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Combinatorial algebraic remarks on Dirac ensembles and related matrix models

Noncommutative Geometry, Free Probability Theory and Random Matrix Theory, Western University, June 13-17, 2022

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* TEAM Fundacja na Rzecz Nauiki Polskiej (Poland), † ERC, indirectly & DFG-Structures Excellence Cluster (Germany)

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A. Schenkel, O. Arizmendi, for discussion, comments and/or bibliography.

Outline

this talk ↓	noncomm. geometry	free prob.	random matrix theory
motivation	no	no	no
introduction	yes	no	no
fuzzy geometries	yes	?	yes
Dirac ensembles	yes	no	yes
renormalisation & free algebra	no	some	yes
gauge theories*	yes	no	after ‘quantisation’
outlook	yes	hopefully	yes

if time allows

Motivation

- From physics to NCG: The Standard Model from the Spectral Action

$$
\begin{gathered}
-\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4}g^2_s f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu + \\
\tfrac{1}{2}ig^2_s(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu + \bar{G}^a\partial^2 G^a + g_s f^{abc}\partial_\mu \bar{G}^a G^b g^c_\mu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - \\
M^2 W^+_\mu W^-_\mu - \tfrac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c^2_w}M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \\
\tfrac{1}{2}\partial_\mu H\partial_\mu H - \tfrac{1}{2}m^2_h H^2 - \partial_\mu\phi^+\partial_\mu\phi^- - M^2\phi^+\phi^- - \tfrac{1}{2}\partial_\mu\phi^0\partial_\mu\phi^0 - \\
\tfrac{1}{2c^2_w}M\phi^0\phi^0 - \beta_h[\tfrac{2M^2}{g^2} + \tfrac{2M}{g}H + \tfrac{1}{2}(H^2+\phi^0\phi^0+2\phi^+\phi^-)] + \tfrac{2M^4}{g^2}\alpha_h - \\
igc_w[\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + \\
Z^0_\mu(W^+_\nu\partial_\nu W^-_\mu - W^-_\nu\partial_\nu W^+_\mu)] - igs_w[\partial_\nu A_\mu(W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - A_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + A_\mu(W^+_\nu\partial_\nu W^-_\mu - \\
W^-_\nu\partial_\nu W^+_\mu)] - \tfrac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \tfrac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + \\
g^2 c^2_w(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s^2_w(A_\mu W^+_\mu A_\nu W^-_\nu - \\
A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w[A_\mu Z^0_\nu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - \\
2A_\mu Z^0_\mu W^+_\nu W^-_\nu] - g\alpha[H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-] - \tfrac{1}{8}g^2\alpha_h[H^4 + \\
(\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2] - \\
gMW^+_\mu W^-_\mu H - \tfrac{1}{2}g\tfrac{M}{c^2_w}Z^0_\mu Z^0_\mu H - \tfrac{1}{2}ig[W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - \\
W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)] + \tfrac{1}{2}g[W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) - \\
W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)] + \tfrac{1}{2}g\tfrac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) - \\
ig\tfrac{s^2_w}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_w MA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) - \\
ig\tfrac{1-2c^2_w}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + igs_w A_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \\
\tfrac{1}{4}g^2 W^+_\mu W^-_\mu[H^2 + (\phi^0)^2 + 2\phi^+\phi^-] - \tfrac{1}{4}g^2\tfrac{1}{c^2_w}Z^0_\mu Z^0_\mu[H^2 + (\phi^0)^2 + \\
2(2s^2_w - 1)^2\phi^+\phi^-] - \tfrac{1}{2}g^2\tfrac{s^2_w}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) -
\end{gathered}
$$

$$
\begin{gathered}
\tfrac{1}{2}ig^2\tfrac{s^2_w}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \tfrac{1}{2}g^2 s_w A_\mu\phi^0(W^+_\mu\phi^- + \\
W^-_\mu\phi^+) + \tfrac{1}{2}ig^2 s_w A_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\tfrac{s_w}{c_w}(2c^2_w - \\
1)Z^0_\mu A_\mu\phi^+\phi^- - g^1 s^2_w A_\mu A_\mu\phi^+\phi^- - \bar{e}^\lambda(\gamma\partial + m^\lambda_e)e^\lambda - \bar{\nu}^\lambda\gamma\partial\nu^\lambda - \\
\bar{u}^\lambda_j(\gamma\partial + m^\lambda_u)u^\lambda_j - \bar{d}^\lambda_j(\gamma\partial + m^\lambda_d)d^\lambda_j + igs_w A_\mu[-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \\
\tfrac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \tfrac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)] + \tfrac{ig}{4c_w}Z^0_\mu[(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + \\
(\bar{e}^\lambda\gamma^\mu(4s^2_w - 1 - \gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(\tfrac{4}{3}s^2_w - 1 - \gamma^5)u^\lambda_j) + (\bar{d}^\lambda_j\gamma^\mu(1 - \\
\tfrac{8}{3}s^2_w - \gamma^5)d^\lambda_j)] + \tfrac{ig}{2\sqrt{2}}W^+_\mu[(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(1 + \\
\gamma^5)C_{\lambda\kappa}d^\kappa_j)] + \tfrac{ig}{2\sqrt{2}}W^-_\mu[(\bar{e}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\lambda\kappa}\gamma^\mu(1 + \\
\gamma^5)u^\lambda_j)] + \tfrac{ig}{2\sqrt{2}}\tfrac{m^\lambda_e}{M}[-\phi^+(\bar{\nu}^\lambda(1-\gamma^5)e^\lambda) + \phi^-(\bar{e}^\lambda(1+\gamma^5)\nu^\lambda)] - \\
\tfrac{g}{2}\tfrac{m^\lambda_e}{M}[H(\bar{e}^\lambda e^\lambda) + i\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda)] + \tfrac{ig}{2M\sqrt{2}}\phi^+[-m^\kappa_d(\bar{u}^\lambda_j C_{\lambda\kappa}(1 - \\
\gamma^5)d^\kappa_j) + m^\lambda_u(\bar{u}^\lambda_j C_{\lambda\kappa}(1+\gamma^5)d^\kappa_j] + \tfrac{ig}{2M\sqrt{2}}\phi^-[m^\lambda_d(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1 + \\
\gamma^5)u^\kappa_j) - m^\kappa_u(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1-\gamma^5)u^\kappa_j] - \tfrac{g}{2}\tfrac{m^\lambda_u}{M}H(\bar{u}^\lambda_j u^\lambda_j) - \\
\tfrac{g}{2}\tfrac{m^\lambda_d}{M}H(\bar{d}^\lambda_j d^\lambda_j) + \tfrac{ig}{2}\tfrac{m^\lambda_u}{M}\phi^0(\bar{u}^\lambda_j\gamma^5 u^\lambda_j) - \tfrac{ig}{2}\tfrac{m^\lambda_d}{M}\phi^0(\bar{d}^\lambda_j\gamma^5 d^\lambda_j)
\end{gathered}
$$

ghosts
shouldn't
be
here

...this 'fits' in

$$\mathrm{Tr}(f(D/\Lambda)) + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle$$

Num. of generations and $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \longrightarrow$ NCG $\longrightarrow$ *Classical* Lagrangian of the Standard Model

[Chamseddine-Connes-Marcolli *ATMP* '07 (Euclidean); J. Barrett *J. Math. Phys.* '07 (Lorenzian)]

Motivation

- From physics to NCG: The Standard Model from the Spectral Action

$\Upsilon_{\cdot} \in M_3(\mathbb{C})$

$$D_F = \left(\begin{array}{cccc|cccccc|cccc|cccc}
0 & 0 & \Upsilon_\nu^* & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_R^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Upsilon_e^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Upsilon_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \Upsilon_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^* & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^* & \otimes 1_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Upsilon_u & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Upsilon_d & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\Upsilon_R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_\nu^T & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_e^T & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^T & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^T & \otimes 1_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_u & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_d & 0 & 0 &
\end{array}\right) \in M_{96}(\mathbb{C})_{\text{s.a.}}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

fermionic Sp. Action

$\langle J\psi, D_F \psi \rangle$

so one zero less

among $\sim 10^4$ entries

$\Rightarrow$ unseen

particle

interaction

but here, the

zeroes come from

geometry!

Num. of generations and $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \longrightarrow$ NCG $\longrightarrow$ *Classical* Lagrangian of the Standard Model

[Chamseddine-Connes-Marcolli *ATMP* '07 (Euclidean); J. Barrett *J. Math. Phys.* '07 (Lorenzian)]

Towards a quantum theory of noncommutative spaces

« *The far distant goal is to set up a functional integral evaluating spectral observables* $\mathscr{S}$ $\quad \langle \mathscr{S} \rangle = \int \mathscr{S}\, \mathrm{e}^{-\operatorname{Tr} f(D/\Lambda) - \frac{1}{2}\langle J\psi, D\psi\rangle + \rho(e,D)}\, \mathrm{d}e\, \mathrm{d}\psi\, \mathrm{d}D$ »

[Eq. 1.892, Connes Marcolli, *NCG, QFT and motives*, 2007]

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$$\text{functional integral} \xrightarrow[\text{paradigm shift}]{} \text{operator integral}$$

$$\int_{\text{metric}} \mathrm{e}^{-\frac{1}{\hbar} S_{\text{EH}}[g]} \mathrm{d}g \xrightarrow{\text{Einstein-Hilbert} \longrightarrow \text{spectral}} \int_{\text{Dirac}} \mathrm{e}^{-\frac{1}{\hbar} \operatorname{Tr} f(D)} \mathrm{d}D$$

(hard to define for manifolds)

$f : \mathbb{R} \longrightarrow \mathbb{R}$ with $f(D) \longrightarrow \infty$ at large argument

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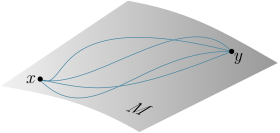
$f : \mathbb{R} \to \mathbb{R}$ with $f(D) \to \infty$ at large argument

- Possible application to (Euclidean) quantum gravity



Towards a quantum theory of noncommutative spaces

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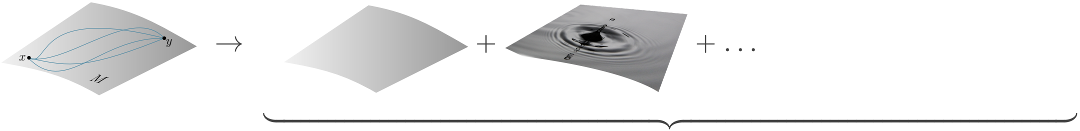

Quantum superposition of geometries

Towards a quantum theory of noncommutative spaces

> « *The far distant goal is to set up a functional integral evaluating spectral observables* $\mathscr{S}$ $\quad \langle \mathscr{S} \rangle = \int \mathscr{S}\, \mathrm{e}^{-\operatorname{Tr} f(D/\Lambda) - \frac{1}{2}\langle J\psi, D\psi\rangle + \rho(e,D)}\, \mathrm{d}e\, \mathrm{d}\psi\, \mathrm{d}D$ »

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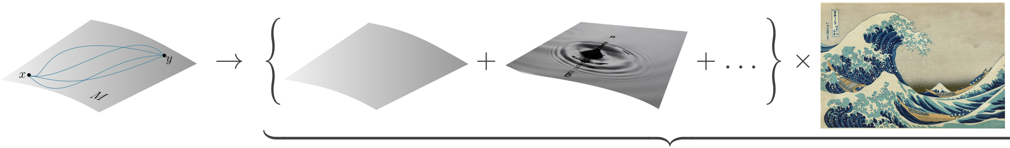

[Hokusai]

Quantum superposition of geometries (small perturbations+ instantons)

- Origin of noncommutative topology

Connes' *noncommutative (nc) geometry* = nc topology

[Gelfand, Najmark *Mat. Sbornik* '43] + metric [A. Connes, *NCG* '94]

{compact Hausdorff topological spaces} $\simeq$ {unital *commutative* C*-algebras}

• Origin of noncommutative topology

Connes' *noncommutative (nc) geometry* = nc topology

[Gelfand, Najmark *Mat. Sbornik* '43] + metric [A. Connes, *NCG* '94]

$$\begin{tikzcd}
\{\text{compact Hausdorff topological spaces}\} \arrow[r, phantom, "\simeq"] \arrow[d, hook] & \{\text{unital } \textit{commutative } \text{C}^*\text{-algebras}\} \arrow[d, hook] \\
\{\text{`noncommutative topological spaces'}\} \arrow[r, phantom, "\simeq"] & \{\text{unital } \cancel{\textit{commutative}} \text{ C}^*\text{-algebras}\}
\end{tikzcd}$$

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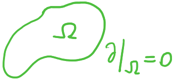
\end{tikzcd}$$

- arguably, the 1st NCG-theorem is *Weyl's law* (1911) on the rate of growth of the Laplace spectrum (ordered $\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \ldots$) of $\Omega \subset \mathbb{R}^d$

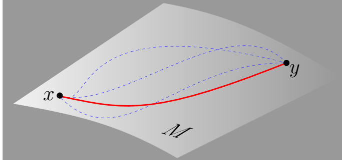


$$\#\{i : \lambda_i \leqslant \Lambda\} = \frac{\mathrm{vol}(\text{unit ball})}{(2\pi)^d} \mathrm{vol}\,\Omega \cdot \Lambda^{d/2} + \mathrm{o}(\Lambda^d)$$

  From this, you cannot answer positively Marek Kac' 1966-question. But you can 'hear the shape of Ω' knowing a *spectral triple*. [A, Connes, *JNCG* 2013] (and from it [Connes-van Suijlekom, *CMP* 2021] can hear an MP3; our story today is not entirely unrelated.)

Replace spin manifold (M, g) by $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

Connes' geodesic distance



$\gamma : \mathbb{R} \to M$

$\inf_{\gamma \text{ as above}} \{\int_\gamma \mathrm{d}s\} = d(x, y)$

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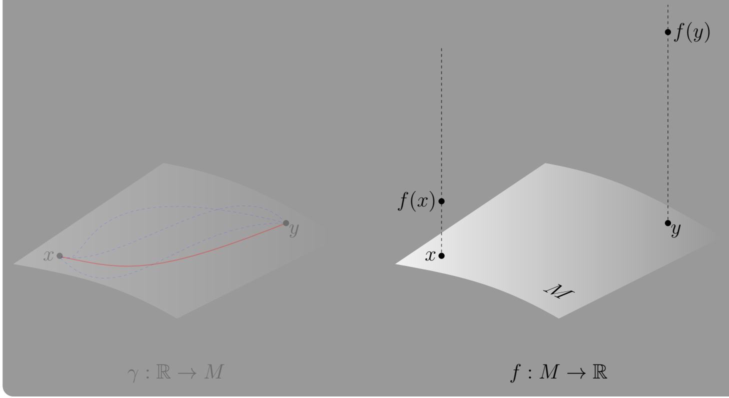

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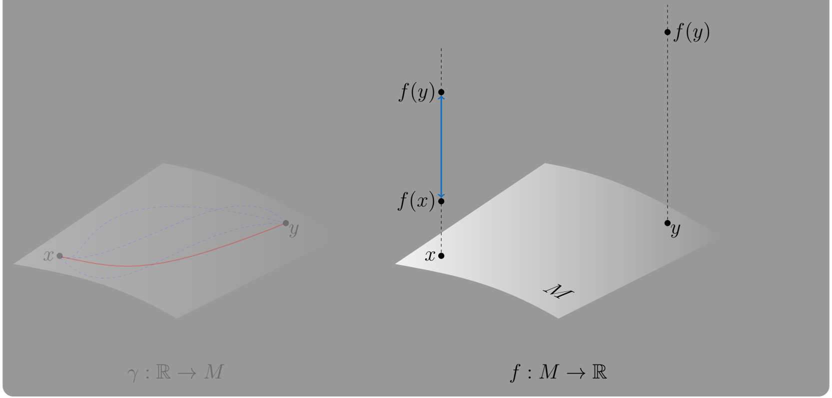

$$|f(x) - f(y)|$$

Replace spin manifold (M, g) by $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

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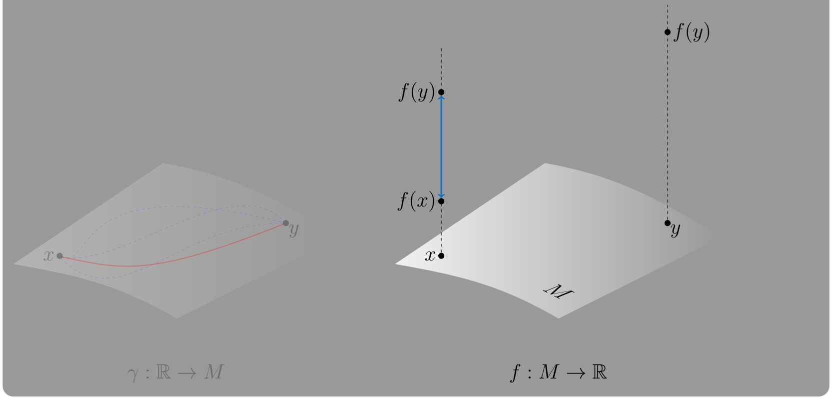

$$\sup_{f\in C^\infty(M)} \{ |f(x) - f(y)| \; : \; ||D_M f - f D_M|| \leqslant 1\}$$

Replace spin manifold (M, g) by $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

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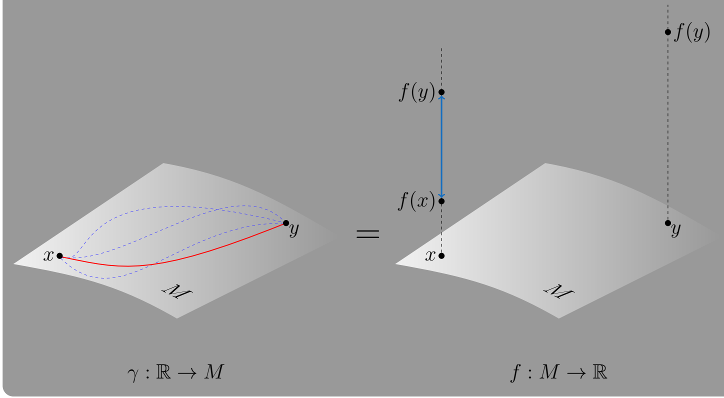

$$\inf_{\gamma \text{ as above}} \{\int_\gamma \mathrm{d}s\} = d(x, y) = \sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| \ : \ ||D_M f - f D_M|| \leqslant 1\}$$

Commutative spectral triples

A spin manifold M yields (A_M, H_M, D_M)

- $A_M = C^\infty(M)$ is a comm. $*$-algebra
- $H_M := L^2(M, \mathbb{S})$ a repr. of A_M
- $D_M = -\mathrm{i}\gamma^\mu(\partial_\mu + \omega_\mu)$ is self-adjoint
- for each $a \in A_M$, $[D_M, a]$ is bounded, and in fact $[D_M, x^\mu] = -\mathrm{i}\gamma^\mu$

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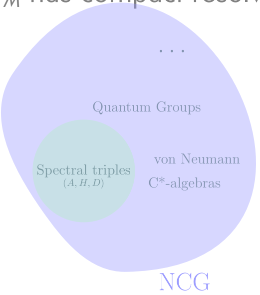
- for each $a \in A_M$, $[D_M, a]$ is bounded, and in fact $[D_M, x^\mu] = -\mathrm{i}\gamma^\mu$
- D_M has compact resolvent . . .



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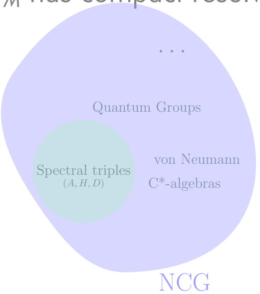
- for each $a \in A_M$, $[D_M, a]$ is bounded, and in fact $[D_M, x^\mu] = -\mathrm{i}\gamma^\mu$
- D_M has compact resolvent …



A *spectral triple* (A, H, D) consists of

- a $*$-algebra A
- a representation H of A
- a self-adjoint operator D on H with compact resolvent and such that $[D, a]$ is bounded for each $a \in A$

Commutative spectral triples

A spin manifold M yields (A_M, H_M, D_M)

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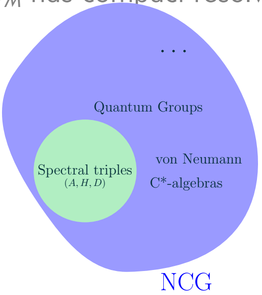

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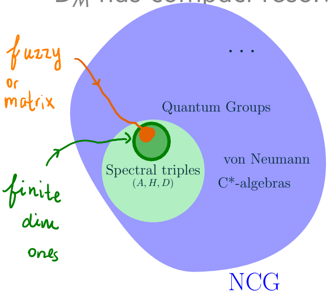

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II. Fuzzy Geometries and Multimatrix Models

A *fuzzy geometry* of signature (p, q), so $\eta = \mathrm{diag}(+_p, -_q)$, consists of

- $A = M_N(\mathbb{C})$
- $H = \mathbb{S} \otimes M_N(\mathbb{C})$, with $\mathbb{S}$ a $\mathbb{C}\ell(p, q)$-module

. . . (axioms for D omitted, `go to axioms` $\triangledown$) . . .

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multi-index J monot. increasing, $|J|$ odd [J. Barrett, *J. Math. Phys.* 2015], $H_J^* = H_J$, $L_J^* = -L_J$

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- Examples: [J. Barrett, L. Glaser, *J. Phys. A* 2016]
  - $D_{(1,1)} = \gamma^1 \otimes [L, \ \cdot\ ] + \gamma^2 \otimes \{H, \ \cdot\ \}$
  - $D_{(0,4)} = \sum_\mu \gamma^\mu \otimes [L_\mu, \ \cdot\ ] + \gamma^{\hat\mu} \otimes \{H_{\hat\mu}, \ \cdot\ \}$

  so we will get double traces from $\mathrm{Tr}_H = \mathrm{Tr}_{\mathbb{S}} \otimes \mathrm{Tr}_{M_N(\mathbb{C})} = \mathrm{Tr}_{\mathbb{S}} \otimes \mathrm{Tr}_N^{\otimes 2}$

Notation: $\mathrm{Tr}_V X$ is the trace on operators $X : V \to V$, $\mathrm{Tr}_V 1 = \dim V$. So $\mathrm{Tr}_N 1 = N$ but $\mathrm{Tr}_{M_N(\mathbb{C})}(1) = N^2$.

- A tool to organize the fuzzy spectral action is chord diagrams:

$$\mathrm{Tr}_{\mathbb{S}}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}) = \dim\mathbb{S}(\overbrace{\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}}^{\theta^{\mu_1\mu_2\mu_3\mu_4}} + \overbrace{(-)\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}}^{\xi^{\mu_1\mu_2\mu_3\mu_4}} + \overbrace{\eta^{\mu_2\mu_3}\eta^{\mu_1\mu_4}}^{\zeta^{\mu_1\mu_2\mu_3\mu_4}})$$

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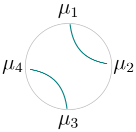

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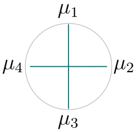

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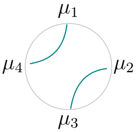

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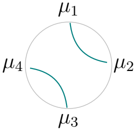
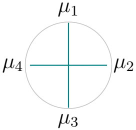
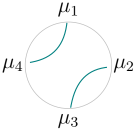
$\theta^{\mu_1\mu_2\mu_3\mu_4} =$ 

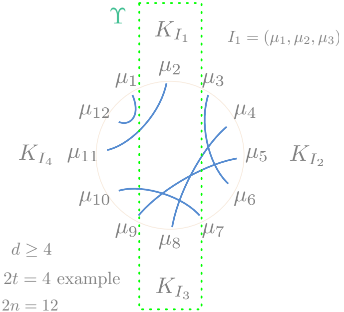
$\xi^{\mu_1\mu_2\mu_3\mu_4} =$ 

$\zeta^{\mu_1\mu_2\mu_3\mu_4} =$ 

- for dimension-d geometries, the combinatorial formula [CP '19] reads

$$\frac{1}{\dim\mathbb{S}}\mathrm{Tr}(D^{2t}) = \overbrace{\sum_{I_1,\ldots,I_{2t}\in\Lambda_d^-}}^{\text{if } J\in\Lambda_d,\ \mathrm{d}x^J\neq 0 \text{ on } \mathbb{R}^d} \Bigg\{ \overbrace{\sum_{\substack{\chi\in\mathrm{CD}_{2n}\\ 2n=\sum_i |I_i|}} \chi^{I_1\ldots I_{2t}}}^{\text{decorated chord diags}}$$

$$\times \Bigg(\sum_{\Upsilon\in\mathscr{P}_{2t}} \mathrm{sgn}(I_\Upsilon) \times \mathrm{Tr}_N\left(K_{I_{\Upsilon^c}}\right) \times \mathrm{Tr}_N\left[(K^T)_{I_\Upsilon}\right]\Bigg)\Bigg\}$$



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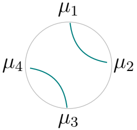
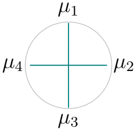
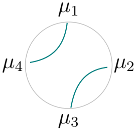
$\theta^{\mu_1\mu_2\mu_3\mu_4} =$ 

$\xi^{\mu_1\mu_2\mu_3\mu_4} =$ 

$\zeta^{\mu_1\mu_2\mu_3\mu_4} =$ 

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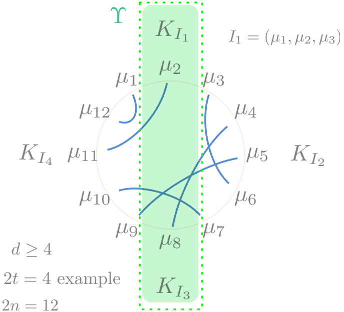


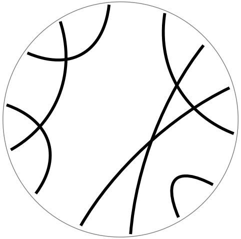
$d \geq 4$

$2t = 4$ example

$2n = 12$

trivial partitions yield $N \cdot \mathrm{Tr}_N(P)$ we all know and love...

Warning: These chord diagrams are not Feynman diagrams, they just determine the classical action.



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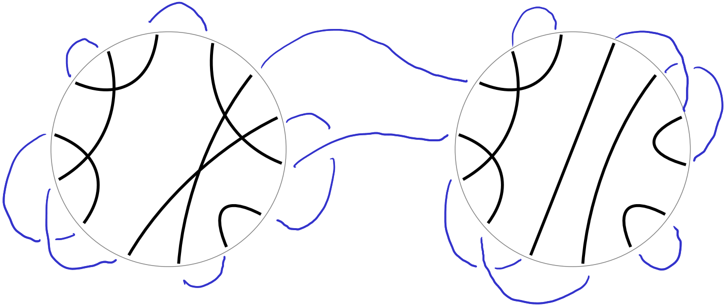
$$\sum_{\text{Colours}}$$



Feynman diags roughly

d=2

* blue ones make sense only if the black ends color agree, in general ill-defined

Let's go to the quantum theory, but before some remarks...

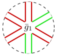
Multimatrix models with multitraces & ribbon graphs

- The chord-diagram description holds in general dimension and signature [CP '19]

$$\mathcal{Z} = \int_{\text{Dirac}} \mathrm{e}^{-\operatorname{Tr}_H f(D)} \mathrm{d}D \quad (\hbar = 1)$$
$$= \int_{M_{p,q}} \mathrm{e}^{-N \operatorname{Tr}_N P - \operatorname{Tr}_N^{\otimes 2}(Q_{(1)} \otimes Q_{(2)})} \mathrm{d}\mathbb{X}_{\text{Leb}}$$

- $\mathbb{X} \in M_{p,q}$ = products of $\mathfrak{su}(N)$ and $\mathcal{H}_N$
- $\mathrm{d}\mathbb{X}_{\text{Leb}}$ is the Lebesgue measure on $M_{p,q}$
- P, $Q_{(i)}$ are certain nc-polynomials
- $\mathcal{Z}_{\text{formal}}$ is the gen. func. of colored ribbon graphs (maps)

$\bar{g}_1 \operatorname{Tr}_N(ABBBAB) \quad \leftrightarrow$



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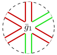

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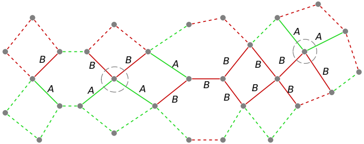
- Multitrace random matrices:
  - 'touching interactions' [Klebanov, *PRD* '95]
  - wormholes [Ambjørn-Jurkiewicz-Loll-Vernizzi, *JHEP* '01]
  - stuffed maps [G. Borot *AHP-D* '14]
  - trace polynomials [G. Cébron, *J.Funct.Anal.* '13]

    [D. Jekel-W. Li-D. Shlyakhtenko, 2021; A. Guionnet...,]
  - AdS/CFT [Witten, `hep-th/0112258`]

- Ribbon graphs: Enumeration of maps [Brezin, Itzykson, Parisi, Zuber, *CMP* '78], here 'face-worded'



III. Functional renormalisation in random matrices

Motivation from '2D-Quantum Gravity'

discrete surfaces	↔	matrix integrals $\mathcal{Z}(\lambda)$ [B. Eynard, *Counting Surfaces* '16]
smooth surface	↔	$\langle \text{area} \rangle$ finite & infinitesimal mesh $\mathfrak{a}$ $\langle \text{area} \rangle_g \sim \frac{\mathfrak{a}^2 (2-2g)}{\lambda/\lambda_\mathrm{c} - 1}$
all topologies	↔	$\mathcal{Z}(\lambda) = \sum_g N^{2-2g} \underset{\substack{\wr \\ (\lambda_\mathrm{c} - \lambda)^{(2-2g)/\theta}}}{\mathcal{Z}_g(\lambda)}$
⇑		
double-scaling limit		$N(\lambda_\mathrm{c} - \lambda)^{1/\theta} = C$
lin. RG-flow near a fixed point	↔	$\lambda(N) = \lambda_\mathrm{c} + (N/C)^{-\theta}$ $\theta = -(\partial\beta/\partial\lambda)\vert_{\lambda_\mathrm{c}}$ [Eichhorn-Koslowski, PRD, '13]

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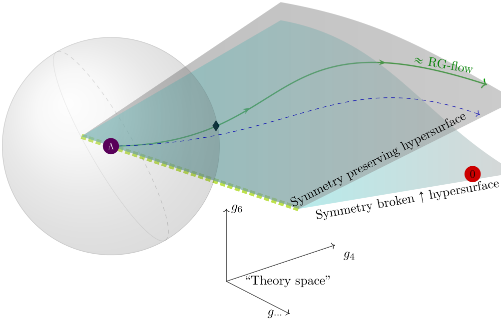
[CP 20] →



Λ	Chosen bare action $S = \Gamma_{N=\Lambda}$
0	Full effective action $\Gamma = \Gamma_{N=0}$
◆	Interpolating action $\Gamma_{N=\Lambda-\rho}$ (projected & truncated)
⟶	RG-flow with truncation and projection
⋯	Moduli of Dirac operators ↪ theory space
⇢	RG-flow without truncation nor projection
$g_{\dots}$	Rest of coupling constants

Homogeneisation of notation

- Let's write $\mathbb{E}[\,\bullet\,]$ for $\langle \bullet \rangle$. Wick's theorem [L. Isserlis *Biometrika* 1918]: for zero-mean x_i's, ...

$$\mathbb{E}[x_{j_1} \cdots x_{j_{2n}}] = \sum_{\substack{\pi \in P_2(2n) \\ \text{(pairings)}}} \prod_{(p,q) \in \pi} \mathbb{E}[x_{j_p} x_{j_q}]$$

- k the number of Hermitian matrices of size N, $X_1^{(N)}, \ldots, X_k^{(N)}$ [Piotr Śniady's notation]

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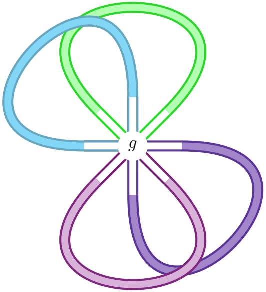
- Ribbon graphs. For $\mathbb{E}[(X_\mu^{(N)})_{i,j}(X_\rho^{(N)})_{l,m}] = \delta_{\mu\rho}\delta_{im}\delta_{jl}$ $\qquad \mu,\rho = 1,\ldots,k;\ i,\ldots,m = 1,\ldots,N$

$$(gN)\cdot\mathbb{E}\Big[\,\mathrm{Tr}_N\left(X_1^{(N)}X_2^{(N)}X_1^{(N)}X_2^{(N)}X_3^{(N)}X_4^{(N)}X_3^{(N)}X_4^{(N)}\right)\Big] =$$

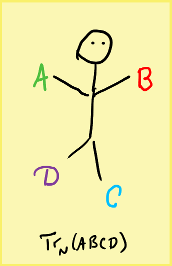


$$\sim N^{-2}$$

- Some Feynman graphs of multimatrix ϕ^4-theory...

Several-loop graph

[pic by 'Princi19skydiver', Wikipedia]



One-loop graph

[pic by 'Wojciech Kielar', Wikipedia]

foot-foot, foot-hand, ok.
but letters must coincide!

(ignore the head)

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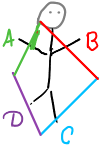

One-loop graph

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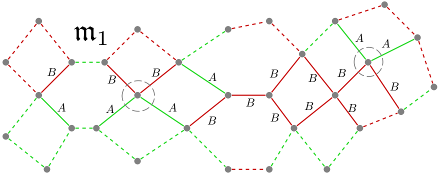
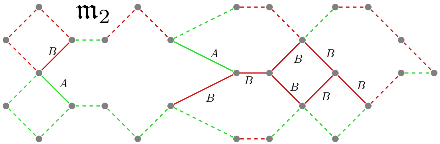
- advantage of functional renormalisation: 1-loops only





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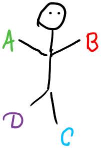

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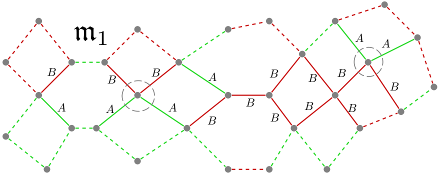
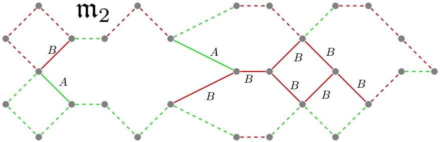


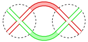


- *Question (sloppy version):* Given an operator $\mathrm{Tr}_N\, w$, in $w \in \mathbb{C}^{(N)}_{\langle k \rangle}$, find (up to given degree in the couplings) all 1-loops it might come from, in the above example sense

Functional Renormalisation for k-matrix models (w/multitrace-measures)

Quantum theories 'flow' with energy, here in RG-time $t = \log N$, $1 \ll N < \mathscr{N}$. E.g. for $k = 2$ and with bare action

$$S[A,B] = \mathscr{N}\, \mathrm{Tr}_{\mathscr{N}} \left\{ \frac{1}{2} A^2 + \frac{1}{2} B^2 + g_{A^4} \frac{1}{4} A^4 + g_{B^4} \frac{1}{4} B^4 + \frac{1}{2} g_{ABAB} ABAB \right\}$$

radiative corrections 'generate' *effective vertices*. For instance  generates $\mathscr{N}\, \mathrm{Tr}_{\mathscr{N}}(ABBA)$.

$$\Gamma_N[A,B] = \mathrm{Tr}_N \Bigg\{ \overbrace{\frac{Z_A}{2} A^2 + \frac{Z_B}{2} B^2 + \bar{g}_{A^4} \frac{1}{4} A^4 + \bar{g}_{B^4} \frac{1}{4} B^4 + \frac{1}{2} \bar{g}_{ABAB} ABAB}^{\text{operators from the bare action (but with 'running couplings')}} + \overbrace{\frac{1}{2} \bar{g}_{ABBA} ABBA + \frac{1}{2} \bar{g}_{A|A}\, \mathrm{Tr}_N(A) \times A + \ldots}^{\text{radiative corrections}} \Bigg\}$$

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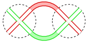
radiative corrections 'generate' *effective vertices*. For instance  generates $\mathcal{N}\,\mathrm{Tr}_{\mathcal{N}}(ABBA)$.

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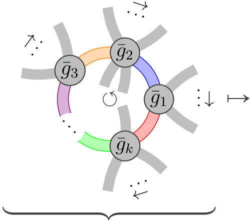
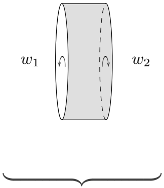
We are interested in *one-loop graphs*. The *effective vertex* O_G^{eff} of such a graph is formed by reading off each word w_i traveling around all ribbon edges (propagators) by both sides:

$$O_G^{\mathrm{eff}} = \overbrace{\mathrm{Tr}_N(w_1)\times\mathrm{Tr}_N(w_2)\times\cdots\times\mathrm{Tr}_N(w_s)}^{\text{from vertices contracted with propagators}}$$
$$\underbrace{\times\,\mathrm{Tr}_N(U_1)\times\mathrm{Tr}_N(U_2)\ldots\times\mathrm{Tr}_N(U_r)}_{\text{from vertices uncontracted with propagators}}$$

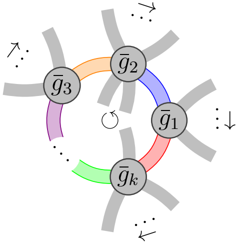


gral. situation (yet, without multitraces)



inner word $w_1 \otimes$ outer word w_2

So the actual question is: find the pre-image of the map



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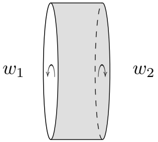
$\longmapsto$



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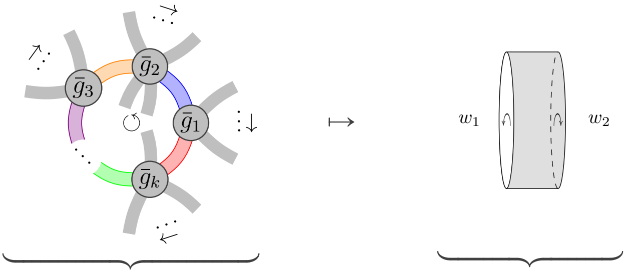

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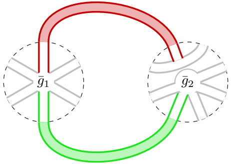
... actually, for multitrace operators: pre-image of any $O = \prod_\alpha \mathrm{Tr}_N\, w_\alpha$:



One then sums over the product of all g_i's appearing in such 1-loops.
These polynomials span the β-function for O.

Two steps

1. Understanding the Func. Renormalisation Eq.

[CP 2007.10914, *Ann. Henri Poincaré* 2021]

- prove Wetterich Equation, or FRGE; its proof determines the algebra that governs geometric series in the Hessian of Γ
- [A. Eichhorn, T. Koslowski, *Phys. Rev. D* '13] oriented us, but the proof of the FRGE dictates an algebra not reported there
- β-equations found for a sextic truncation (48 running operators). For the unique real solution g^* leading to a single relevant direction (positive e.v. of $-(\partial\beta_i/\partial g_j)_{i,j}|_{g*}$) yields an R_N-dependent

  $$g^*_{A^4} = 1.002 \times \Big(g^*_{A^4}\big|_{\text{[Kazakov-Zinn-Justin, \textit{Nucl. Phys. B} '99]}} \Big)$$

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2. Unicity (using a ribbon graph argument)

[CP 2111.02858 *Lett. Math. Phys.* 2022]

- write down Wetterich Equation "$\dot{\Gamma} = \frac{1}{2} \mathrm{Tr}_{M_k(\mathcal{A})} \big\{ \dot{R}_N / [\Gamma^{(2)} + R_N] \big\}$"
- assume an expansion of its rhs in unitary-invariant operators ($\neq$ exact RG)
- impose the one-loop structure and solve for the algebra $\mathcal{A} = \mathcal{A}_{k,N}$
- determine from it the algebra that computes Wetterich equation; it is unique and the one reported in [CP 2007.10914]

- *nc-derivative* $\partial_A : \mathbb{C}_{\langle k \rangle} \to \mathbb{C}_{\langle k \rangle}^{\otimes 2}$ sums over 'replacements of A by $\otimes$' [Turnbull+Rota-Sagan-Stein+Voiculescu]:

$$\partial_A(PAAR) = P \otimes AR + PA \otimes R\,, \ \text{but}$$
$$\partial_A(ALGEBRA) = 1 \otimes LGEBRA + ALGEBR \otimes 1$$

- parenthetically, nc-derivatives allow to compactly write loop-equations (or Dyson-Schwinger eqs.) in random matrix theory

$$\mathbb{E}\left[\left(\frac{1}{N}\,\mathrm{Tr}_N \otimes \frac{1}{N}\,\mathrm{Tr}_N\right)(\partial_X P)\right] = \mathbb{E}[P \mathscr{D}_X V] \qquad P \in \mathbb{C}_{\langle k \rangle}$$

[J. Mingo, R. Speicher *Probab. Math. Stat.* 2013] [A. Guionnet *Jpn. J. Math.* 2016] (cf. S. Azarfar and N. Pagliaroli's talks)

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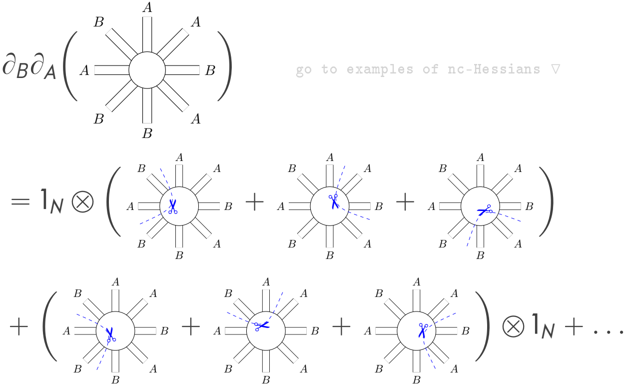
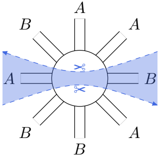


go to examples of nc-Hessians ▽

in ellipsis $\sum_{cuts}$ like



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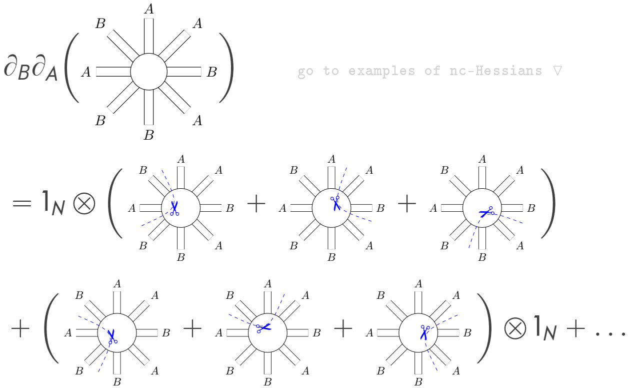

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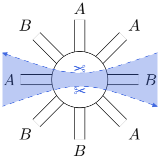

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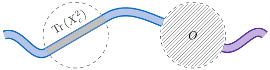
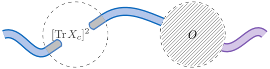
- products of traces ⇒ extend by $\boxtimes$, $\mathcal{A}_k = (\mathbb{C}_{\langle k\rangle} \otimes \mathbb{C}_{\langle k\rangle}) \oplus (\mathbb{C}_{\langle k\rangle} \boxtimes \mathbb{C}_{\langle k\rangle})$





$$\mathrm{Hess}_{a,b}(\mathrm{Tr}\, P \cdot \mathrm{Tr}\, Q) = \mathrm{Tr}\, P \cdot \mathrm{Hess}_{a,b}[\mathrm{Tr}\, Q] + (\partial_{X_a} \mathrm{Tr}\, P) \boxtimes (\partial_{X_b} \mathrm{Tr}\, Q) + (P \leftrightarrow Q)$$

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yields an ordinary unit

generates the constants in the real vect. space

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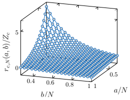

assume

$$= \sum_{k=0}^{\infty} \bar{h}_k(N, \eta_1, \dots, \eta_n) \times \underbrace{\frac{1}{2} (-1)^k \mathrm{Tr}_{\mathcal{M}_k(\mathcal{A})} \left\{ (\mathrm{Hess}\, \Gamma_N^{\mathrm{Int}}[\mathbb{X}])^{\star k} \right\}}_{\text{regulator-independent part}}$$

- $\mathrm{STr} = \mathrm{Tr}_k \otimes \mathrm{Tr}_{\mathcal{A}_k}$. Tadpoles

imply

$$\mathrm{Tr}_{\mathcal{A}_k}(P \otimes Q) = \mathrm{Tr}_N\, P \cdot \mathrm{Tr}_N\, Q, \qquad \mathrm{Tr}_{\mathcal{A}_k}(P \boxtimes Q) = \mathrm{Tr}_N (PQ)$$

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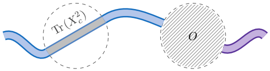

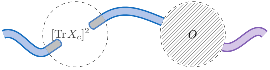

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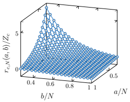
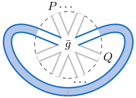
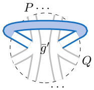
$$\overset{\text{assume}}{=} \sum_{k=0}^{\infty} \bar{h}_k(N, \eta_1, \ldots, \eta_n) \times \underbrace{\frac{1}{2}(-1)^k\,\mathrm{Tr}_{M_k(\mathcal{A})}\left\{(\mathrm{Hess}\,\Gamma_N^{\mathrm{Int}}[\mathbb{X}])^{\star k}\right\}}_{\text{regulator-independent part}}$$



- STr $= \mathrm{Tr}_k \otimes \mathrm{Tr}_{\mathcal{A}_k}$. Tadpoles





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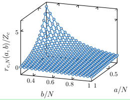
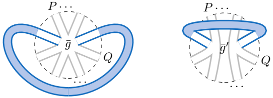


U(N)-invariants

$$\overset{\text{assume}}{=} \sum_{k=0}^{\infty} \bar{h}_k(N, \eta_1, \ldots, \eta_n) \times \underbrace{\frac{1}{2}(-1)^k \mathrm{Tr}_{M_k(\mathcal{A})} \left\{ (\mathrm{Hess}\, \Gamma_N^{\mathrm{Int}}[\mathbb{X}])^{\star k} \right\}}_{\text{regulator-independent part}}$$

Projecting to U(N)-invariants

- STr = $\mathrm{Tr}_k \otimes \mathrm{Tr}_{\mathcal{A}_k}$. Tadpoles*  imply

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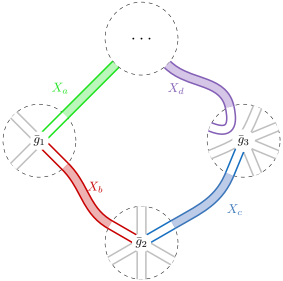
called
* tadpoles, since
usually look like

Finding $\star$

Want: 

$\subset \mathrm{Hess}_{a,b}\, O_1 \star \mathrm{Hess}_{b,c}\, O_2 \star \mathrm{Hess}_{c,d}\, O_3 \star \ldots \star \mathrm{Hess}_{*,a}\, O_\ell$

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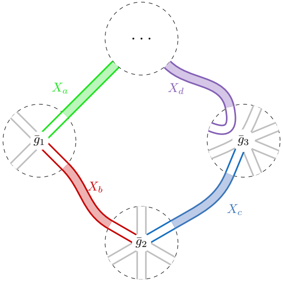
Want: 

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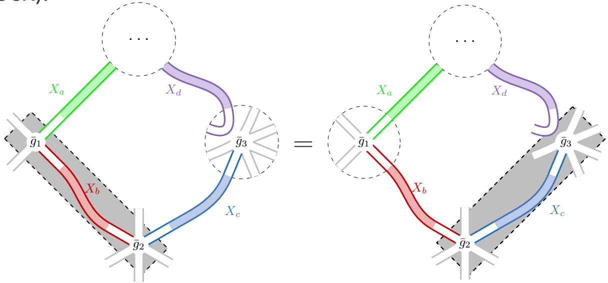
Associativity (trivial check):



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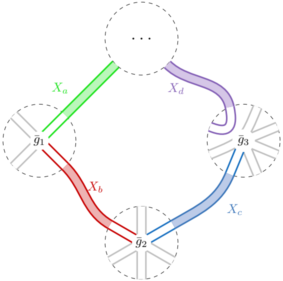

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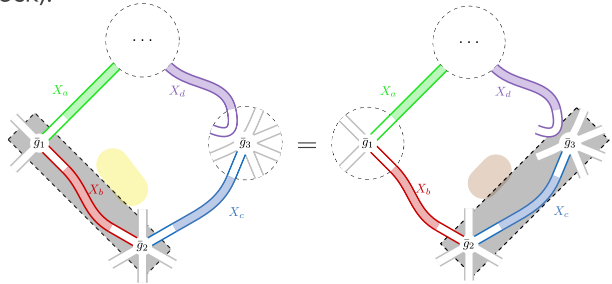

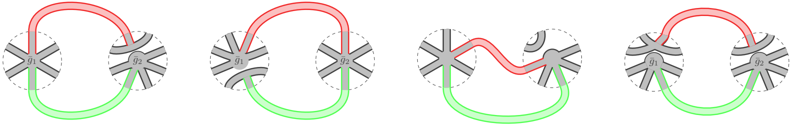

$$\begin{tikzcd}
\text{Case I} \arrow[d] & \text{Case II} \arrow[d] & \text{Case III} \arrow[d] & \text{Case IV} \arrow[d] \\
\mathrm{Tr}_{\mathcal{A}_n}[(U \otimes W) \star (P \boxtimes Q)] \arrow[ddd] & \mathrm{Tr}_{\mathcal{A}_n}[(U \boxtimes W) \star (P \otimes Q)] \arrow[ddd] & \mathrm{Tr}_{\mathcal{A}_n}[(U \otimes W) \star (P \otimes Q)] \arrow[d] & \mathrm{Tr}_{\mathcal{A}_n}[(U \boxtimes W) \star (P \boxtimes Q)] \arrow[dl] \arrow[dd] \\
 & & (U \otimes W) \star (P \otimes Q)/\sim \arrow[dr] \arrow[ddd] \arrow[dddr] & \\
 & & & (U \boxtimes W) \star (P \boxtimes Q)/\sim \arrow[dlll, bend right] \arrow[dll] \arrow[ddl] \arrow[dd] \\
(U \otimes W) \star (P \boxtimes Q) \arrow[drr] \arrow[drrr, bend right] & (U \boxtimes W) \star (P \otimes Q) & & \\
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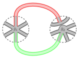
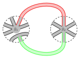
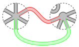
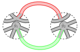
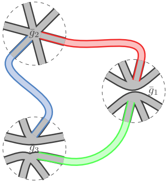
\end{tikzcd}$$

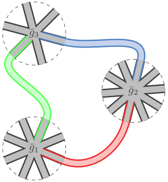


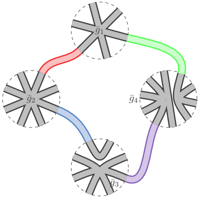












$$\begin{tikzcd}
\text{Case I} \arrow[d] & \text{Case II} \arrow[d] & \text{Case III} \arrow[d] & \text{Case IV} \arrow[d] \\
\mathrm{Tr}_{\mathcal{A}_n}[(U \otimes W) \star (P \boxtimes Q)] \arrow[dd] & \mathrm{Tr}_{\mathcal{A}_n}[(U \boxtimes W) \star (P \otimes Q)] \arrow[dd] & \mathrm{Tr}_{\mathcal{A}_n}[(U \otimes W) \star (P \otimes Q)] \arrow[d] & \mathrm{Tr}_{\mathcal{A}_n}[(U \boxtimes W) \star (P \boxtimes Q)] \arrow[dd] \arrow[dl] \\
 & & (U \otimes W) \star (P \otimes Q)/\sim \arrow[dd] \arrow[ddr] \arrow[dr] & \\
(U \otimes W) \star (P \boxtimes Q) \arrow[drr] \arrow[drrr] & (U \boxtimes W) \star (P \otimes Q) & & (U \boxtimes W) \star (P \boxtimes Q)/\sim \arrow[lll, bend right] \arrow[ll] \arrow[dl] \arrow[d] \\
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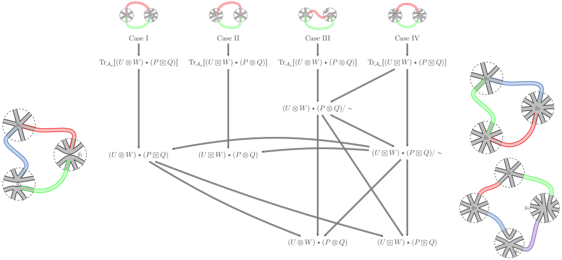
\end{tikzcd}$$

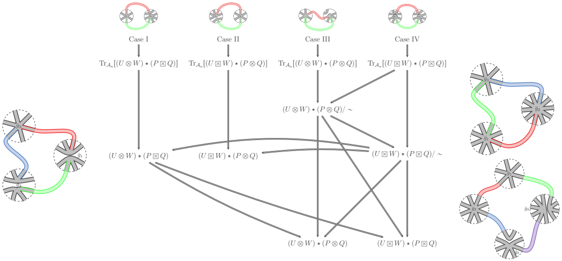


Thm. [CP '22] If the RG-flow is computable in terms of $\mathrm{U}(N)$-invariants, the algebra of Functional Renormalisation is $M_k(\mathcal{A}_{k,N},\star)$ where

$$\mathcal{A}_{k,N} = (\mathbb{C}^{(N)}_{\langle k\rangle}\otimes\mathbb{C}^{(N)}_{\langle k\rangle})\oplus(\mathbb{C}^{(N)}_{\langle k\rangle}\boxtimes\mathbb{C}^{(N)}_{\langle k\rangle})$$

whose product in hom. elements reads:

$$\begin{aligned}(U\otimes W)\star(P\otimes Q) &= PU\otimes WQ\,,\\ (U\boxtimes W)\star(P\otimes Q) &= U\boxtimes PWQ\,,\\ (U\otimes W)\star(P\boxtimes Q) &= WPU\boxtimes Q\,,\\ (U\boxtimes W)\star(P\boxtimes Q) &= \mathrm{Tr}_N(WP)U\boxtimes Q\,,\end{aligned}$$



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$$\mathrm{Tr}_{\mathcal{A}_k}(P\otimes Q)=\mathrm{Tr}_N\,P\cdot\mathrm{Tr}_N\,Q\,,$$
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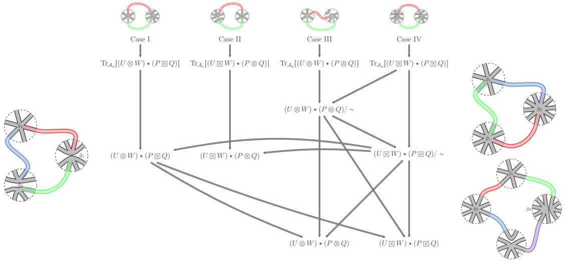

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Remark: To be more precise, any occurrence of the free algebra in $\mathcal{A}_{k,N}$ should be replaced by the algebra of 'trace polynomials' (e.g. $\mathrm{Tr}_N(X_1 X_3) X_2 + N \, \mathrm{Tr}_N(X_2^2)$) [D. Jekel-W. Li-D. Shlyakhtenko, '21]

Example: A Hermitian 3-matrix model. Consider $O_1 = \frac{\bar{g}_1}{2}[\mathrm{Tr}_N(\frac{A^2}{2})]^2$ and $O_2 = \bar{g}_2\,\mathrm{Tr}_N(ABC)$.

$$\mathrm{Hess}_{I,J}\, O_1 = \delta_I^J \delta_I^A \bar{g}_1 \{\underbrace{\mathrm{Tr}_N(A^2/2)\cdot[1_N \otimes 1_N]} + \underbrace{A \boxtimes A}\}\,,$$

where a ‘filled ribbon’ means contracted in the one-loop graph, and ‘white ribbon’ uncontracted.

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A B C

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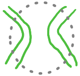
$O_1 \sim$



(uncontracted vertex)

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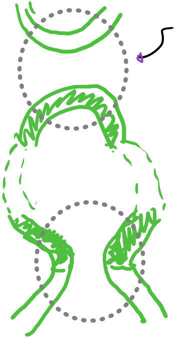

counts as a single vertex even if as graph disconnected

Recalling also Gaëtan's lecture, multi-traces ⇒ maps are stuffed.

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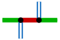

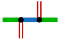

$$\mathrm{Hess}\, O_2 = \bar{g}_2 \begin{bmatrix} 0 & C \otimes 1_N & B \otimes 1_N \\ 1_N \otimes C & 0 & A \otimes 1_N \\ 1_N \otimes B & 1_N \otimes A & 0 \end{bmatrix} \Rightarrow [\mathrm{Hess}\, O_2]^{\star 2} = \bar{g}_2^2 \begin{bmatrix} \overbrace{C \otimes C} + \overbrace{B \otimes B} & B \otimes A & C \otimes A \\ A \otimes B & A \otimes A + C \otimes C & C \otimes B \\ A \otimes C & B \otimes C & B \otimes B + A \otimes A \end{bmatrix}.$$

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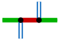

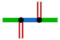

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Extracting coefficients

$$[\bar g_1 \bar g_2^2]\,\mathrm{Tr}_{M_3(\mathcal{A})}\{\mathrm{Hess}\, O_1 \star [\mathrm{Hess}\, O_2]^{\star 2}\} = \mathrm{Tr}_N(A^2/2)\times\left[(\mathrm{Tr}_N\, C)^2 + (\mathrm{Tr}_N\, B)^2\right] + \mathrm{Tr}_N(ACAC + ABAB)\,,$$

which are effective vertices of the four one-loop graphs that can be formed with the contractions of (the filled ribbon half-edges of) any of {, } with any of {, }.

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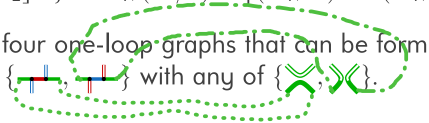

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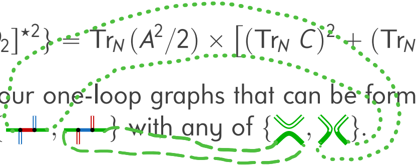

Why not using graphs? Soon, nc-Hessians get bulky: In [CP '20] 48 such operators run

these entries are not in $\mathbb{C}_{\langle 2\rangle}$ / cyclicity

Operator	Its nc-Hessian
$\mathrm{Tr}(A)\,\mathrm{Tr}(A^3)$	$3\cdot\begin{pmatrix} \mathrm{Tr}\,A\cdot(A\otimes 1+1\otimes A) +1\boxtimes A^2+A^2\boxtimes 1 & 0 \\ 0 & 0 \end{pmatrix}$
$\mathrm{Tr}\,A\,\mathrm{Tr}(AAABB)$	$\begin{pmatrix} (\mathrm{Tr}(A)1\otimes(ABB)+\mathrm{Tr}(A)1\otimes(BBA)+\mathrm{Tr}(A)(ABB)\otimes 1+\mathrm{Tr}(A)(BBA)\otimes 1+1\boxtimes(AABB)+1\boxtimes(ABBA)+1\boxtimes(BBAA)+(AABB)\boxtimes 1+(ABBA)\boxtimes 1+(BBAA)\boxtimes 1+\mathrm{Tr}(A)A\otimes B^2+\mathrm{Tr}(A)B^2\otimes A) & \mathrm{Tr}(A)(1\otimes(AAB)+(BAA)\otimes 1+A\otimes AB+B\otimes A^2+A^2\otimes B+BA\otimes A)+1\boxtimes(A^3B)+1\boxtimes(BA^3) \\ \mathrm{Tr}(A)(1\otimes(BAA)+(AAB)\otimes 1+A\otimes BA+B\otimes A^2+A^2\otimes B+AB\otimes A)+(A^3B)\boxtimes 1+(BA^3)\boxtimes 1 & \mathrm{Tr}(A)(1\otimes(A^3)+(A^3)\otimes 1) \end{pmatrix}$

Table: Some Hessians. Here $\mathrm{Tr}=\mathrm{Tr}_N$.

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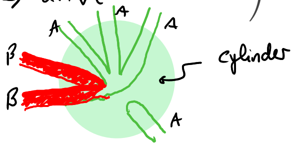
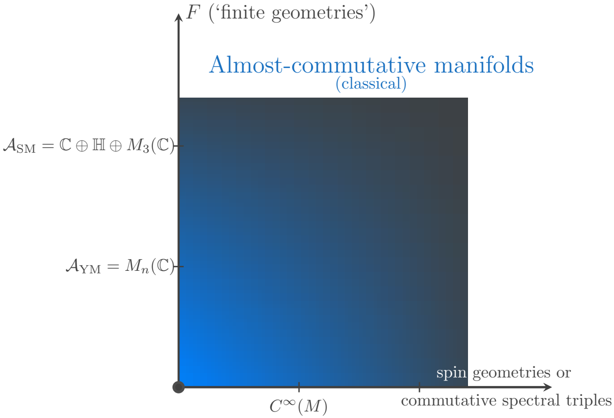
empty ⇒ alive



III. Matrix gauge theory



F ('finite geometries')

Almost-commutative manifolds

(classical)

$\mathcal{A}_{\mathrm{SM}} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$

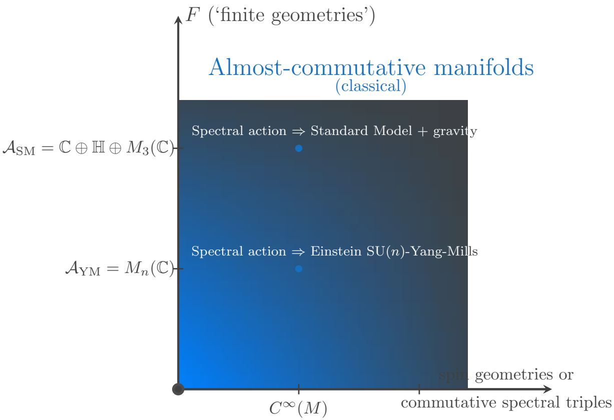
$\mathcal{A}_{\mathrm{YM}} = M_n(\mathbb{C})$

$C^\infty(M)$

spin geometries or

commutative spectral triples

III. Matrix gauge theory



F ('finite geometries')

Almost-commutative manifolds
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Spectral action $\Rightarrow$ Standard Model + gravity

$\mathcal{A}_{\mathrm{SM}} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$

Spectral action $\Rightarrow$ Einstein SU(n)-Yang-Mills

$\mathcal{A}_{\mathrm{YM}} = M_n(\mathbb{C})$

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spin geometries or
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$$\mathcal{Z}_{\mathsf{AC}} \overset{?}{=} \int_{\mathsf{Dirac}} \mathrm{e}^{-\frac{1}{\hbar}\,\mathsf{Tr}\, f(D)} \mathrm{d}D$$

(hard for almost-commutative manifolds)

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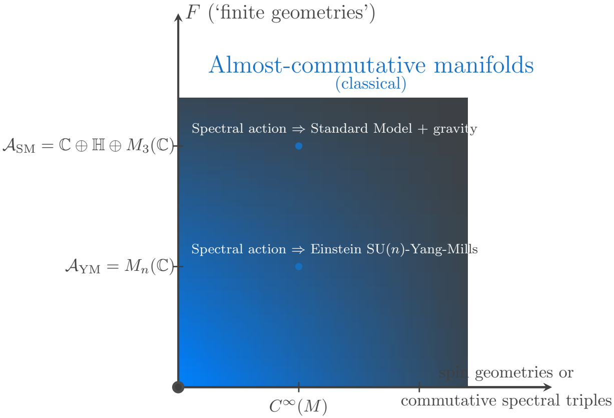

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Definition [CP 2105.01025] We define a *gauge matrix spectral triple* $G_\ell \times F$ as the spectral triple product of a fuzzy geometry G_ℓ with a finite geometry $F = (A_F, H_F, D_F)$, $\dim A_F < \infty$.

Lemma-Definition [CP 2105.01025] Consider a gauge matrix spectral triple $G_\ell \times F$ with

$$F = (M_n(\mathbb{C}), M_n(\mathbb{C}), D_F)$$

and G_ℓ Riemannian ($d = 4$) fuzzy geometry on $M_N(\mathbb{C})$, whose fluctuated Dirac op. is

$$D_\omega = \sum_{\mu=0}^{3} \overbrace{\gamma^\mu \otimes (\ell_\mu + \mathscr{a}_\mu) + \gamma^{\hat{\mu}} \otimes (\mathscr{x}_\mu + \mathscr{s}_\mu)}^{D_{\text{gauge}}} + \overbrace{\gamma \otimes \Phi}^{D_{\text{Higgs}}} , \qquad \mathscr{a}_\mu = \text{‘gauge potential’}, \ \mathscr{x}_\mu = \text{spin connection?}$$

The *field strength* is given by

$$\mathscr{F}_{\mu\nu} := [\overbrace{\ell_\mu + \mathscr{a}_\mu}^{\mathscr{d}_\mu}, \ell_\nu + \mathscr{a}_\nu] =: [\mathsf{F}_{\mu\nu}, \ \cdot \ ]$$

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$$D_\omega = \sum_{\mu=0}^{3} \overbrace{\gamma^\mu \otimes (\ell_\mu + \mathscr{a}_\mu) + \gamma^{\hat\mu} \otimes (\mathscr{x}_\mu + \mathscr{s}_\mu)}^{D_{\text{gauge}}} + \overbrace{\gamma \otimes \Phi}^{D_{\text{Higgs}}} , \qquad \mathscr{a}_\mu = \text{‘gauge potential’}, \ \mathscr{x}_\mu = \text{spin connection?}$$

The *field strength* is given by

$$\mathscr{F}_{\mu\nu} := [\overbrace{\ell_\mu + \mathscr{a}_\mu}^{\mathscr{d}_\mu}, \ell_\nu + \mathscr{a}_\nu] =: [\mathsf{F}_{\mu\nu}, \ \cdot\ ]$$

Lemma The gauge group $\mathrm{G}(A) \cong \mathrm{PU}(N) \times \mathrm{PU}(n)$ acts as follows

$$\mathsf{F}_{\mu\nu} \mapsto \mathsf{F}^{u}_{\mu\nu} = u\mathsf{F}_{\mu\nu}u^* \text{ for all } u \in \mathrm{G}(A)$$

Lemma-Definition [CP 2105.01025] Consider a gauge matrix spectral triple $G_{\not\ell} \times F$ with

$$F = (M_n(\mathbb{C}), M_n(\mathbb{C}), D_F)$$

and $G_{\not\ell}$ Riemannian ($d = 4$) fuzzy geometry on $M_N(\mathbb{C})$, whose fluctuated Dirac op. is

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The content of the Spectral Action ...

Meaning	Random matrix case, flat $d = 4$ Riem.	Smooth operator
	Tr = trace of ops. $M_N \otimes M_n \to M_N \otimes M_n$	
Derivation	$\ell_\mu = [L_\mu \otimes 1_n, \ \cdot \ ]$	∂_i
Gauge potential	$\mathscr{a}_\mu = [A_\mu, \ \cdot \ ]$	$\mathbb{A}_i$
Covariant derivative	$\mathscr{d}_\mu = \ell_\mu + \mathscr{a}_\mu$	$\mathbb{D}_i = \partial_i + \mathbb{A}_i$

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Covariant derivative	$\mathscr{d}_\mu = \mathscr{l}_\mu + \mathscr{a}_\mu$	$\mathbb{D}_i = \partial_i + \mathbb{A}_i$
Field strength	$[\mathscr{d}_\mu, \mathscr{d}_\nu] = \overbrace{[\mathscr{l}_\mu, \mathscr{l}_\nu]}^{\not\equiv 0} + [\mathscr{l}_\mu, \mathscr{a}_\nu] - [\mathscr{l}_\nu, \mathscr{a}_\mu] + [\mathscr{a}_\mu, \mathscr{a}_\nu]$	$[\mathbb{D}_i, \mathbb{D}_j] = \overbrace{[\partial_i, \partial_j]}^{\equiv 0} + \partial_i \mathbb{A}_j - \partial_j \mathbb{A}_i + [\mathbb{A}_i, \mathbb{A}_j]$
Yang-Mills action	$-\frac{1}{4} \operatorname{Tr}(\mathscr{F}_{\mu\nu} \mathscr{F}^{\mu\nu})$	$-\frac{1}{4} \int_M \operatorname{Tr}_{\mathfrak{su}(n)} (\mathbb{F}_{ij} \mathbb{F}^{ij}) \mathrm{vol}$
Higgs field	Φ	h
Higgs potential	$\operatorname{Tr}(f_2 \Phi^2 + f_4 \Phi^4)$	$\int_M (f_2 \vert h \vert^2 + f_4 \vert h \vert^4) \mathrm{vol}$
Gauge-Higgs coupling	$-\operatorname{Tr}(\mathscr{d}_\mu \Phi \mathscr{d}^\mu \Phi)$	$-\int_M \vert \mathbb{D}_i h \vert^2 \mathrm{vol}$

Meaning	Random matrix case, flat $d=4$ Riem. Tr = trace of ops. $M_N \otimes M_n \to M_N \otimes M_n$	Smooth operator	
Derivation	$\ell_\mu = [L_\mu \otimes 1_n, \ \cdot\ ]$	∂_i	
Gauge potential	$a_\mu = [A_\mu, \ \cdot\ ]$	$\mathbb{A}_i$	
Covariant derivative	$d_\mu = \ell_\mu + a_\mu$	$\mathbb{D}_i = \partial_i + \mathbb{A}_i$	
Field strength	$[d_\mu, d_\nu] = \overbrace{[\ell_\mu, \ell_\nu]}^{\not\equiv 0} +$ $[\ell_\mu, a_\nu] - [\ell_\nu, a_\mu] + [a_\mu, a_\nu]$	$[\mathbb{D}_i, \mathbb{D}_j] = \overbrace{[\partial_i, \partial_j]}^{\equiv 0} +$ $\partial_i \mathbb{A}_j - \partial_j \mathbb{A}_i + [\mathbb{A}_i, \mathbb{A}_j]$	$+$
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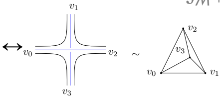
Propagators and $\sim (\ell_\mu)_{ij}(\ell_\nu)_{jm}(\ell^\mu)_{ml}(\ell^\nu)_{li} \leftrightarrow$



Conclusion

- spectral triple $\equiv$ spin manifold mod. commutativity of the ‘algebra of functions’
- spin $M \times \{\text{finite spectral triple}\} \equiv$ almost-commutative

  (reproduces classical Standard Model, but hard to quantize)

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[Connes Marcolli, *NCG, QFT and motives*, ’07, next sceenshot]

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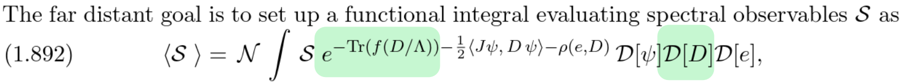
The far distant goal is to set up a functional integral evaluating spectral observables $\mathcal{S}$ as

$$\langle \mathcal{S} \rangle = \mathcal{N} \int \mathcal{S}\, e^{-\mathrm{Tr}(f(D/\Lambda)) - \frac{1}{2}\langle J\psi,\, D\,\psi\rangle - \rho(e,D)}\, \mathcal{D}[\psi]\mathcal{D}[D]\mathcal{D}[e], \tag{1.892}$$
»

- The matrix algebra $M_k(\mathcal{A}_{k,N})$ where functional renormalisation for random matrices (k-martix model) takes place was provided. $\mathcal{A}_{k,N}$ is a bigger relative of $\mathbb{C}^{(N)}_{\langle k \rangle}$

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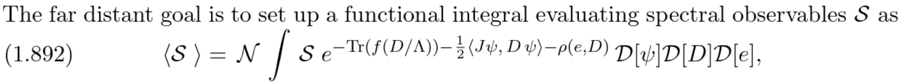

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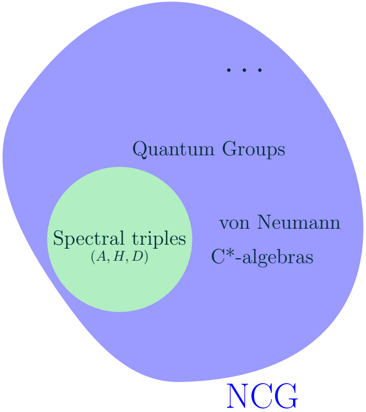
thank you!

References: [CP 1912.13288] [CP 2007.10914] [CP 2102.06999] [CP 2105.01025] [CP 2111.02858]



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Quantum Groups

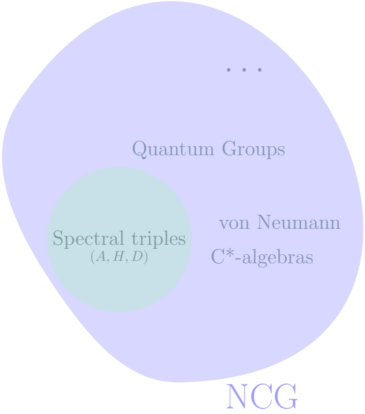
Spectral triples

(A, H, D)

von Neumann

C*-algebras

NCG



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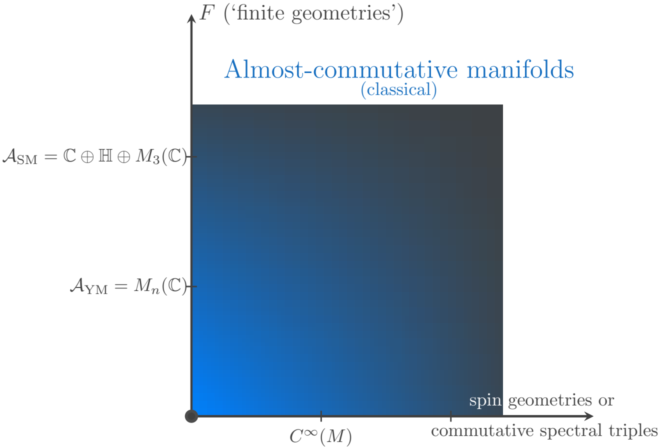


F ('finite geometries')

Almost-commutative manifolds

(classical)

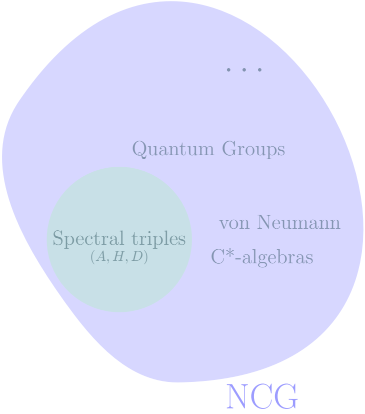
$\mathcal{A}_{\mathrm{SM}} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$

$\mathcal{A}_{\mathrm{YM}} = M_n(\mathbb{C})$

$C^\infty(M)$

spin geometries or

commutative spectral triples



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Quantum Groups

Spectral triples

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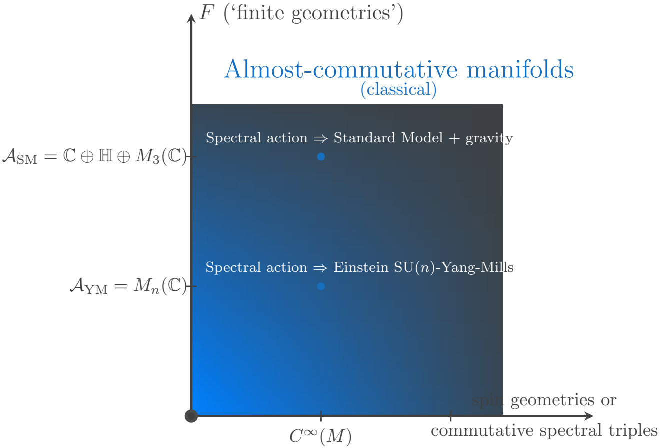

F (‘finite geometries’)

Almost-commutative manifolds

(classical)

Spectral action ⇒ Standard Model + gravity

$\mathcal{A}_{\text{SM}} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$

Spectral action ⇒ Einstein SU(n)-Yang-Mills

$\mathcal{A}_{\text{YM}} = M_n(\mathbb{C})$

spin geometries or

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NCG toolkit in high energy physics

- On a spectral triple (A, H, D) the (bosonic) classical action is given by

  $$S(D) = \mathrm{Tr}_H\, f(D/\Lambda)$$ [Chamseddine-Connes *CMP* '97]

  for a bump function f, Λ a scale. It's computed with heat kernel expansion

  [P. Gilkey, *J. Diff. Geom.* '75]

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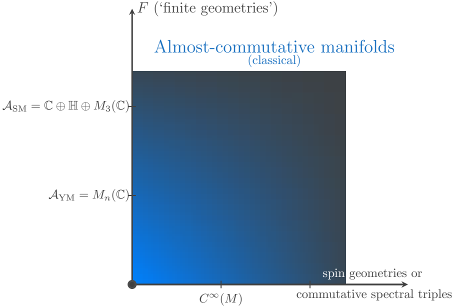

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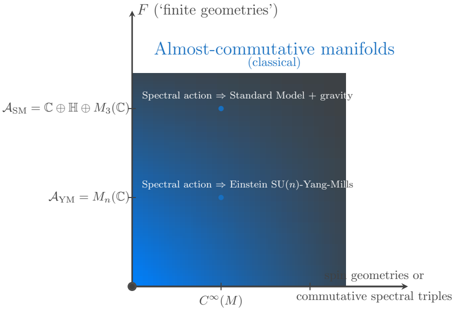

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- let's sketch *connections*: if S^G is a G-invariant functional on M

$$S^G \rightsquigarrow S^{\mathrm{Maps}(M,G)}$$

$$\mathrm{d} \rightsquigarrow \mathrm{d} + \mathbb{A} \qquad \mathbb{A} \in \Omega^1(M) \otimes \mathfrak{g}$$

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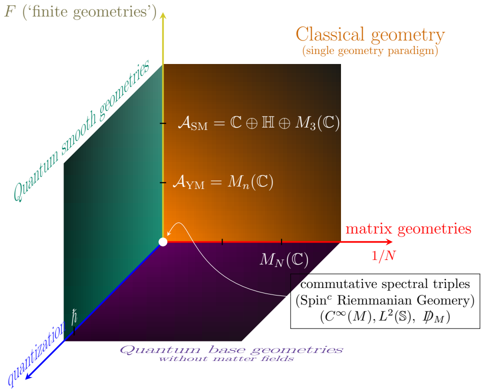
- given (A, H, D) and a Morita equivent algebra B (i.e. $\mathrm{End}_A(E) \cong B$) yields new $(B, E \otimes_A H, D')$. For $A = B$, in fact a tower

  $$\left\{(A, H, D + \omega \pm J\omega J^{-1})\right\}_{\omega\in\Omega^1_D(A)}$$

  $$D_\omega \mapsto \mathrm{Ad}(u) D_\omega \mathrm{Ad}(u)^* = D_{\omega_u}$$

  $$\omega \mapsto \omega_u = u\omega u^* + u[D, u^*] \qquad u \in \mathcal{U}(A)$$ `skip cube`

Organisation



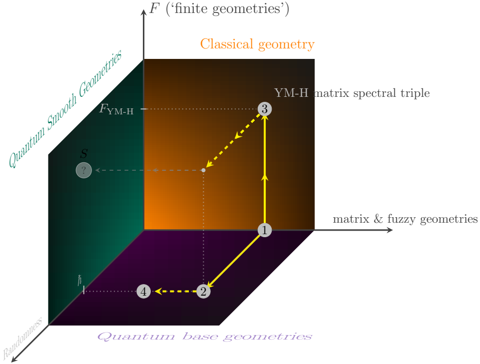
Aim: Make sense of

$$\mathcal{Z} = \int_{\text{Dirac}} \mathrm{e}^{-\operatorname{Tr}_H f(D)} \mathrm{d}D$$

- *Plane* $(\hbar, 1/N, 0)$ *of 'base geometries'*
- *Plane* $(\hbar, 0, F) = \lim_{N\to\infty}(\hbar, 1/N, F)$
- *Plane* $(0, 1/N, F) = \lim_{\hbar\to 0}(\hbar, 1/N, F)$ *of classical geometries*

[CP 2105.01025]

Organisation



1 Matrix Geometries
[J. Barrett, *J. Math. Phys.* 2015]

2 Dirac ensembles [J. Barrett, L. Glaser, *J. Phys. A* 2016] and how to compute the spectral action
[CP 1912.13288]

3 Gauge matrix spectral triples *(this talk)*
[CP 2105.01025]

4 Functional Renormalisation [CP 2007.10914] and
[CP 2111.02858]

Operator	Its noncommutative Hessian
$\operatorname{Tr}(A)\operatorname{Tr}(A^3)$	$3 \cdot \begin{pmatrix} \operatorname{Tr} A \cdot (A \otimes 1 + 1 \otimes A) + 1 \boxtimes A^2 + A^2 \boxtimes 1 & 0 \\ 0 & 0 \end{pmatrix}$
$\operatorname{Tr}(ABAB)$	$2 \cdot \begin{pmatrix} B \otimes B & (1 \otimes BA + AB \otimes 1) \\ (1 \otimes AB + BA \otimes 1) & A \otimes A \end{pmatrix}$
$\operatorname{Tr} A \operatorname{Tr}(AAABB)$	$\begin{pmatrix} (\operatorname{Tr}(A) 1 \otimes (ABB) + \operatorname{Tr}(A) 1 \otimes (BBA) + \operatorname{Tr}(A)(ABB) \otimes 1 + \operatorname{Tr}(A)(BBA) \otimes 1 + 1 \boxtimes (AABB) + 1 \boxtimes (ABBA) + 1 \boxtimes (BBAA) + (AABB) \boxtimes 1 + (ABBA) \boxtimes 1 + (BBAA) \boxtimes 1 + \operatorname{Tr}(A) A \otimes B^2 + \operatorname{Tr}(A) B^2 \otimes A) & \operatorname{Tr}(A)(1 \otimes (AAB) + (BAA) \otimes 1 + A \otimes AB + B \otimes A^2 + A^2 \otimes B + BA \otimes A) + 1 \boxtimes (A^3 B) + 1 \boxtimes (BA^3) \\ \operatorname{Tr}(A)(1 \otimes (BAA) + (AAB) \otimes 1 + A \otimes BA + B \otimes A^2 + A^2 \otimes B + AB \otimes A) + (A^3 B) \boxtimes 1 + (BA^3) \boxtimes 1 & \operatorname{Tr}(A)(1 \otimes (A^3) + (A^3) \otimes 1) \end{pmatrix}$

Table: Some Hessians order operators. Here $\operatorname{Tr} = \operatorname{Tr}_N$.

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$\operatorname{Tr}(A)\operatorname{Tr}(A^3)$	$3\cdot\begin{pmatrix} \operatorname{Tr}A\cdot(A\otimes 1+1\otimes A) +1\boxtimes A^2+A^2\boxtimes 1 & 0 \\ 0 & 0 \end{pmatrix}$
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$\operatorname{Tr}A\operatorname{Tr}(AAABB)$	$\begin{pmatrix} (\operatorname{Tr}(A)1\otimes(ABB)+\operatorname{Tr}(A)1\otimes(BBA)+\operatorname{Tr}(A)(ABB)\otimes 1+\operatorname{Tr}(A)(BBA)\otimes 1+1\boxtimes(AABB)+1\boxtimes(ABBA)+1\boxtimes(BBAA)+(AABB)\boxtimes 1+(ABBA)\boxtimes 1+(BBAA)\boxtimes 1+\operatorname{Tr}(A)A\otimes B^2+\operatorname{Tr}(A)B^2\otimes A) & \operatorname{Tr}(A)(1\otimes(AAB)+(BAA)\otimes 1+A\otimes AB+B\otimes A^2+A^2\otimes B+BA\otimes A)+1\boxtimes(A^3B)+1\boxtimes(BA^3) \\ \operatorname{Tr}(A)(1\otimes(BAA)+(AAB)\otimes 1+A\otimes BA+B\otimes A^2+A^2\otimes B+AB\otimes A)+(A^3B)\boxtimes 1+(BA^3)\boxtimes 1 & \operatorname{Tr}(A)(1\otimes(A^3)+(A^3)\otimes 1) \end{pmatrix}$

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β-functions of NCG two-matrix models, signature $\eta = \mathrm{diag}(e_1, e_2)$

$$2h_1(a_4 + c_{22} + 2d_{2|02} + 6d_{2|2}) = \eta_a$$

$$2h_1(b_4 + c_{22} + 6d_{02|02} + 2d_{2|02}) = \eta_b$$

$$-h_1[e_a(a_4 - c_{1111}) + 2d_{1|12} + 6d_{1|3}] + d_{1|1}(\eta + 1) = \beta(d_{1|1})$$

$$-h_1[e_b(b_4 - c_{1111}) + 6d_{01|03} + 2d_{01|21}] + d_{01|01}(\eta + 1) = \beta(d_{01|01})$$

The next block encompasses the connected quartic couplings:

$$\begin{aligned} &h_2\left(4a_4^2 + 4c_{22}^2\right) + a_4(2\eta + 1) \\ &-h_1(24a_6 e_a + 4c_{42} e_b + 4d_{02|4} e_b + 4d_{2|4} e_a) = \beta(a_4) \end{aligned}$$

$$\begin{aligned} &h_2\left(4b_4^2 + 4c_{22}^2\right) + b_4(2\eta + 1) \\ &-h_1(24b_6 e_b + 4c_{24} e_a + 4d_{02|04} e_b + 4d_{2|04} e_a) = \beta(b_4) \end{aligned}$$

$$\begin{aligned} &-h_1\left(2e_a c_{1212} + e_b 2c_{2121} + 3e_a c_{24} + 3e_b c_{42} + e_a d_{02|22} + e_b d_{2|22}\right) \\ &+h_2\left(2a_4 c_{22} + 2b_4 c_{22} + 2e_a e_b c_{1111}^2 + 2e_a e_b c_{22}^2\right) + c_{22}(2\eta + 1) = \beta(c_{22}) \end{aligned}$$

$$\begin{aligned} &8e_a e_b c_{1111} c_{22} h_2 + c_{1111}(2\eta + 1) \\ &+h_1\left(4e_a c_{1311} + 4e_b c_{3111} + 2e_a d_{02|1111} + 2e_b d_{2|1111}\right) = \beta(c_{1111}) \end{aligned}$$

back to main presentation ⊲⊲

$$2h_2(6a_4a_6 + e_a e_b c_{22}c_{42}) + a_6(3\eta + 2) = \beta(a_6)$$

$$2h_2(6b_4b_6 + e_a e_b c_{22}c_{24}) + b_6(3\eta + 2) = \beta(b_6)$$

$$\begin{aligned} 4h_2\{a_4 c_{3111} + e_a e_b[c_{22}(c_{1311} + 2c_{3111}) \\ -c_{1111}(2c_{2121} + c_{42})]\} + c_{3111}(3\eta + 2) = \beta(c_{3111}) \end{aligned}$$

$$\begin{aligned} 2h_2[2a_4 c_{2121} + e_a e_b(-2c_{1111}c_{3111} \\ +4c_{2121}c_{22} + c_{22}c_{24})] + c_{2121}(3\eta + 2) = \beta(c_{2121}) \end{aligned}$$

$$\begin{aligned} 2h_2[a_4c_{24} + 3b_4c_{24} + 2e_a e_b(c_{22}(3b_6 + c_{2121} + c_{24} + c_{42}) \\ -c_{1111}c_{1311})] + c_{24}(3\eta + 2) = \beta(c_{24}) \end{aligned}$$

$$\begin{aligned} 4h_2\{b_4 c_{1311} + e_a e_b[c_{22}(2c_{1311} + c_{3111}) \\ -c_{1111}(2c_{1212} + c_{24})]\} + c_{1311}(3\eta + 2) = \beta(c_{1311}) \end{aligned}$$

$$\begin{aligned} 2h_2[2b_4 c_{1212} + e_a e_b(c_{22}(4c_{1212} + c_{42}) \\ -2c_{1111}c_{1311})] + c_{1212}(3\eta + 2) = \beta(c_{1212}) \end{aligned}$$

$$\begin{aligned} 2h_2[3a_4c_{42} + 2e_a e_b(3a_6c_{22} - c_{1111}c_{3111} + c_{1212}c_{22} \\ +c_{22}c_{24} + c_{22}c_{42}) + b_4c_{42}] + c_{42}(3\eta + 2) = \beta(c_{42}) \end{aligned}$$

back to main presentation ⋘

Matrix or Fuzzy Geometries `GO TO characterization` $\lll$

Definition ("condensed" from [J. Barrett, *J. Math. Phys.* 2015]).
A *fuzzy geometry* of *signature* $(p, q) \in \mathbb{Z}^2_{\geqslant 0}$ is given by

- a simple matrix algebra $\mathcal{A}$ – we take always $\mathcal{A} = M_N(\mathbb{C})$
- a Hermitian $\mathbb{C}\ell(p, q)$-module $\mathbb{S}$ with a *chirality* γ. That is a linear map $\gamma : \mathbb{S} \to \mathbb{S}$ satisfying $\gamma^* = \gamma$ and $\gamma^2 = 1$
- a Hilbert space $\mathcal{H} = \mathbb{S} \otimes M_N(\mathbb{C})$ with inner product $\langle v \otimes R, w \otimes S \rangle = (v, w) \operatorname{Tr}_N (R^* S)$ for each $R, S \in M_N(\mathbb{C})$, being $(\cdot\,, \cdot)$ the inner product of $\mathbb{S}$
- a left-$\mathcal{A}$ representation $\rho(a)(v \otimes R) = v \otimes (aR)$ on $\mathcal{H}$, $a \in \mathcal{A}$ and $v \otimes R \in \mathcal{H}$

. . .

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- three signs $\epsilon, \epsilon', \epsilon'' \in \{-1, +1\}$ determined through $s := q - p$ by the following table:

$s \equiv q - p \pmod 8$	0	1	2	3	4	5	6	7
ϵ	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
ϵ'	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$
ϵ''	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$+$

- a real structure $J = C \otimes *$, where $*$ is complex conjugation and C is an anti-unitarity on $\mathbb{S}$ satisfying $C^2 = \epsilon$ and $C\gamma^\mu = \epsilon'\gamma^\mu C$ for all the gamma matrices $\mu = 1, \ldots, p + q$.
- a self-adjoint operator D on $\mathcal{H}$ satisfying the *order-one condition*

$$[\,[D, \rho(a)]\,, J\rho(b)J^{-1}] = 0 \qquad \text{for all } a, b \in \mathcal{A}$$

- a chirality $\Gamma = \gamma \otimes 1_{\mathcal{A}}$ for $\mathcal{H}$, where γ is the chirality of $\mathbb{S}$. The signs above impose: