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Combinatorial algebraic remarks on Dirac ensembles and related matrix models

*Noncommutative Geometry, Free Probability Theory and
Random Matrix Theory, Western University, June 13-17, 2022*

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* TEAM Fundacja na Rzecz Nauki Polskiej (Poland), † ERC, indirectly & DFG-Structures Excellence Cluster (Germany)

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A. Schenkel, O. Arizmendi, for discussion, comments and/or bibliography.

Outline

this talk ↓	noncomm. geometry	free prob.	random matrix theory
motivation	no	no	no
introduction	yes	no	no
fuzzy geometries	yes	?	yes
Dirac ensembles	yes	no	yes
renormalisation & free algebra	no	some	yes
gauge theories	yes	no	after 'quantisation'
outlook	yes	hopefully	yes

Motivation

- From physics to NCG: The Standard Model from the Spectral Action

$$\begin{aligned}
 & -\frac{1}{2}\partial_\nu g_\mu^a \partial_\nu g_\mu^a - g_s f^{abc} \partial_\mu g_\nu^a g_\nu^b g_\nu^c - \frac{1}{4}g_s^2 f^{abc} f^{ade} g_\mu^b g_\nu^c g_\mu^d g_\nu^e + \\
 & \frac{1}{2}ig_s^2 (\bar{g}_i^a \gamma^\mu g_j^a) g_\mu^a + G^a \partial^2 G^a + g_s f^{abc} \partial_\mu G^a G^b g_\mu^c - \partial_\nu W_\mu^+ \partial_\nu W_\mu^- - \\
 & M^2 W_\mu^+ W_\mu^- - \frac{1}{2}\partial_\nu Z_\mu^0 \partial_\nu Z_\mu^0 - \frac{1}{2}M^2 Z_\mu^0 Z_\mu^0 - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \\
 & \frac{1}{2}\partial_\mu H \partial_\mu H - \frac{1}{2}m_h^2 H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+ \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \\
 & \frac{1}{2c_w} M \phi^0 \phi^0 - \beta_h [\frac{2M^2}{g^2} + \frac{2M}{g} H + \frac{1}{2}(H^2 + \phi^0 \phi^0 + 2\phi^+ \phi^-)] + \frac{2M^4}{g^2} \alpha_h - \\
 & ig_{c_w} [\partial_\nu Z_\mu^0 (W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-) - Z_\nu^0 (W_\mu^+ \partial_\nu W_\mu^- - W_\mu^- \partial_\nu W_\mu^+) + \\
 & Z_\mu^0 (W_\nu^+ \partial_\nu W_\mu^- - W_\nu^- \partial_\nu W_\mu^+) - ig_{s_w} [\partial_\nu A_\mu (W_\mu^+ W_\nu^- - \\
 & W_\nu^+ W_\mu^-) - A_\nu (W_\mu^+ \partial_\nu W_\mu^- - W_\nu^- \partial_\nu W_\mu^+) + A_\mu (W_\nu^+ \partial_\nu W_\mu^- - \\
 & W_\nu^- \partial_\nu W_\mu^+) - \frac{1}{2}g^2 W_\mu^+ W_\nu^- W_\nu^+ W_\mu^- + \frac{1}{2}g^2 W_\mu^- W_\nu^+ W_\nu^- W_\mu^+ + \\
 & g^2 c_w^2 (Z_\mu^0 W_\nu^+ Z_\nu^0 W_\mu^- - Z_\mu^0 Z_\nu^0 W_\mu^+ W_\nu^-) + g^2 s_w^2 (A_\mu W_\nu^+ A_\nu W_\mu^- - \\
 & A_\mu A_\nu W_\nu^+ W_\mu^-) + g^2 s_w c_w [A_\mu Z_\nu^0 (W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-) - \\
 & 2A_\mu Z_\mu^0 W_\nu^+ W_\nu^-] - g\alpha [H^3 + H\phi^0 \phi^0 + 2H\phi^+ \phi^-] - \frac{1}{8}g^2 \alpha_h [H^4 + \\
 & (\phi^0)^4 + 4(\phi^0)^2 \phi^+ \phi^- + 4H^2 \phi^+ \phi^- + 2(\phi^0)^2 H^2] - \\
 & gM W_\mu^+ W_\mu^- H - \frac{1}{2}ig \frac{M}{c_w} Z_\mu^0 Z_\mu^0 H - \frac{1}{2}ig [W_\mu^+ (\phi^0 \partial_\mu \phi^- - \phi^- \partial_\mu \phi^0) - \\
 & W_\mu^- (\phi^0 \partial_\mu \phi^+ - \phi^+ \partial_\mu \phi^0)] + \frac{1}{2}g [W_\mu^+ (H \partial_\mu \phi^- - \phi^- \partial_\mu H) - \\
 & W_\mu^- (H \partial_\mu \phi^+ - \phi^+ \partial_\mu H)] + \frac{1}{2}g \frac{1}{c_w} (Z_\mu^0 (H \partial_\mu \phi^0 - \phi^0 \partial_\mu H) - \\
 & ig \frac{s_w}{c_w} M Z_\mu^0 (W_\mu^+ \phi^- - W_\mu^- \phi^+) + ig s_w M A_\mu (W_\mu^+ \phi^- - W_\mu^- \phi^+) - \\
 & ig \frac{1-2c_w^2}{2c_w} Z_\mu^0 (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) + ig s_w A_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) - \\
 & \frac{1}{4}g^2 W_\mu^+ W_\mu^- [H^2 + (\phi^0)^2 + 2\phi^+ \phi^-] - \frac{1}{4}g^2 \frac{1}{c_w} Z_\mu^0 Z_\mu^0 [H^2 + (\phi^0)^2 + \\
 & 2(2s_w^2 - 1)^2 \phi^+ \phi^-] - \frac{1}{2}g^2 \frac{s_w^2}{c_w} Z_\mu^0 \phi^0 (W_\mu^+ \phi^- + W_\mu^- \phi^+) -
 \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{2}ig^2 \frac{s_w^2}{c_w} Z_\mu^0 H (W_\mu^+ \phi^- - W_\mu^- \phi^+) + \frac{1}{2}g^2 s_w A_\mu \phi^0 (W_\mu^+ \phi^- + \\
 & W_\mu^- \phi^+) + \frac{1}{2}ig^2 s_w A_\mu H (W_\mu^+ \phi^- - W_\mu^- \phi^+) - g^2 \frac{s_w}{c_w} (2c_w^2 - \\
 & 1) Z_\mu^0 A_\mu \phi^+ \phi^- - g^1 s_w^2 A_\mu A_\mu \phi^+ \phi^- - \bar{e}^\lambda (\gamma \partial + m_e^\lambda) e^\lambda - \bar{\nu}^\lambda \gamma \partial \nu^\lambda - \\
 & \bar{u}_j^\lambda (\gamma \partial + m_u^\lambda) u_j^\lambda - \bar{d}_j^\lambda (\gamma \partial + m_d^\lambda) d_j^\lambda + ig s_w A_\mu [- (\bar{e}^\lambda \gamma^\mu e^\lambda) + \\
 & \frac{2}{3} (\bar{u}_j^\lambda \gamma^\mu u_j^\lambda) - \frac{1}{3} (\bar{d}_j^\lambda \gamma^\mu d_j^\lambda)] + \frac{ig}{4c_w} Z_\mu^0 [(\bar{\nu}^\lambda \gamma^\mu (1 + \gamma^5) \nu^\lambda) + \\
 & (\bar{e}^\lambda \gamma^\mu (4s_w^2 - 1 - \gamma^5) e^\lambda) + (\bar{u}_j^\lambda \gamma^\mu (3s_w^2 - 1 - \gamma^5) u_j^\lambda) + (\bar{d}_j^\lambda \gamma^\mu (1 - \\
 & \frac{8}{3}s_w^2 - \gamma^5) d_j^\lambda)] + \frac{ig}{2\sqrt{2}} W_\mu^+ [(\bar{\nu}^\lambda \gamma^\mu (1 + \gamma^5) e^\lambda) + (\bar{u}_j^\lambda \gamma^\mu (1 + \\
 & \gamma^5) C_{\lambda k} d_k^\lambda)] + \frac{ig}{2\sqrt{2}} W_\mu^- [(\bar{e}^\lambda \gamma^\mu (1 + \gamma^5) \nu^\lambda) + (\bar{d}_j^\lambda C_{\lambda k}^\dagger \gamma^\mu (1 + \\
 & \gamma^5) u_j^\lambda)] + \frac{ig}{2\sqrt{2}} \frac{m_\lambda^2}{M} [-\phi^+ (\bar{\nu}^\lambda (1 - \gamma^5) e^\lambda) + \phi^- (\bar{e}^\lambda (1 + \gamma^5) \nu^\lambda)] - \\
 & \frac{g}{2} \frac{m_\lambda^2}{M} [H (\bar{e}^\lambda e^\lambda) + i\phi^0 (\bar{e}^\lambda \gamma^5 e^\lambda)] + \frac{ig}{2M\sqrt{2}} \phi^+ [-m_d^\lambda (\bar{u}_j^\lambda C_{\lambda k} (1 - \\
 & \gamma^5) d_j^\lambda) + m_u^\lambda (\bar{u}_j^\lambda C_{\lambda k} (1 + \gamma^5) d_j^\lambda)] + \frac{ig}{2M\sqrt{2}} \phi^- [m_d^\lambda (\bar{d}_j^\lambda C_{\lambda k}^\dagger (1 + \\
 & \gamma^5) u_j^\lambda) - m_u^\lambda (\bar{d}_j^\lambda C_{\lambda k}^\dagger (1 - \gamma^5) u_j^\lambda) - \frac{g}{2} \frac{m_\lambda^2}{M} H (\bar{u}_j^\lambda u_j^\lambda) - \\
 & \frac{g}{2} \frac{m_\lambda^2}{M} H (\bar{d}_j^\lambda d_j^\lambda) + \frac{ig}{2} \frac{m_\lambda^2}{M} \phi^0 (u_j^\lambda \gamma^5 u_j^\lambda) - \frac{ig}{2} \frac{m_\lambda^2}{M} \phi^0 (\bar{d}_j^\lambda \gamma^5 \bar{d}_j^\lambda)
 \end{aligned}$$

...this 'fits' in

$$\text{Tr}(f(D/\Lambda)) + \frac{1}{2} \langle J\tilde{\xi}, D_A \tilde{\xi} \rangle$$

Num. of generations and $\mathbb{C} \oplus \mathbb{H} \oplus \mathcal{M}_3(\mathbb{C}) \rightsquigarrow$  \rightsquigarrow Classical Lagrangian of the Standard Model

[Chamseddine-Connes-Marcolli ATMP '07 (Euclidean); J. Barrett J. Math. Phys. '07 (Lorentzian)]

Motivation

- From physics to NCG: The Standard Model from the Spectral Action

$$D_F = \begin{pmatrix} 0 & 0 & \Upsilon_\nu^* & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_R^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Upsilon_c^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Upsilon_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Upsilon_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Upsilon_u & 0 & 0 & 0 & 0 & \otimes 1_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Upsilon_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Upsilon_R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_\nu^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_c^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\Upsilon}_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\Upsilon}_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^T \otimes 1_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\Upsilon}_u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\Upsilon}_d & 0 \end{pmatrix} \in M_{96}(\mathbb{C})_{\text{s.a.}}$$

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Towards a quantum theory of noncommutative spaces

« The far distant goal is to set up a functional integral evaluating spectral

observables \mathcal{S} $\langle \mathcal{S} \rangle = \int \mathcal{S} e^{-\text{Tr} f(D/\Lambda) - \frac{1}{2} \langle J\psi, D\psi \rangle + \rho(e, D)} de d\psi dD$ »

[Eq. 1.892, Connes Marcolli, *NCG, QFT and motives*, 2007]

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functional integral $\xrightarrow{\text{paradigm shift}}$ operator integral

$$\int_{\text{metric}} e^{-\frac{1}{\hbar} S_{\text{EH}}[g]} dg \xrightarrow{\text{Einstein-Hilbert} \rightarrow \text{spectral}} \int_{\text{Dirac}} e^{-\frac{1}{\hbar} \text{Tr} f(D)} dD$$

(hard to define for manifolds)

$f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(D) \rightarrow \infty$ at large argument

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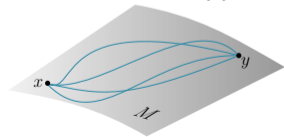
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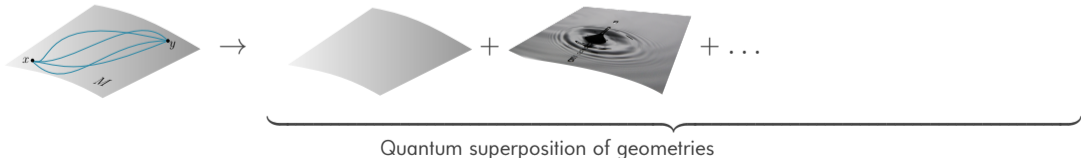
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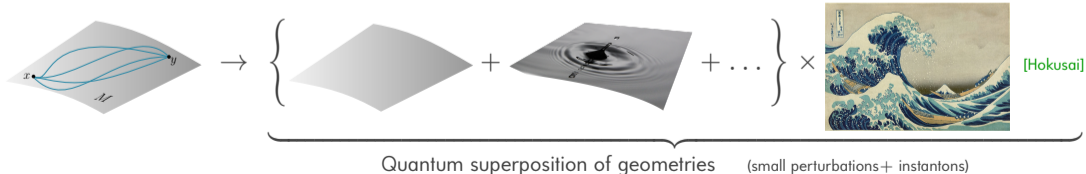
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- Origin of noncommutative topology

Connes' *noncommutative (nc) geometry* = nc topology

[Gelfand, Najmark *Mat. Sbornik* '43] + metric [A. Connes, *NCG* '94]

{compact Hausdorff topological spaces} \simeq {unital *commutative* C^* -algebras}

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- arguably, the 1st NCG-theorem is *Weyl's law* (1911) on the rate of growth of the Laplace spectrum (ordered $\lambda_0 \leq \lambda_1 \leq \lambda_2 \dots$) of $\Omega \subset \mathbb{R}^d$

$$\#\{i : \lambda_i \leq \Lambda\} = \frac{\text{vol}(\text{unit ball})}{(2\pi)^d} \text{vol } \Omega \cdot \Lambda^{d/2} + o(\Lambda^d)$$

From this, you cannot answer positively Marek Kac' 1966-question.

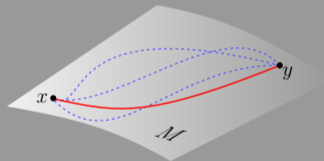
But you can 'hear the shape of Ω ' knowing a *spectral triple*.

[A, Connes, *JNCG* 2013] (and from it [Connes-van Suijlekom, *CMP* 2021] can hear an MP3;

our story today is not entirely unrelated.)

Replace spin manifold (M, g) by $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

Connes' geodesic distance

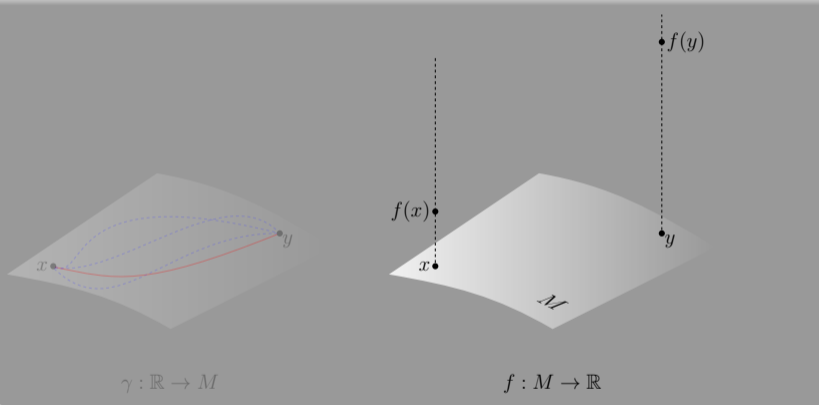


$$\gamma: \mathbb{R} \rightarrow M$$

$$\inf_{\gamma \text{ as above}} \left\{ \int_{\gamma} ds \right\} = d(x, y)$$

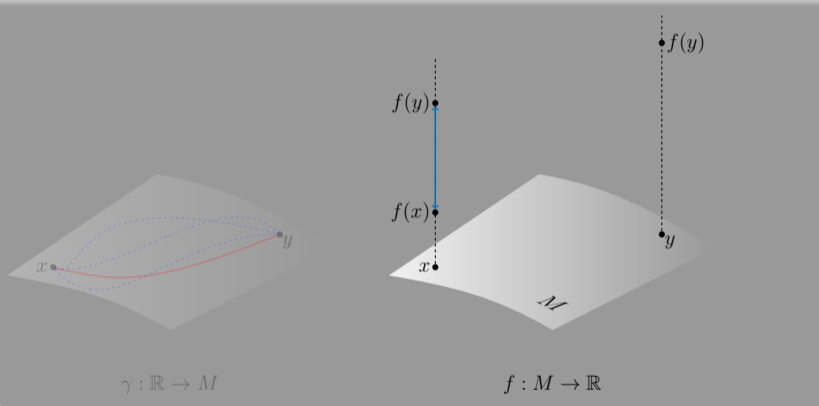
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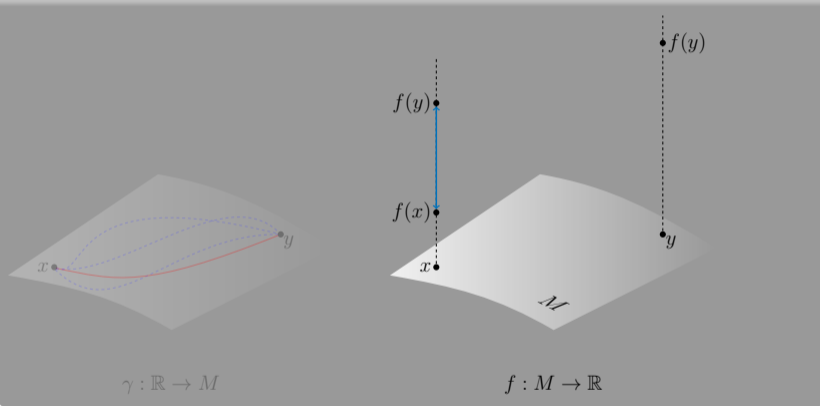
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$$|f(x) - f(y)|$$

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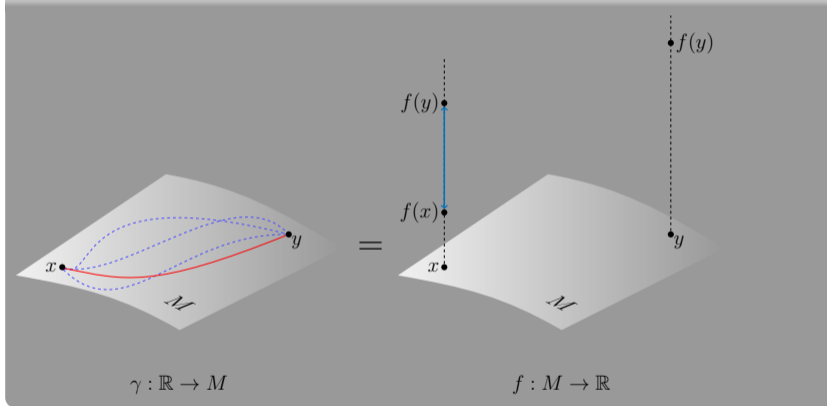
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$$\sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| : \|D_M f - f D_M\| \leq 1 \}$$

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$$\inf_{\gamma \text{ as above}} \left\{ \int_{\gamma} ds \right\} = d(x, y) = \sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| : \|D_M f - f D_M\| \leq 1 \}$$

Commutative spectral triples

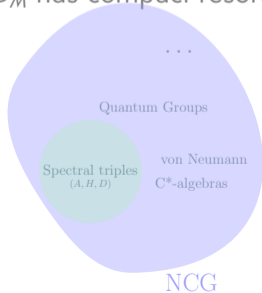
A spin manifold M yields (A_M, H_M, D_M)

- $A_M = C^\infty(M)$ is a comm. $*$ -algebra
- $H_M := L^2(M, \mathbb{S})$ a repr. of A_M
- $D_M = -i\gamma^\mu(\partial_\mu + \omega_\mu)$ is self-adjoint
- for each $a \in A_M$, $[D_M, a]$ is bounded, and in fact $[D_M, x^\mu] = -i\gamma^\mu$

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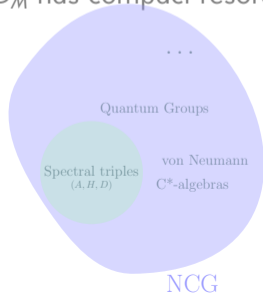
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A *spectral triple* (A, H, D) consists of

- a $*$ -algebra A
- a representation H of A
- a self-adjoint operator D on H with compact resolvent and such that $[D, a]$ is bounded for each $a \in A$



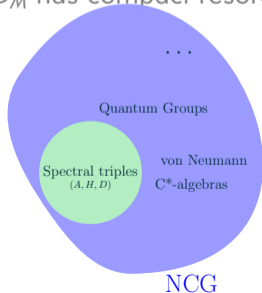
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II. Fuzzy Geometries and Multimatrix Models

A *fuzzy geometry* of signature (p, q) , so $\eta = \text{diag}(+p, -q)$, consists of

- $A = M_N(\mathbb{C})$
- $H = \mathbb{S} \otimes M_N(\mathbb{C})$, with \mathbb{S} a $\text{Cl}(p, q)$ -module
... (axioms for D omitted, go to axioms ∇) ...

II. Fuzzy Geometries and Multimatrix Models

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... (axioms for D omitted, go to axioms ∇) ...
- Fixing conventions for γ 's, characterisation of D in even dimensions:

$$D = \sum_J \Gamma_{\text{s.a.}}^J \otimes \{H_J, \cdot\} + \sum_J \Gamma_{\text{anti.}}^J \otimes [L_J, \cdot]$$

multi-index J monot. increasing, $|J|$ odd [J. Barrett, *J. Math. Phys.* 2015], $H_J^* = H_J$, $L_J^* = -L_J$

II. Fuzzy Geometries and Multimatrix Models

A *fuzzy geometry* of signature (p, q) , so $\eta = \text{diag}(+p, -q)$, consists of

- $A = M_N(\mathbb{C})$
- $H = \mathbb{S} \otimes M_N(\mathbb{C})$, with \mathbb{S} a $\text{Cl}(p, q)$ -module
... (axioms for D omitted, go to axioms ∇) ...
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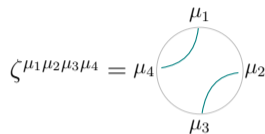
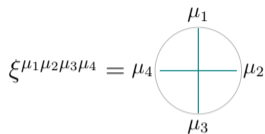
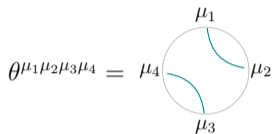
- Examples: [J. Barrett, L. Glaser, *J. Phys. A* 2016]
 - $D_{(1,1)} = \gamma^1 \otimes [L, \cdot] + \gamma^2 \otimes \{H, \cdot\}$
 - $D_{(0,4)} = \sum_{\mu} \gamma^{\mu} \otimes [L_{\mu}, \cdot] + \gamma^{\hat{\mu}} \otimes \{H_{\hat{\mu}}, \cdot\}$

so we will get double traces from $\text{Tr}_H = \text{Tr}_{\mathbb{S}} \otimes \text{Tr}_{M_N(\mathbb{C})} = \text{Tr}_{\mathbb{S}} \otimes \text{Tr}_N^{\otimes 2}$

Notation: $\text{Tr}_V X$ is the trace on operators $X : V \rightarrow V$, $\text{Tr}_V 1 = \dim V$. So $\text{Tr}_N 1 = N$ but $\text{Tr}_{M_N(\mathbb{C})}(1) = N^2$.

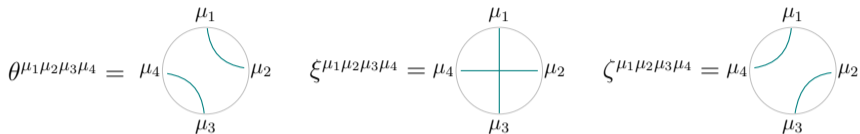
- A tool to organize the fuzzy spectral action is **chord diagrams**:

$$\text{Tr}_{\mathbb{S}}(\gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4}) = \dim \mathbb{S}(\overbrace{\eta^{\mu_1 \mu_2} \eta^{\mu_3 \mu_4}}^{\theta^{\mu_1 \mu_2 \mu_3 \mu_4}} + \overbrace{(-)\eta^{\mu_1 \mu_3} \eta^{\mu_2 \mu_4}}^{\xi^{\mu_1 \mu_2 \mu_3 \mu_4}} + \overbrace{\eta^{\mu_2 \mu_3} \eta^{\mu_1 \mu_4}}^{\zeta^{\mu_1 \mu_2 \mu_3 \mu_4}})$$



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- for dimension- d geometries, the combinatorial formula [CP '19] reads

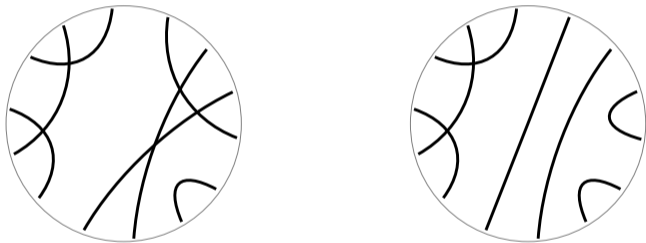
$$\frac{1}{\dim \mathbb{S}} \text{Tr}(D^{2t}) = \sum_{\substack{\text{if } J \in \Lambda_d, dx^J \neq 0 \text{ on } \mathbb{R}^d \\ h_1, \dots, l_t \in \Lambda_d^-}} \left\{ \sum_{\substack{\chi \in \text{CD}_{2n} \\ 2n = \sum_i |l_i|}} \chi^{h_1 \dots l_t} \right. \\ \left. \times \left(\sum_{\Upsilon \in \mathcal{P}_{2t}} \text{sgn}(l_\Upsilon) \times \text{Tr}_N(K_{l_\Upsilon}) \times \text{Tr}_N[(K^T)_{l_\Upsilon}] \right) \right\}$$

$d \geq 4$
 $2t = 4$ example
 $2n = 12$

Warning: These chord diagrams are **not** Feynman diagrams, they just determine the classical action.



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Let's go to the quantum theory, but before some remarks...

Multimatrix models with multitraces & ribbon graphs

- The chord-diagram description holds in general dimension and signature [CP '19]

$$\begin{aligned} \mathcal{Z} &= \int_{\text{Dirac}} e^{-\text{Tr}_H f(D)} dD \quad (\hbar = 1) \\ &= \int_{\mathcal{M}_{p,q}} e^{-N \text{Tr}_N P - \text{Tr}_N^{\otimes 2} (Q_{(1)} \otimes Q_{(2)})} d\mathbb{X}_{\text{Leb}} \end{aligned}$$

- $\mathbb{X} \in \mathcal{M}_{p,q} =$ products of $\mathfrak{su}(N)$ and \mathcal{H}_N
- $d\mathbb{X}_{\text{Leb}}$ is the Lebesgue measure on $\mathcal{M}_{p,q}$
- $P, Q_{(i)}$ are certain nc-polynomials
- $\mathcal{Z}_{\text{formal}}$ is the gen. func. of colored ribbon graphs (maps)

$$\bar{g}_1 \text{Tr}_N (ABBBAB) \leftrightarrow$$



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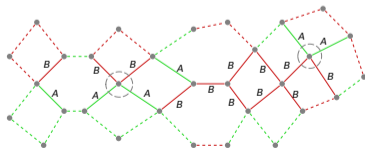
$$\bar{g}_2 \text{Tr}_N^{\otimes 2} (AABABA \otimes AA) \leftrightarrow \text{Diagram 2}$$



- Multitrace random matrices:

- 'touching interactions' [Klebanov, PRD '95]
- wormholes [Ambjørn-Jurkiewicz-Loll-Vernizzi, JHEP '01]
- stuffed maps [G. Borot AHP-D '14]
- trace polynomials [G. Cébron, J.Funct.Anal. '13]
- [D. Jekel-W. Li-D. Shlyakhtenko, 2021; A. Guionnet...,]
- AdS/CFT [Witten, hep-th/0112258]

- Ribbon graphs: Enumeration of maps [Brezin, Itzykson, Parisi, Zuber, CMP '78], here 'face-worded'



III. Functional renormalisation in random matrices

Motivation from '2D-Quantum Gravity'

discrete surfaces \leftrightarrow matrix integrals $\mathcal{Z}(\lambda)$
 [B. Eynard, *Counting Surfaces* '16]

smooth surface \leftrightarrow $\langle \text{area} \rangle$ finite
 $\&$ infinitesimal mesh a
 $\langle \text{area} \rangle_g \sim \frac{a^2(2-2g)}{\lambda/\lambda_c - 1}$

all topologies \leftrightarrow $\mathcal{Z}(\lambda) = \sum_g N^{2-2g} \mathcal{Z}_g(\lambda)$
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double-scaling limit $N(\lambda_c - \lambda)^{1/\theta} = C$

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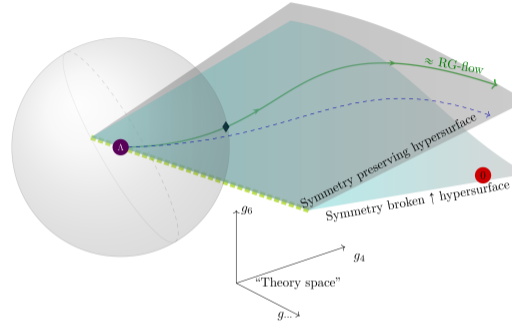
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- A Chosen bare action $S = \Gamma_{N=\Lambda}$
- B Full effective action $\Gamma = \Gamma_{N=0}$
- ◆ Interpolating action $\Gamma_{N=\Lambda-\rho}$ (projected & truncated)
- RG-flow with truncation and projection
- ⋯ Moduli of Dirac operators \leftrightarrow theory space
- - - RG-flow without truncation nor projection
- $g \dots$ Rest of coupling constants

[CP 20] \rightarrow

Homogeneisation of notation

- Let's write $\mathbb{E}[\cdot]$ for $\langle \cdot \rangle$. Wick's theorem [L. Isserlis *Biometrika* 1918]: for zero-mean x_i 's, ...

$$\mathbb{E}[x_{i_1} \cdots x_{i_{2n}}] = \sum_{\substack{\pi \in P_2(2n) \\ \text{(pairings)}}} \prod_{(p,q) \in \pi} \mathbb{E}[x_{i_p} x_{i_q}]$$

- k the number of Hermitian matrices of size N , $X_1^{(N)}, \dots, X_k^{(N)}$ [Piotr Śniady's notation]

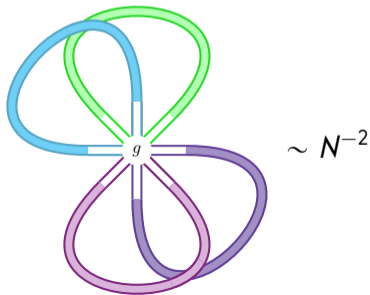
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- Ribbon graphs. For $\mathbb{E}[(X_\mu^{(N)})_{i,j} (X_\rho^{(N)})_{l,m}] = \delta_{\mu\rho} \delta_{im} \delta_{jl}$ $\mu, \rho = 1, \dots, k; i, \dots, m = 1, \dots, N$

$$(gN) \cdot \mathbb{E} \left[\text{Tr}_N \left(X_1^{(N)} X_2^{(N)} X_1^{(N)} X_2^{(N)} X_3^{(N)} X_4^{(N)} X_3^{(N)} X_4^{(N)} \right) \right] =$$



- Some Feynman graphs of multimatrix ϕ^4 -theory...

Several-loop graph



[pic by 'Princi19skydiver', Wikipedia]

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[pic by 'Wojciech Kielar', Wikipedia]

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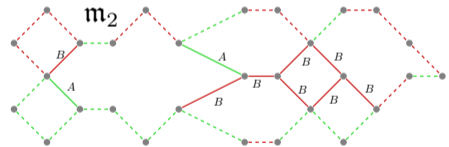
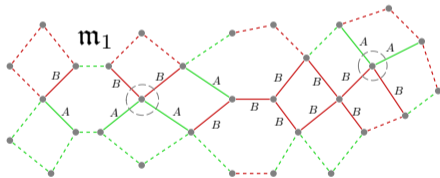
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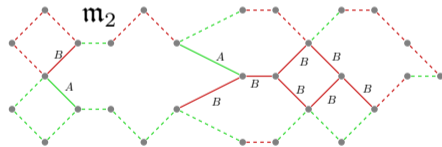
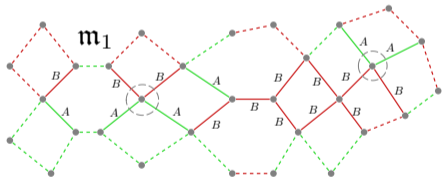
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- Question (sloppy version):* Given an operator $\text{Tr}_N w$, in $w \in \mathbb{C}^{\binom{N}{k}}$, find (up to given degree in the couplings) all 1-loops it might come from, in the above example sense

Functional Renormalisation for k -matrix models (w/multitrace-measures)

Quantum theories 'flow' with energy, here in RG-time $t = \log N$, $1 \ll N < \mathcal{N}$. E.g. for $k = 2$ and with bare action

$$S[A, B] = \mathcal{N} \operatorname{Tr}_{\mathcal{N}} \left\{ \frac{1}{2} A^2 + \frac{1}{2} B^2 + g_{A^4} \frac{1}{4} A^4 + g_{B^4} \frac{1}{4} B^4 + \frac{1}{2} g_{ABAB} ABAB \right\}$$

radiative corrections 'generate' *effective vertices*. For instance  generates $\mathcal{N} \operatorname{Tr}_{\mathcal{N}}(ABBA)$.

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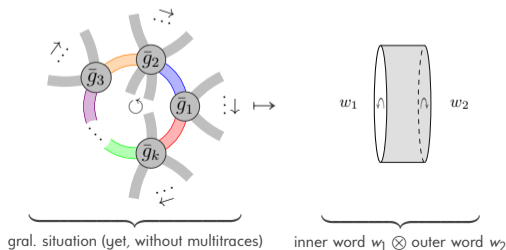
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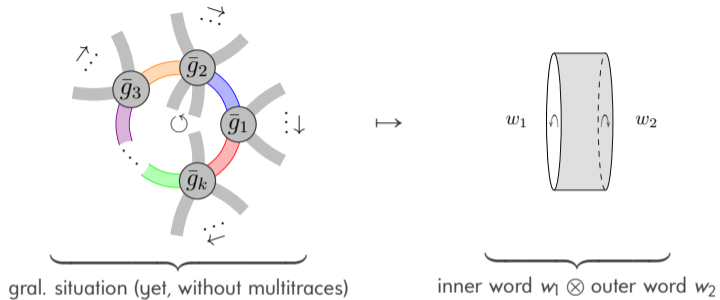
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We are interested in *one-loop graphs*. The *effective vertex* O_G^{eff} of such a graph is formed by reading off each word w_i traveling around all ribbon edges (propagators) by both sides:

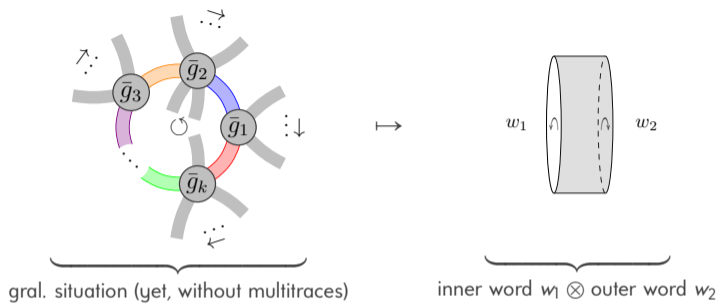
$$O_G^{\text{eff}} = \overbrace{\operatorname{Tr}_N(w_1) \times \operatorname{Tr}_N(w_2) \times \dots \times \operatorname{Tr}_N(w_s)}^{\text{from vertices contracted with propagators}} \times \underbrace{\operatorname{Tr}_N(U_1) \times \operatorname{Tr}_N(U_2) \dots \times \operatorname{Tr}_N(U_r)}_{\text{from vertices uncontracted with propagators}}$$



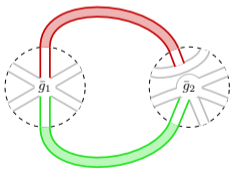
So the actual question is: find the pre-image of the map



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... actually, for multitrace operators: pre-image of any $O = \prod_{\alpha} \text{Tr}_N w_{\alpha}$:



One then sums over the product of all g_i 's appearing in such 1-loops.
 These polynomials span the β -function for O .

Two steps

1. Understanding the Func. Renormalisation Eq.

[CP 2007.10914, *Ann. Henri Poincaré* 2021]

- prove Wetterich Equation, or FRGE; its proof determines the algebra that governs geometric series in the Hessian of Γ
- [A. Eichhorn, T. Koslowski, *Phys. Rev. D* '13] oriented us, but the proof of the FRGE dictates an algebra not reported there
- β -equations found for a sextic truncation (48 running operators). For the unique real solution g^* leading to a single relevant direction (positive e.v. of $-(\partial\beta_i/\partial g_j)_{i,j}|_{g^*}$) yields an R_N -dependent

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2. Unicity (using a ribbon graph argument)

[CP 2111.02858 *Left. Math. Phys.* 2022]

- write down Wetterich Equation
“ $\dot{\Gamma} = \frac{1}{2} \text{Tr}_{\mathcal{M}_k(\mathcal{A})} \{ \dot{R}_N / [\Gamma^{(2)} + R_N] \}$ ”
- assume an expansion of its rhs in unitary-invariant operators (\neq exact RG)
- impose the one-loop structure and solve for the algebra $\mathcal{A} = \mathcal{A}_{k,N}$
- determine from it the algebra that computes Wetterich equation; it is unique and the one reported in [CP 2007.10914]

- *nc-derivative* $\partial_A : \mathbb{C}\langle k \rangle \rightarrow \mathbb{C}\langle k \rangle^{\otimes 2}$ sums over ‘replacements of A by \otimes ’

[Turnbull+Rota-Sagan-Stein+Voiculescu]:

$$\partial_A(PAAR) = P \otimes AR + PA \otimes R, \text{ but}$$

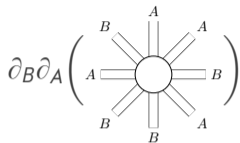
$$\partial_A(ALGEBRA) = 1 \otimes LGEBRA + ALGEBR \otimes 1$$

- parenthetically, nc-derivatives allow to compactly write loop-equations (or Dyson-Schwinger eqs.) in random matrix theory

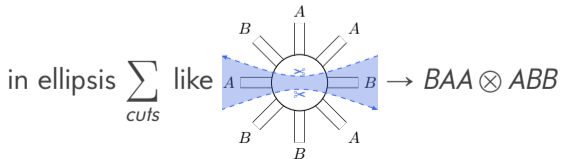
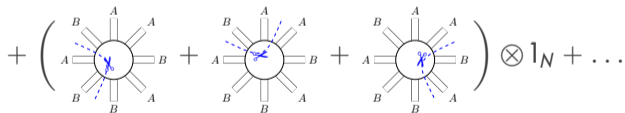
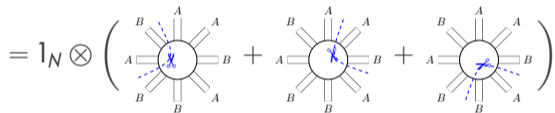
$$\mathbb{E} \left[\left(\frac{1}{N} \text{Tr}_N \otimes \frac{1}{N} \text{Tr}_N \right) (\partial_X P) \right] = \mathbb{E}[P \mathcal{D}_X V] \quad P \in \mathbb{C}\langle k \rangle$$

[J. Mingo, R. Speicher *Probab. Math. Stat.* 2013] [A. Guionnet *Jpn. J. Math.* 2016] (cf. S. Azarfar and N. Pagliaroli’s talks)

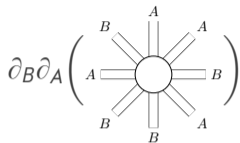
- $W \in \mathbb{C}_{\langle k \rangle}$, the *nc-Hessian* $\text{Hess Tr}_N W \in M_k(\mathbb{C}_{\langle k \rangle} \otimes \mathbb{C}_{\langle k \rangle})$ has entries are $\text{Hess}_{b,a} \text{Tr} W = (\partial_{X_b} \circ \partial_{X_a}) \text{Tr}_N W$. Are computed by 'cuts': e.g. $W = \text{ABAABABB}$ [CP '21]



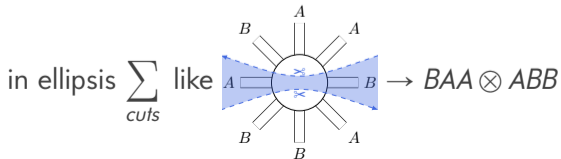
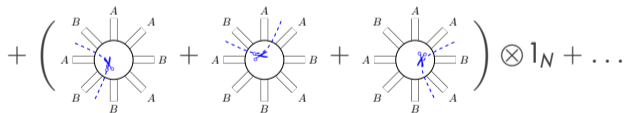
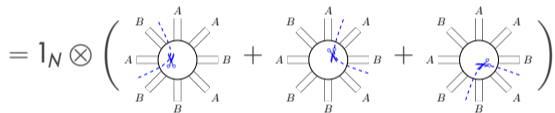
go to examples of nc-Hessians ▽



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- products of traces \Rightarrow extend by \boxtimes , $\mathcal{A}_k = (\mathbb{C}_{\langle k \rangle} \otimes \mathbb{C}_{\langle k \rangle}) \oplus (\mathbb{C}_{\langle k \rangle} \boxtimes \mathbb{C}_{\langle k \rangle})$



$$\text{Hess}_{a,b}(\text{Tr } P \cdot \text{Tr } Q) = \text{Tr } P \cdot \text{Hess}_{a,b}[\text{Tr } Q] + (\partial_{X_a} \text{Tr } P) \boxtimes (\partial_{X_b} \text{Tr } Q) + (P \leftrightarrow Q)$$

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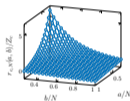


$$\text{Hess}_{a,b}(\text{Tr } P \cdot \text{Tr } Q) = \text{Tr } P \cdot \text{Hess}_{a,b}[\text{Tr } Q] + (\partial_{X_a} \text{Tr } P) \boxtimes (\partial_{X_b} \text{Tr } Q) + (P \leftrightarrow Q)$$

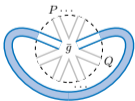
- Wetterich Eq. governs the functional RG $t = \log N$

$$\partial_t \Gamma_N[\mathbb{X}] = \frac{1}{2} \text{STr} \left\{ \frac{\partial_t R_N}{\text{Hess } \Gamma_N[\mathbb{X}] + R_N} \right\}$$

$\stackrel{\text{assume}}{=} \sum_{k=0}^{\infty} \bar{h}_k(N, \eta_1, \dots, \eta_n) \times \underbrace{\frac{1}{2} (-1)^k \text{Tr}_{M_k(\mathcal{A})} \{ (\text{Hess } \Gamma_N^{\text{Int}}[\mathbb{X}])^{*k} \}}_{\text{regulator-independent part}}$



- $\text{STr} = \text{Tr}_k \otimes \text{Tr}_{\mathcal{A}_k}$. Tadpoles



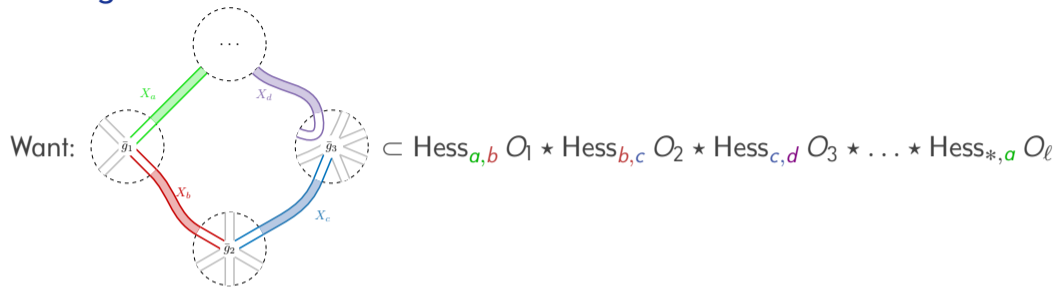
$$\text{Tr}_{\mathcal{A}_k}(P \otimes Q) = \text{Tr}_N P \cdot \text{Tr}_N Q,$$



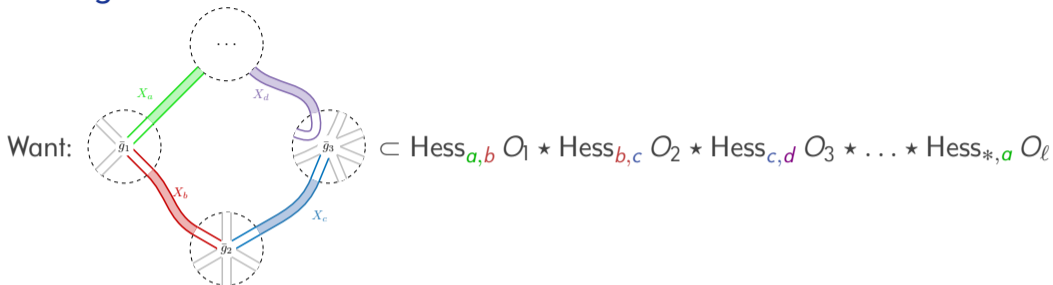
imply

$$\text{Tr}_{\mathcal{A}_k}(P \boxtimes Q) = \text{Tr}_N(PQ)$$

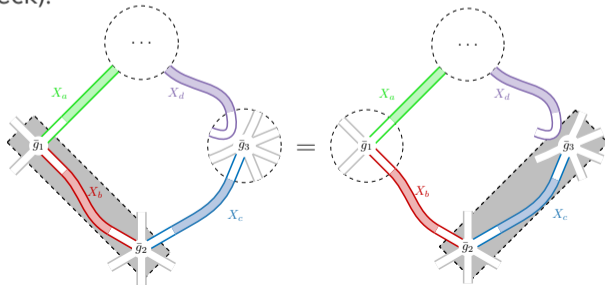
Finding \star

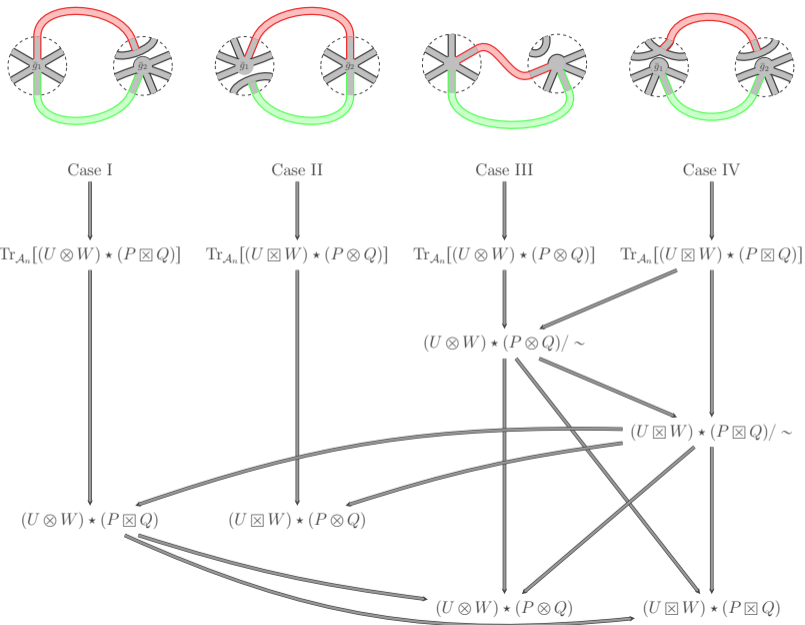


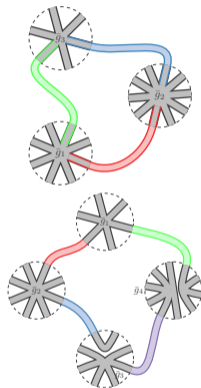
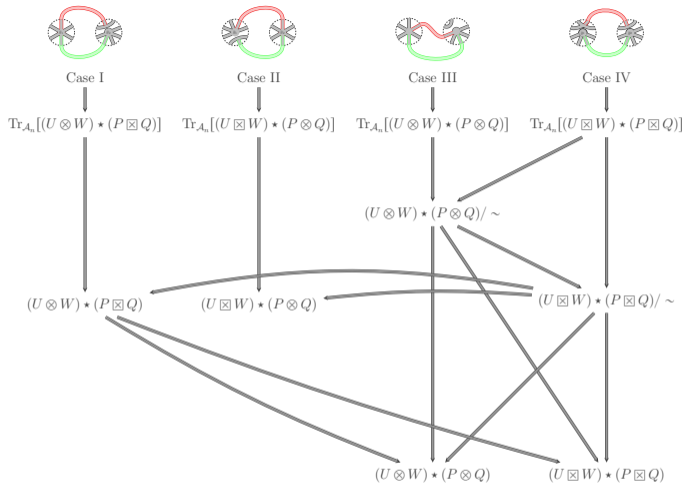
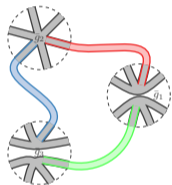
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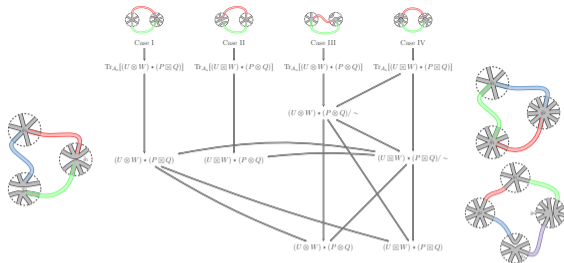


Associativity (trivial check):







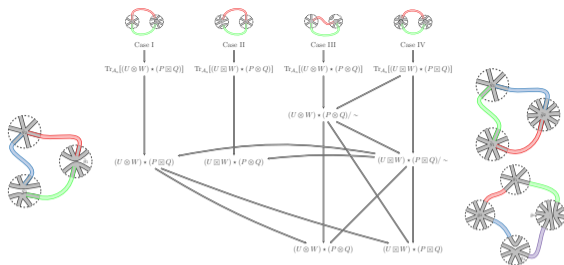


Thm. [CP '22] If the RG-flow is computable in terms of $U(N)$ -invariants, the algebra of Functional Renormalisation is $\mathcal{M}_k(\mathcal{A}_{k,N}, \star)$ where

$$\mathcal{A}_{k,N} = (\mathbb{C}_{\langle k \rangle}^{(N)} \otimes \mathbb{C}_{\langle k \rangle}^{(N)}) \oplus (\mathbb{C}_{\langle k \rangle}^{(N)} \boxtimes \mathbb{C}_{\langle k \rangle}^{(N)})$$

whose product in hom. elements reads:

$$\begin{aligned} (U \otimes W) \star (P \otimes Q) &= PU \otimes WQ, \\ (U \boxtimes W) \star (P \otimes Q) &= U \boxtimes PWQ, \\ (U \otimes W) \star (P \boxtimes Q) &= WPU \boxtimes Q, \\ (U \boxtimes W) \star (P \boxtimes Q) &= \text{Tr}_N(WP)U \boxtimes Q, \end{aligned}$$



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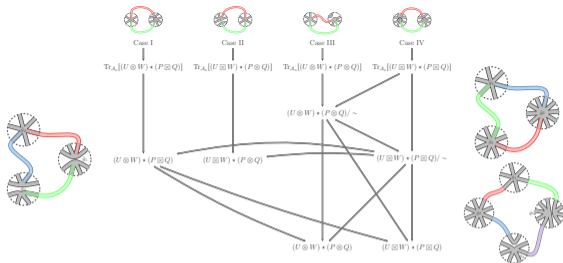
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Remark: To be more precise, any occurrence of the free algebra in $\mathcal{A}_{k,N}$ should be replaced by the algebra of 'trace polynomials' (e.g. $\text{Tr}_N(X_1 X_3) X_2 + N \text{Tr}_N(X_2^2)$) [D. Jekel-W. Li-D. Shlyakhtenko, '21]

Example: A Hermitian 3-matrix model. Consider $O_1 = \frac{\bar{g}_1}{2} [\text{Tr}_N(\frac{A^2}{2})]^2$ and $O_2 = \bar{g}_2 \text{Tr}_N(ABC)$.

$$\text{Hess}_{I,J} O_1 = \delta_I^J \delta_I^A \bar{g}_1 \left\{ \underbrace{\text{Tr}_N(A^2/2) \cdot [1_N \otimes 1_N]}_{\text{filled ribbon}} + \underbrace{A \boxtimes A}_{\text{white ribbon}} \right\},$$

where a 'filled ribbon' means contracted in the one-loop graph, and 'white ribbon' uncontracted.

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$$\text{Hess } O_2 = \bar{g}_2 \begin{bmatrix} 0 & C \otimes 1_N & B \otimes 1_N \\ 1_N \otimes C & 0 & A \otimes 1_N \\ 1_N \otimes B & 1_N \otimes A & 0 \end{bmatrix} \Rightarrow [\text{Hess } O_2]^{\star 2} = \bar{g}_2^2 \begin{bmatrix} \overbrace{C \otimes C + B \otimes B}^{\text{diagrams}} & B \otimes A & C \otimes A \\ A \otimes B & A \otimes A + C \otimes C & C \otimes B \\ A \otimes C & B \otimes C & B \otimes B + A \otimes A \end{bmatrix}.$$

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Extracting coefficients

$$[\bar{g}_1 \bar{g}_2^2] \text{Tr}_{M_3(\mathcal{A})} \{ \text{Hess } O_1 \star [\text{Hess } O_2]^{\star 2} \} = \text{Tr}_N(A^2/2) \times [(\text{Tr}_N C)^2 + (\text{Tr}_N B)^2] + \text{Tr}_N(ACAC + ABAB),$$

which are effective vertices of the four one-loop graphs that can be formed with the contractions of (the filled ribbon half-edges of) any of $\left\{ \begin{array}{c} \text{---} \text{---} \\ | \quad | \\ \text{---} \text{---} \end{array} \right\}, \left\{ \begin{array}{c} \text{---} \text{---} \\ | \quad | \\ \text{---} \text{---} \end{array} \right\}$ with any of $\left\{ \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \right\}, \left\{ \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right\}$.

Why not using graphs? Soon, nc-Hessians get bulky: In [CP '20] 48 such operators run

Operator	Its nc-Hessian
$\text{Tr}(A) \text{Tr}(A^3)$	$3 \cdot \begin{pmatrix} \text{Tr} A \cdot (A \otimes 1 + 1 \otimes A) & 0 \\ +1 \boxtimes A^2 + A^2 \boxtimes 1 & \\ 0 & 0 \end{pmatrix}$
$\text{Tr} A \text{Tr}(AAABB)$	$\begin{pmatrix} (\text{Tr}(A)1 \otimes (ABB) + \text{Tr}(A)1 \otimes (BBA) + \text{Tr}(A)(ABB) \otimes 1 + \text{Tr}(A)(BBA) \otimes 1 + 1 \boxtimes (AABB) + 1 \boxtimes (ABBA) + 1 \boxtimes (BBAA) + (AABB) \boxtimes 1 + (ABBA) \boxtimes 1 + (BBAA) \boxtimes 1 + \text{Tr}(A)A \otimes B^2 + \text{Tr}(A)B^2 \otimes A) & \text{Tr}(A)(1 \otimes (AAB) + (BAA) \otimes 1 + A \otimes AB + B \otimes A^2 + A^2 \otimes B + BA \otimes A) + 1 \boxtimes (A^3 B) + 1 \boxtimes (BA^3) \\ \text{Tr}(A)(1 \otimes (BAA) + (AAB) \otimes 1 + A \otimes BA + B \otimes A^2 + A^2 \otimes B + AB \otimes A) + (A^3 B) \boxtimes 1 + (BA^3) \boxtimes 1 & \text{Tr}(A)(1 \otimes (A^3) + (A^3) \otimes 1) \end{pmatrix}$

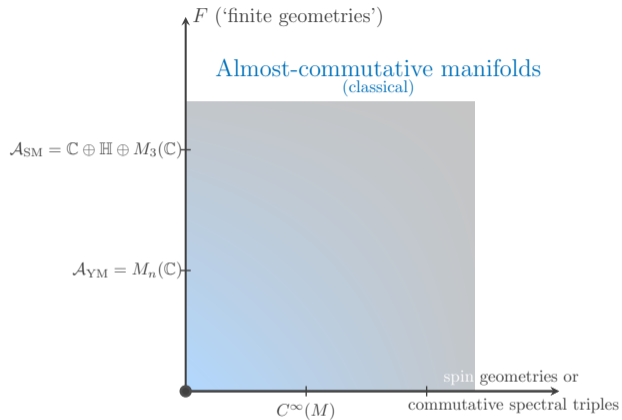
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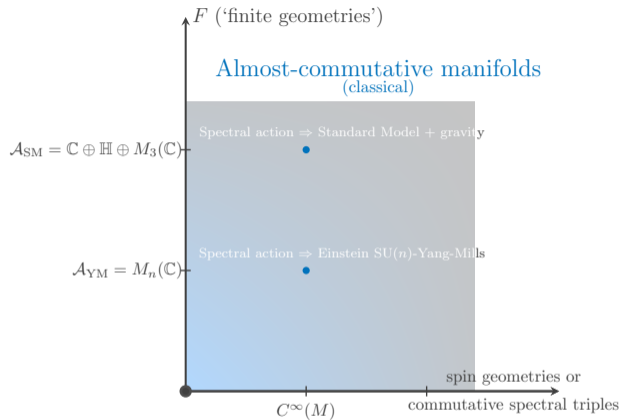
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III. Matrix gauge theory



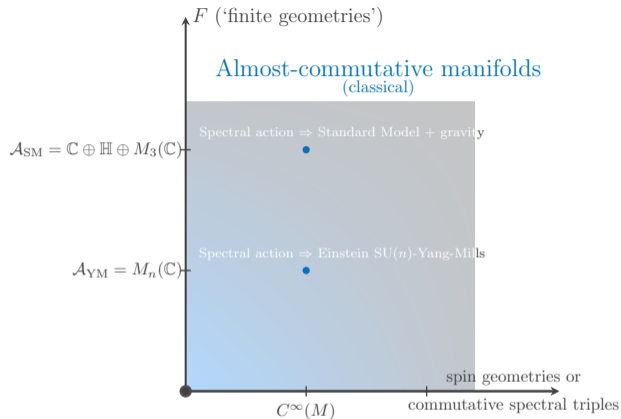
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Definition [CP 2105.01025] We define a *gauge matrix spectral triple* $G_\ell \times F$ as the spectral triple product of a fuzzy geometry G_ℓ with a finite geometry $F = (A_F, H_F, D_F)$, $\dim A_F < \infty$.

Lemma-Definition [CP 2105.01025] Consider a gauge matrix spectral triple $G_\ell \times F$ with

$$F = (M_n(\mathbb{C}), M_n(\mathbb{C}), D_F)$$

and G_ℓ Riemannian ($d = 4$) fuzzy geometry on $M_N(\mathbb{C})$, whose **fluctuated** Dirac op. is

$$D_\omega = \sum_{\mu=0}^3 \overbrace{\gamma^\mu \otimes (\ell_\mu + a_\mu) + \gamma^{\hat{\mu}} \otimes (x_\mu + s_\mu)}^{D_{\text{gauge}}} + \overbrace{\gamma \otimes \Phi}^{D_{\text{Higgs}}}, \quad a_\mu = \text{'gauge potential'}, x_\mu = \text{spin connection?}$$

The *field strength* is given by

$$\mathcal{F}_{\mu\nu} := \overbrace{[\ell_\mu + a_\mu, \ell_\nu + a_\nu]}^{d_\mu} =: [F_{\mu\nu}, \cdot]$$

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The content of the Spectral Action ...

Meaning

Random matrix case, flat $d = 4$ Riem.

Smooth operator

Tr = trace of ops. $M_N \otimes M_n \rightarrow M_N \otimes M_n$

Derivation

$$\ell_\mu = [L_\mu \otimes 1_n, \cdot]$$

$$\partial_i$$

Gauge potential

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Covariant derivative	$d_\mu = \ell_\mu + a_\mu$	$\mathbb{D}_i = \partial_i + \mathbb{A}_i$
Field strength	$[d_\mu, d_\nu] = \overbrace{[\ell_\mu, \ell_\nu]}^{\neq 0} + [\ell_\mu, a_\nu] - [\ell_\nu, a_\mu] + [a_\mu, a_\nu]$	$[\mathbb{D}_i, \mathbb{D}_j] = \overbrace{[\partial_i, \partial_j]}^{\equiv 0} + \partial_i \mathbb{A}_j - \partial_j \mathbb{A}_i + [\mathbb{A}_i, \mathbb{A}_j]$
Yang-Mills action	$-\frac{1}{4} \text{Tr}(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu})$	$-\frac{1}{4} \int_{\mathcal{M}} \text{Tr}_{\text{su}(n)}(\mathbb{F}_{ij} \mathbb{F}^{ij}) \text{vol}$
Higgs field	Φ	h
Higgs potential	$\text{Tr}(f_2 \Phi^2 + f_4 \Phi^4)$	$\int_{\mathcal{M}} (f_2 h ^2 + f_4 h ^4) \text{vol}$
Gauge-Higgs coupling	$-\text{Tr}(d_\mu \Phi d^\mu \Phi)$	$-\int_{\mathcal{M}} \mathbb{D}_i h ^2 \text{vol}$

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+

Yang-Mills action

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Higgs field

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$$h$$

Higgs potential

$$\text{Tr}(f_2 \Phi^2 + f_4 \Phi^4)$$

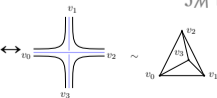
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$$-\text{Tr}(d_\mu \Phi d^\mu \Phi)$$

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Propagators and $\sim (\ell_\mu)_{ij} (\ell_\nu)_{jm} (\ell^\mu)_{ml} (\ell^\nu)_{li}$



Conclusion

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[Connes Marcolli, *NCG, QFT and motives*, '07, next screenshot]

« The far distant goal is to set up a functional integral evaluating spectral observables \mathcal{S} as
(1.892) $\langle \mathcal{S} \rangle = \mathcal{N} \int \mathcal{S} e^{-\text{Tr}(f(D/\Lambda)) - \frac{1}{2} \langle J\psi, D\psi \rangle - \rho(e,D)} \mathcal{D}[\psi] \mathcal{D}[D] \mathcal{D}[e],$ »

- The matrix algebra $M_k(\mathcal{A}_{k,N})$ where functional renormalisation for random matrices (k -matrix model) takes place was provided. $\mathcal{A}_{k,N}$ is a bigger relative of $\mathbb{C}^{\binom{N}{k}}$

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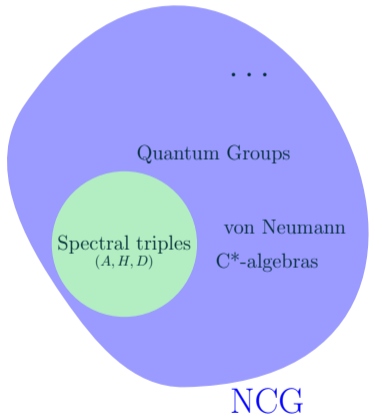
[Connes Marcolli, *NCG, QFT and motives*, '07, next screenshot]

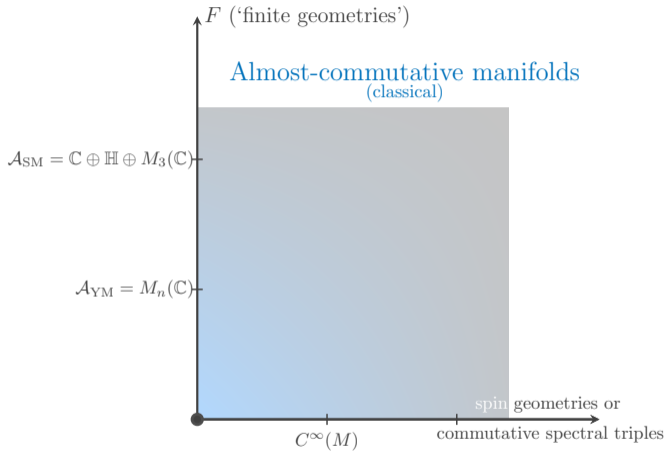
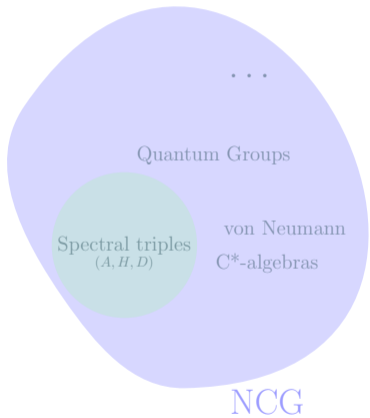
« The far distant goal is to set up a functional integral evaluating spectral observables \mathcal{S} as
(1.892) $\langle \mathcal{S} \rangle = \mathcal{N} \int \mathcal{S} e^{-\text{Tr}(f(D/\Lambda)) - \frac{1}{2} \langle J\psi, D\psi \rangle - \rho(e,D)} \mathcal{D}[\psi] \mathcal{D}[D] \mathcal{D}[e],$ »

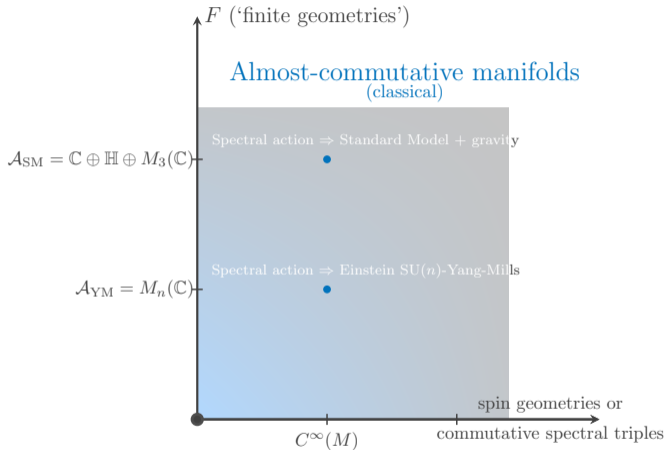
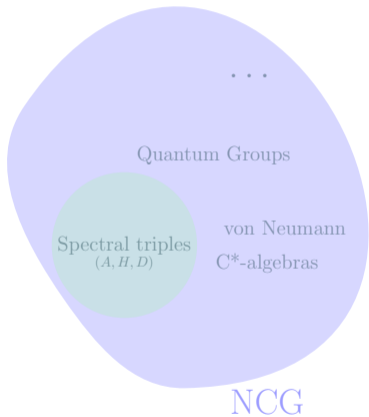
- The matrix algebra $M_k(\mathcal{A}_{k,N})$ where functional renormalisation for random matrices (k -matrix model) takes place was provided. $\mathcal{A}_{k,N}$ is a bigger relative of $\mathbb{C}^{\binom{N}{k}}$

thank you!

References: [CP 1912.13288] [CP 2007.10914] [CP 2102.06999] [CP 2105.01025] [CP 2111.02858]







NCG toolkit in high energy physics

- On a spectral triple (A, H, D) the (bosonic) classical action is given by

$$S(D) = \text{Tr}_H f(D/\Lambda) \quad [\text{Chamseddine-Connes CMP '97}]$$

for a bump function f , Λ a scale. It's computed with heat kernel expansion

[P. Gilkey, *J. Diff. Geom.* '75]

NCG toolkit in high energy physics

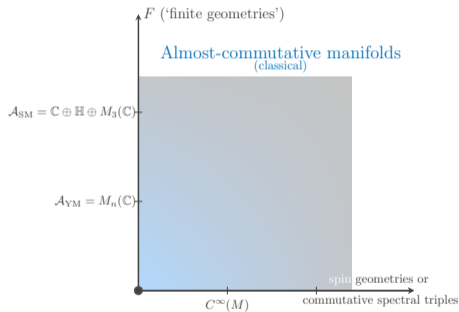
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[P. Gilkey, *J. Diff. Geom.* '75]

- Realistic, classical models come from *almost-commutative manifolds* $M \times F$, where F is a finite-dim. spectral triple $(C^\infty(A_F), H_M \otimes H_F, D_M \otimes 1_F + \gamma_5 \otimes D_F)$



NCG toolkit in high energy physics

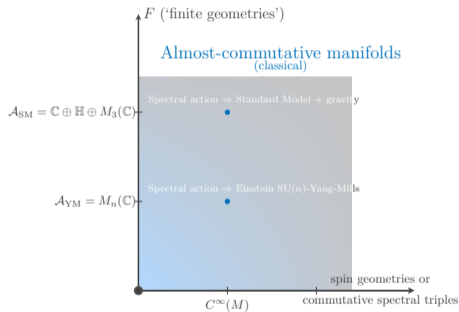
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- let's sketch *connections*: if S^G is a G -invariant functional on \mathcal{M}

$$S^G \rightsquigarrow S^{\text{Maps}(\mathcal{M}, G)}$$

$$d \rightsquigarrow d + \mathbb{A} \quad \mathbb{A} \in \Omega^1(\mathcal{M}) \otimes \mathfrak{g}$$

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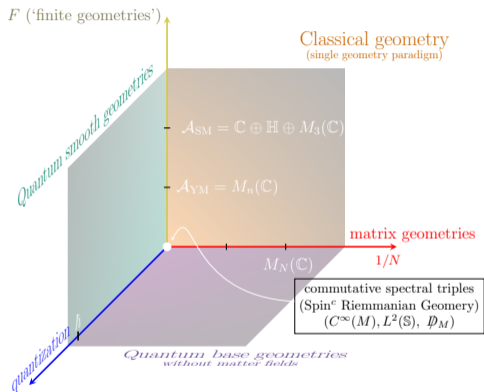
- given (A, H, D) and a Morita equivalent algebra B (i.e. $\text{End}_A(E) \cong B$) yields new $(B, E \otimes_A H, D')$. For $A = B$, in fact a tower

$$\{(A, H, D + \omega \pm J\omega J^{-1})\}_{\omega \in \Omega_D^1(A)}$$

$$D_\omega \mapsto \text{Ad}(u)D_\omega \text{Ad}(u)^* = D_{\omega_u}$$

$$\omega \mapsto \omega_u = u\omega u^* + u[D, u^*] \quad u \in \mathcal{U}(A) \quad \text{skip cube}$$

Organisation



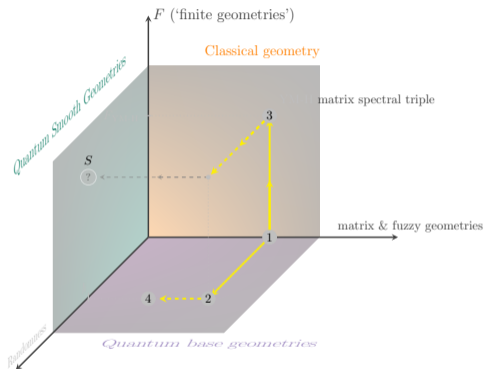
Aim: Make sense of

$$\mathcal{Z} = \int_{\text{Dirac}} e^{-\text{Tr}_H f(D)} dD$$

- Plane $(\hbar, 1/N, 0)$ of 'base geometries'
- Plane $(\hbar, 0, F) = \lim_{N \rightarrow \infty} (\hbar, 1/N, F)$
- Plane $(0, 1/N, F) = \lim_{\hbar \rightarrow 0} (\hbar, 1/N, F)$ of classical geometries

[CP 2105.01025]

Organisation



1 Matrix Geometries

[J. Barrett, *J. Math. Phys.* 2015]

2 Dirac ensembles [J. Barrett, L. Glaser, *J. Phys. A* 2016] and how to compute the spectral action

[CP 1912.13288]

3 Gauge matrix spectral triples (*this talk*)

[CP 2105.01025]

4 Functional Renormalisation [CP 2007.10914] and

[CP 2111.02858]

Operator	Its noncommutative Hessian
$\text{Tr}(A) \text{Tr}(A^3)$	$3 \cdot \begin{pmatrix} \text{Tr} A \cdot (A \otimes 1 + 1 \otimes A) & 0 \\ +1 \boxtimes A^2 + A^2 \boxtimes 1 & \\ 0 & 0 \end{pmatrix}$
$\text{Tr}(ABAB)$	$2 \cdot \begin{pmatrix} B \otimes B & (1 \otimes BA + AB \otimes 1) \\ (1 \otimes AB + BA \otimes 1) & A \otimes A \end{pmatrix}$
$\text{Tr} A \text{Tr}(AAABB)$	$\begin{pmatrix} (\text{Tr}(A)1 \otimes (ABB) + \text{Tr}(A)1 \otimes (BBA) + \text{Tr}(A)(ABB) \otimes 1 + \text{Tr}(A)(BBA) \otimes 1 + 1 \boxtimes (AABB) + 1 \boxtimes (ABBA) + 1 \boxtimes (BBAA) + (AABB) \boxtimes 1 + (ABBA) \boxtimes 1 + (BBAA) \boxtimes 1 + \text{Tr}(A)A \otimes B^2 + \text{Tr}(A)B^2 \otimes A) & \text{Tr}(A)(1 \otimes (AAB) + (BAA) \otimes 1 + A \otimes AB + B \otimes A^2 + A^2 \otimes B + BA \otimes A) + 1 \boxtimes (A^3 B) + 1 \boxtimes (BA^3) \\ \text{Tr}(A)(1 \otimes (BAA) + (AAB) \otimes 1 + A \otimes BA + B \otimes A^2 + A^2 \otimes B + AB \otimes A) + (A^3 B) \boxtimes 1 + (BA^3) \boxtimes 1 & \text{Tr}(A)(1 \otimes (A^3) + (A^3) \otimes 1) \end{pmatrix}$

Table: Some Hessians order operators. Here $\text{Tr} = \text{Tr}_N$.

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β -functions of NCG two-matrix models, signature $\eta = \text{diag}(\mathbf{e}_1, \mathbf{e}_2)$

$$2h_1(\mathbf{a}_4 + \mathbf{c}_{22} + 2\mathbf{d}_{2|02} + 6\mathbf{d}_{2|2}) = \eta_{\mathbf{a}}$$

$$2h_1(\mathbf{b}_4 + \mathbf{c}_{22} + 6\mathbf{d}_{02|02} + 2\mathbf{d}_{2|02}) = \eta_{\mathbf{b}}$$

$$-h_1[\mathbf{e}_{\mathbf{a}}(\mathbf{a}_4 - \mathbf{c}_{1111}) + 2\mathbf{d}_{1|12} + 6\mathbf{d}_{1|3}] + \mathbf{d}_{1|1}(\eta + 1) = \beta(\mathbf{d}_{1|1})$$

$$-h_1[\mathbf{e}_{\mathbf{b}}(\mathbf{b}_4 - \mathbf{c}_{1111}) + 6\mathbf{d}_{01|03} + 2\mathbf{d}_{01|21}] + \mathbf{d}_{01|01}(\eta + 1) = \beta(\mathbf{d}_{01|01})$$

The next block encompasses the connected quartic couplings:

$$h_2(4\mathbf{a}_4^2 + 4\mathbf{c}_{22}^2) + \mathbf{a}_4(2\eta + 1)$$

$$-h_1(24\mathbf{a}_6\mathbf{e}_{\mathbf{a}} + 4\mathbf{c}_{42}\mathbf{e}_{\mathbf{b}} + 4\mathbf{d}_{02|4}\mathbf{e}_{\mathbf{b}} + 4\mathbf{d}_{2|4}\mathbf{e}_{\mathbf{a}}) = \beta(\mathbf{a}_4)$$

$$h_2(4\mathbf{b}_4^2 + 4\mathbf{c}_{22}^2) + \mathbf{b}_4(2\eta + 1)$$

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$$-h_1(2\mathbf{e}_{\mathbf{a}}\mathbf{c}_{1212} + \mathbf{e}_{\mathbf{b}}2\mathbf{c}_{2121} + 3\mathbf{e}_{\mathbf{a}}\mathbf{c}_{24} + 3\mathbf{e}_{\mathbf{b}}\mathbf{c}_{42} + \mathbf{e}_{\mathbf{a}}\mathbf{d}_{02|22} + \mathbf{e}_{\mathbf{b}}\mathbf{d}_{2|22})$$

$$+h_2(2\mathbf{a}_4\mathbf{c}_{22} + 2\mathbf{b}_4\mathbf{c}_{22} + 2\mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}}\mathbf{c}_{1111}^2 + 2\mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}}\mathbf{c}_{22}^2) + \mathbf{c}_{22}(2\eta + 1) = \beta(\mathbf{c}_{22})$$

$$8\mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}}\mathbf{c}_{1111}\mathbf{c}_{22}h_2 + \mathbf{c}_{1111}(2\eta + 1)$$

$$+h_1(4\mathbf{e}_{\mathbf{a}}\mathbf{c}_{1311} + 4\mathbf{e}_{\mathbf{b}}\mathbf{c}_{3111} + 2\mathbf{e}_{\mathbf{a}}\mathbf{d}_{02|1111} + 2\mathbf{e}_{\mathbf{b}}\mathbf{d}_{2|1111}) = \beta(\mathbf{c}_{1111})$$

$$2h_2(6a_4a_6 + e_a e_b c_{22}c_{42}) + a_6(3\eta + 2) = \beta(a_6)$$

$$2h_2(6b_4b_6 + e_a e_b c_{22}c_{24}) + b_6(3\eta + 2) = \beta(b_6)$$

$$4h_2\{a_4c_{3111} + e_a e_b [c_{22}(c_{1311} + 2c_{3111}) - c_{1111}(2c_{2121} + c_{42})]\} + c_{3111}(3\eta + 2) = \beta(c_{3111})$$

$$2h_2[2a_4c_{2121} + e_a e_b (-2c_{1111}c_{3111} + 4c_{2121}c_{22} + c_{22}c_{24})] + c_{2121}(3\eta + 2) = \beta(c_{2121})$$

$$2h_2[a_4c_{24} + 3b_4c_{24} + 2e_a e_b (c_{22}(3b_6 + c_{2121} + c_{24} + c_{42}) - c_{1111}c_{1311})] + c_{24}(3\eta + 2) = \beta(c_{24})$$

$$4h_2\{b_4c_{1311} + e_a e_b [c_{22}(2c_{1311} + c_{3111}) - c_{1111}(2c_{1212} + c_{24})]\} + c_{1311}(3\eta + 2) = \beta(c_{1311})$$

$$2h_2[2b_4c_{1212} + e_a e_b (c_{22}(4c_{1212} + c_{42}) - 2c_{1111}c_{1311})] + c_{1212}(3\eta + 2) = \beta(c_{1212})$$

$$2h_2[3a_4c_{42} + 2e_a e_b (3a_6c_{22} - c_{1111}c_{3111} + c_{1212}c_{22} + c_{22}c_{24} + c_{22}c_{42}) + b_4c_{42}] + c_{42}(3\eta + 2) = \beta(c_{42})$$

Matrix or Fuzzy Geometries GO TO characterization \Leftarrow

Definition (“condensed” from [J. Barrett, *J. Math. Phys.* 2015]).

A *fuzzy geometry* of *signature* $(p, q) \in \mathbb{Z}_{\geq 0}^2$ is given by

- a simple matrix algebra \mathcal{A} - we take always $\mathcal{A} = M_N(\mathbb{C})$
- a Hermitian $\mathbb{C}\ell(p, q)$ -module \mathbb{S} with a *chirality* γ . That is a linear map $\gamma : \mathbb{S} \rightarrow \mathbb{S}$ satisfying $\gamma^* = \gamma$ and $\gamma^2 = 1$
- a Hilbert space $\mathcal{H} = \mathbb{S} \otimes M_N(\mathbb{C})$ with inner product $\langle v \otimes R, w \otimes S \rangle = (v, w) \text{Tr}_N(R^* S)$ for each $R, S \in M_N(\mathbb{C})$, being (\cdot, \cdot) the inner product of \mathbb{S}
- a left- \mathcal{A} representation $\rho(a)(v \otimes R) = v \otimes (aR)$ on \mathcal{H} , $a \in \mathcal{A}$ and $v \otimes R \in \mathcal{H}$

...

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- three signs $\epsilon, \epsilon', \epsilon'' \in \{-1, +1\}$ determined through $s := q - p$ by the following table:

$s \equiv q - p \pmod{8}$	0	1	2	3	4	5	6	7
ϵ	+	+	-	-	-	-	+	+
ϵ'	+	-	+	+	+	-	+	+
ϵ''	+	+	-	+	+	+	-	+

- a real structure $J = C \otimes *$, where $*$ is complex conjugation and C is an anti-unitarity on \mathbb{S} satisfying $C^2 = \epsilon$ and $C\gamma^\mu = \epsilon'\gamma^\mu C$ for all the gamma matrices $\mu = 1, \dots, p + q$.
- a self-adjoint operator D on \mathcal{H} satisfying the *order-one condition*

$$[[D, \rho(a)], J\rho(b)J^{-1}] = 0 \quad \text{for all } a, b \in \mathcal{A}$$
- a chirality $\Gamma = \gamma \otimes 1_{\mathcal{A}}$ for \mathcal{H} , where γ is the chirality of \mathbb{S} . The signs above impose: