AN INVITATION TO DIRAC ENSEMBLES IN (RANDOM FINITE) NONCOMMUTATIVE GEOMETRY

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- *Noncommutative geometry* or NCG [Co94] trades geometry by algebra. If this algebra is noncommutative one can study the geometry of broader class of spaces (fractals, Penrose tilings,...)
- we focus on some NCG-methods in high energy physics; for details on this first half-page of motivation see [vS15, Mar18]. Other physical NCG-applications, e.g. to the quantum Hall effect and to topological insulators are not treated here, cf. [BvES94] and [BKR17] respectively
- the NCG-analogue of (spin^c) Riemannian geometry is called spectral triple $(\mathcal{A}, \mathcal{H}, D)$, which consists of a $*$-algebra $\mathcal{A}$ represented on a Hilbert space $\mathcal{H}$ and a self-adjoint operator D on $\mathcal{H}$ obeying several axioms. Given a «nice» spin manifold M, one gets a spectral triple $(C^\infty(M), L^2(S), D_M)$ from the algebra $C^\infty(M)$ of smooth functions on M, the Hilbert space of square-integrable spinors and the canonical Dirac (ess. self-adj.) operator D_M on $L^2(S)$
- any commutative spectral triple is a manifold, due to Connes' reconstruction theorem (hard fact, here very imprecisely formulated). In the broader landscape where noncommutativity is allowed, the several concepts of dimension (metric, KO-theoretical...) need not to agree [Mar18]. There is, further, a *spectral dimension* set SpDim obtained from poles of ζ-functions $\zeta_D(z) = \mathrm{Tr}(|D|^{-z})$ of the Dirac operator. For each $s \in \mathrm{SpDim}$ there is a well-defined volume and $⨏$, its integration. In terms of these, the asymptotic expansion (here only in powers, but log-terms can be present [KS12] for other geometries) of the Chamseddine-Connes *spectral action* in an «energy» parameter $N \gg 1$ reads:

$$\mathrm{Tr}\, f(D/N) \sim \sum_{s \in \mathrm{SpDim} \cap \mathbb{R}_+} f_s N^s ⨏ |D|^{-s} + f(0)\zeta_D(0) + \ldots \quad (f = \text{Laplace-Stieltjes t. of a measure on } \mathbb{R}_+)$$

- for a 4-manifold the coefficients of the moments $f_s = \int_0^\infty f(v) v^{s-1} \mathrm{d}v$ of f are

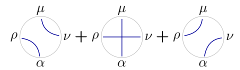
$$N^4 ⨏ |D|^{-4} = c_4(N)\mathrm{vol}(M) \qquad \text{[cosmological constant]}$$
$$N^2 ⨏ |D|^{-2} = c_2(N) ⨏ R \qquad \text{[Einstein-Hilbert]}$$
$$\zeta_D(0) = c_0 ⨏ (R^*R^*) + c_0' ⨏ C^2 \qquad \text{[Gauß-Bonnet + conformal gravity]}$$

  The Cartesian product of a *finite spectral triple* (i.e. one whose algebra is finite-dimensional algebra) with a commutative one allows to geometrically derive the (Lagrangian of the) Standard Model [CM07] on a Riemannian manifold, for a suitable finite spectral triple. *However, the resulting theory is classical*. As initiated by [BG16], there also with computer simulations, the aim is to define the partition function $\mathcal{Z}_{\text{NCG}} = \int_{\text{Dirac}} \mathrm{e}^{-\frac{1}{\hbar}\mathrm{Tr}\, f(D/N)} \mathrm{d}D$ mentioned in [CM07, §18]. On a first approach, we circumvent analytic subtleties during quantization by using fuzzy geometries
- A *fuzzy geometry of signature* $\eta = \mathrm{diag}(\overbrace{+1, \ldots, +1}^{p}, \overbrace{-1, \ldots, -1}^{q})$ is based on $\mathcal{A} = M_N(\mathbb{C})$ and a Hilbert space $\mathcal{H} =$ (irreducible $\mathbb{C}\ell(p,q)$-module $\mathbb{S}$) $\otimes M_N(\mathbb{C})$. Letting $\{A, B\}_\pm = AB \pm BA$, the spectral triple axioms force [Bar15] the Dirac operator to be

$$D = \sum_\mu \gamma^\mu \otimes \{X_\mu, \cdot\,\}_{\epsilon_\mu} + \sum_{\mu,\nu,\rho} \underbrace{\gamma^\mu\gamma^\nu\gamma^\rho}_{=:\gamma^I \; I=(\mu,\nu,\rho)} \otimes \{X_{\mu\nu\rho}, \cdot\,\}_{\epsilon_{\mu\nu\rho}} + \ldots, \qquad X_\mu, X_I, \ldots \in M_N(\mathbb{C})$$

- the traces of products of γ's can be organized diagrammatically, e.g.

$$\mathrm{Tr}_{\mathbb{S}}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\rho) = \dim \mathbb{S} \cdot \Big\{ \ldots \Big\}$$



  where each labelled line μ—ν in a *chord diagram* χ amounts to $\eta^{\mu\nu}$. The diagram's value is the product of all its chords, with a general sign $(-1)^{\#\{\text{crossings in } \chi\}}$. Generally, $\mathrm{Tr}_{\mathbb{S}}(\gamma^{I_1}\gamma^{I_2}\cdots)$ leads to (multi)indices $\chi^{I_1 I_2 \cdots}$. For $f(x) = \sum_m f_m x^m$, there is an expansion in chord diagrams

$$\mathrm{Tr}\, f(D) = \sum_m f_{2m} \sum_{\substack{I_1, \ldots, I_{2m} \\ \chi \in \{n\text{-chorded diags}\} \\ n = |I_1| + \ldots + |I_{2m}|}} \chi^{I_1 \ldots I_{2m}} \Big\{ \mathrm{Tr}_N \left[X_{I_1} X_{I_2} \cdots X_{I_{2m}} \pm \text{ word backwards} \right] + \text{ double traces} \Big\}$$

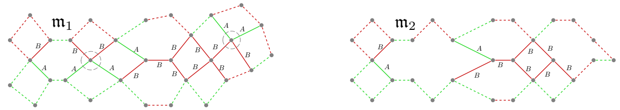
- in terms of the k-tuple $\mathbb{X} = (X_1, \ldots, X_k)$, $k = 2^{p+q-1}$, the spectral action takes the form

$$\mathrm{Tr}\, f(D) = \mathrm{Tr}_N^{\otimes 2} \{1_N \otimes P + Q_{(1)} \otimes Q_{(2)}\} \quad \text{where } P, Q_1, Q_2 \in \mathbb{C}_{\langle k \rangle} = \mathbb{C}\langle \mathbb{X} \rangle\,.$$

  For 2-dimensional fuzzy geometries $(p + q = 2)$, allowed monomials are:

$$P \in \mathrm{span}\{\cancel{A}, \cancel{B}, A^2, B^2, \cancel{AB}, ABAB, AABB, AAABAB, \cancel{ABABAB}, \ldots\}$$
$$Q_1 \otimes Q_2 \in \mathrm{span}\{A \otimes A, B \otimes B, B \otimes ABA, BA \otimes BA \ldots\} \quad \text{(insertions of } \otimes \text{ in the words above)}$$

  which is obvious, as chord diagrams select these polynomials. However, for the spectral action of 4-dimensional $(p+q = 4)$ fuzzy geometries determined in [Pér19] the allowed NC-polynomials are less predictable. The «*quantum spectral action*» becomes a random k-matrix model $\mathcal{Z} = \int \mathrm{e}^{-\mathrm{Tr}\, f(D)} \mathrm{d}\mathbb{X}_{\text{Leb}}$ over Hermitian and anti-Hermitian $N \times N$ matrices. This partition function generates «worded» maps; below, two planar maps $\mathfrak{m}_1$ and $\mathfrak{m}_2$ in the alphabet consisting of A and B are shown:



(ignore dashed circles by now)

- a *gauge matrix spectral triple* = fuzzy spectral triple $\times$ finite spectral triple; the most general *fluctuated* Dirac operator is (with $A_\mu \in \Omega^1_D(M_N(\mathbb{C})) = \{\sum_i a_i[D, b_i] \mid a_i, b_i \in \mathcal{A}\}, c \in M_n(\mathbb{C})_{\text{s.a}}$)

$$D = \sum_\mu \gamma^\mu \otimes \Big(\overbrace{[L_\mu \otimes 1_n, \cdot\,]}^{\ell_\mu} + \overbrace{[A_\mu \otimes c, \cdot\,]}^{a_\mu} \Big) + \gamma \otimes \Phi + \overbrace{\cancel{\textstyle\sum_{\mu,\nu,\sigma} \gamma^\mu\gamma^\nu\gamma^\sigma \otimes x_{\mu\nu\sigma}}}^{\text{(if flat; room for gravitation)}}$$

- the operators ℓ_μ, a_μ serve to define the fuzzy *field strength* $\mathscr{F}_{\mu\nu} = [d_\mu, d_\nu]$. Here $d_\mu = \ell_\mu + a_\mu$ is seen as fuzzy analogue of smooth covariant derivative $\mathbb{D}_\mu = \partial_\mu + \mathbb{A}_\mu$ (locally $\mathbb{A}_\mu$ is a connection on SU(n)-principal bundle and $\mathbb{F}_{\mu\nu} = [\mathbb{D}_\mu, \mathbb{D}_\nu]$ its curvature)
- physically, «gauge matrix spectral triple» means that can have Yang-Mills on a fuzzy space (all described in Connes' spectral formalism). This is the meaning of

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THEOREM. [Pér21a] *On the Cartesian spectral triple product of a flat Riemannian fuzzy geometry with* $(M_n(\mathbb{C}), M_n(\mathbb{C}), D_F)$ *the spectral action for* $f(x) = \frac{1}{2}\sum_{i=1}^m a_i x^i$ *reads*

$$\frac{1}{4}\operatorname{Tr} f(D) = S^{\ell}_{\rm YM} + S^{\ell}_{\rm H} + S^{\ell}_{\rm g\text{-}H} + S^{\ell}_{\vartheta} + \dots .$$

Each sector is defined as follows (with $f_{\rm e}$ *the even part of* f *and* $\vartheta = \sum_{\mu,\nu}\eta^{\mu\nu}\mathscr{d}_\mu\mathscr{d}_\nu$*):*

$$S^{\ell}_{\rm YM}(\ell, \mathscr{a}) := -\frac{a_4}{4}\operatorname{Tr}_{M_N\otimes M_n}(\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu})$$

$$S^{\ell}_{\rm g\text{-}H}(\ell, \mathscr{a}, \Phi) := -a_4 \operatorname{Tr}_{M_N\otimes M_n}\left(\mathscr{d}_\mu\Phi\mathscr{d}^\mu\Phi\right)$$

$$S^{\ell}_{\rm H}(\Phi) := \operatorname{Tr}_{M_N\otimes M_n} f_{\rm e}(\Phi)$$

$$S^{\ell}_{\vartheta}(\ell, \mathscr{a}) := \operatorname{Tr}_{M_N\otimes M_n} f_{\rm e}(\vartheta^{1/2})$$

- term by term, these are the fuzzy version of $S_{\rm YM}(\mathbb{A}) = -\frac{1}{4}\int_M \operatorname{Tr}_{\mathfrak{su}(n)}(\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu})$vol,the Higgs lagrangian, and gauge-Higgs coupling $S_{\rm g\text{-}H} = -\int_M \mathbb{D}_\mu H\overline{(\mathbb{D}^\mu H)}$vol
- the obtained symmetry of the spectral action or *gauge symmetry* is $\mathcal{G} = \mathrm{PU}(N)\times\mathrm{PU}(n)$, the fuzzy counterpart to the C^∞-gauge group DIFF$(M)\ltimes$ MAPS$[M, \mathrm{SU}(n)]$ of Einstein-Yang-Mills theory. Gauge invariance is due to $\mathscr{F}_{\mu\nu}$=[$\mathsf{f}_{\mu\nu}$, $\cdot$]. The matrix $\mathsf{f}_{\mu\nu}$, which exists by Jacobi identity, is acted upon by the gauge group as $\mathsf{f}_{\mu\nu}\mapsto\mathsf{f}^u_{\mu\nu} = u\mathsf{f}_{\mu\nu}u^*$, $u\in\mathcal{G}$
- the *functional renormalization* flow in the time $t = \log N$ can be used to find *fixed points*—zeroes of the β-functions $\beta_g = \partial_t g(N)$ for each coupling g—that likely signal a phase transition (to a continuum? See [KP21] for other approach). Wetterich equation $\partial_t\Gamma = \frac{1}{2}\mathrm{STr}\{\partial_t R_N/(R_N + \mathrm{Hess}\,\Gamma)\}$ for the effective action Γ (generating function of edge 2-connected graphs, with an infrared regulator R_N) is used to determine the β-functions
- in the formalism for (multi)matrix models [Pér20], the Hessian's entries $\mathrm{Hess}_{a,b} = \partial_a\partial_b$ are in sense of NC-*derivative* $\partial_a : \mathbb{C}_{\langle k\rangle}\to\mathbb{C}_{\langle k\rangle}\otimes\mathbb{C}_{\langle k\rangle}$ given on the basis by

$$X_{j_1}\dots X_{j_p}\mapsto\delta^a_{j_1}1\otimes X_{j_2}\cdots X_{j_p} + \delta^a_{j_2}X_{j_1}\otimes X_{j_3}\cdots X_{j_p} + \dots + \delta^a_{j_p}X_{j_1}\cdots X_{j_{p-1}}\otimes 1$$

and Voiculescu's *cyclic derivative* $\mathscr{D}_b : \mathbb{C}_{\langle k\rangle}\to\mathbb{C}_{\langle k\rangle}$, $\mathscr{D}_b = \partial_b\circ\operatorname{Tr}_N$. Multi-traces cause a larger image of the Hessian's entries, namely

$$\mathcal{B}_{k,N} := (\mathbb{C}_{\langle k\rangle}\otimes\mathbb{C}_{\langle k\rangle})\oplus(\mathbb{C}_{\langle k\rangle}\boxtimes\mathbb{C}_{\langle k\rangle})$$ let us abbreviate this $\mathcal{B}$, which as vector space is $\mathbb{C}^{\otimes 2}_{\langle k\rangle}\oplus\mathbb{C}^{\otimes 2}_{\langle k\rangle}$.

Ribbon graphs together with the one-loop structure of Wetterich equation reveal the algebra for $\mathcal{B}$: for any word P, Q, U, W [Pér21b],

$$(U\otimes W)\star(P\otimes Q) = PU\otimes WQ\,, \qquad (U\boxtimes W)\star(P\otimes Q) = U\boxtimes PWQ\,, \tag{1a}$$

$$(U\otimes W)\star(P\boxtimes Q) = WPU\boxtimes Q\,, \qquad (U\boxtimes W)\star(P\boxtimes Q) = \operatorname{Tr}(WP)U\boxtimes Q\,, \tag{1b}$$

$$\operatorname{Tr}_{\mathcal{B}}(P\boxtimes Q) = \operatorname{Tr}_N(PQ)\,, \qquad \operatorname{Tr}_{\mathcal{B}}(P\otimes Q) = \operatorname{Tr}_N P\times\operatorname{Tr}_N Q\,, \tag{1c}$$

which, together with bilinearity, define the $\boxtimes$ symbol. *Functional renormalization of* k*-matrix models takes place in* $M_k(\mathcal{B})$ in the sense that the geometric series in $\mathrm{Hess}\,\Gamma$ in the rhs of Wetterich equation is computed with the algebra (1) on $k\times k$ matrices with entries in $\mathcal{B}$, and $\mathrm{STr} = \operatorname{Tr}_{M_k(\mathcal{B})}$.

EXAMPLE. Consider two operators $O_1 = \frac{\bar g_1}{2}[\operatorname{Tr}_N(\frac{A^2}{2})]^2$ and $O_2 = \bar g_2\operatorname{Tr}_N(ABC)$ in a Hermitian 3-matrix model. Suppose that we wish to determine the $\bar g_1\bar g_2^2$-coefficient of the rhs of Wetterich equation. Then

$$\mathrm{Hess}_{I,J}\,O_1 = \delta^J_I\delta^A_I\bar g_1\{\overbrace{\operatorname{Tr}_N(A^2/2)[1_N\otimes 1_N]} + \overbrace{A\boxtimes A}\}\,,$$

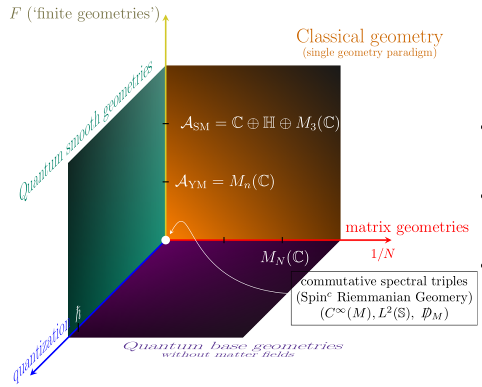
where a «filled half-edge» means that that half-edge is contracted in the (field theoretically) one-loop graph, and an «empty ribbon» that it is not. We also have

$$\mathrm{Hess}\,O_2 = \bar g_2\begin{pmatrix} 0 & C\otimes 1_N & B\otimes 1_N\\ 1_N\otimes C & 0 & A\otimes 1_N\\ 1_N\otimes B & 1_N\otimes A & 0\end{pmatrix}\Rightarrow\left[(\mathrm{Hess}\,O_2)^{\star 2}\right]_{1,1} = \bar g_2^2(\overbrace{C\otimes C} + \overbrace{B\otimes B})\,.$$

We extract the coefficient $[\bar g_1\bar g_2^2]\mathrm{STr}\{\mathrm{Hess}\,O_1[\mathrm{Hess}\,O_2]^{\star 2}\}$ which equals

$$\operatorname{Tr}_{\mathcal{B}}\left\{[\operatorname{Tr}_N(A^2/2)\times(1_N\otimes 1_N) + A\boxtimes A]\star(C\otimes C + B\otimes B)\right\}$$
$$= \operatorname{Tr}_N(A^2/2)\times[\operatorname{Tr}^2_N C + \operatorname{Tr}^2_N B] + \operatorname{Tr}_N(ACAC + ABAB)\,,$$

which are effective vertices of the four one-loop graphs that can be formed with the contractions of (the filled ribbon half-edges of) any of { , } with any of { , }. A less simple situation is the NCG-motivated 2-matrix model (truncated to ~ 40 operators) considered in [Pér20]. Even though we should eventually get rid of the R_N-dependence, it is reassuring to recognize the critical coupling value $1/4\pi$ from the exact Kazakov–Zinn-Justin solution to the $ABAB$ two-matrix model (a simplified version of Di Francesco's *meander matrix model*).



OUTLOOK:

- understand each point in the cube, and find a path to reach the continuum
- develop the BV-formalism to quantize the gauge theory of [Pér21a]
- 'turn on' the spin connection and re-analyze all as a model with gravity

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