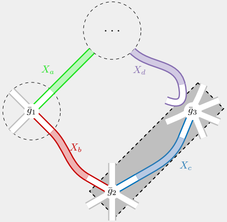


$\subset \mathrm{Hess}_{a,b}\, O_1 \star \mathrm{Hess}_{b,c}\, O_2 \star \mathrm{Hess}_{c,d}\, O_3 \star \dots \star \mathrm{Hess}_{*,a}\, O_\ell$

Associativity (trivial check):



$=$



FUZZY OR MATRIX GEOMETRIES

A *fuzzy geometry* of signature (p, q), so $\eta = \mathrm{diag}(+_p, -_q)$, consists of

- $A = M_N(\mathbb{C})$
- $H = \mathbb{S} \otimes M_N(\mathbb{C})$, with $\mathbb{S}$ a $\mathbb{C}\ell(p, q)$-module

. . . +axioms (omitted) that can be solved for D . . .

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- Fixing conventions for γ's, D in even dimensions: [Barrett, *J. Math. Phys.* '15]

$$D = \sum_J \Gamma^J_{\text{s.a.}} \otimes \{H_J, \ \cdot\ \} + \sum_J \Gamma^J_{\text{anti.}} \otimes [L_J, \ \cdot\ ]$$

multi-index J monot. increasing, $|J|$ odd, $H_J^* = H_J$, $L_J^* = -L_J$

Fuzzy or Matrix Geometries

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⋘back

- Examples:
  - $D_{(1,1)} = \gamma^1 \otimes [L, \ \cdot \ ] + \gamma^2 \otimes \{H, \ \cdot \ \}$
  - $D_{(0,4)} = \sum_\mu \gamma^\mu \otimes [L_\mu, \ \cdot \ ] + \gamma^{\hat{\mu}} \otimes \{H_{\hat{\mu}}, \ \cdot \ \}$ ($\hat{\mu} =$ omit μ from (0123))

so we will get double traces from $\mathrm{Tr}_H = \mathrm{Tr}_{\mathbb{S}} \otimes \mathrm{Tr}_{M_N(\mathbb{C})} = \mathrm{Tr}_{\mathbb{S}} \otimes \mathrm{Tr}_N^{\otimes 2}$

Notation: $\mathrm{Tr}_V\, X$ is the trace of $X : V \to V$, $\mathrm{Tr}_V\, 1 = \dim V$. So $\mathrm{Tr}_N\, 1 = N$ but $\mathrm{Tr}_{M_N^{\mathbb{C}}}\, 1 = N^2$.