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Combinatorial algebraic remarks on Dirac ensembles and related matrix models

Noncommutative Geometry, Free Probability Theory and Random Matrix Theory, Western University, June 13-17, 2022

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Based on 1912.13288*; 2007.10914*; 2102.06999*; 2105.01025*,†; 2111.02858†,*

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Thanks to: M. Khalkhali, A. Sitarz, L Glaser, D. Benedetti, R. Gurău,
A. Schenkel, O. Arizmendi, for discussion, comments and/or bibliography.

Outline

this talk ↓	noncomm. geometry	free prob.	random matrix theory
motivation	no	no	no
introduction	yes	no	no
fuzzy geometries	yes	?	yes
Dirac ensembles	yes	no	yes
renormalisation & free algebra	no	some	yes
gauge theories*	yes	no	after 'quantisation'
outlook	yes	hopefully	yes

*if time allows

Motivation

- **From physics to NCG:** The Standard Model from the Spectral Action

$$-\frac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \frac{1}{4}g_s^2 f^{abc} f^{ade} g^b_\mu g^c_\nu g^d_\mu g^e_\nu +$$
$$\frac{1}{2}ig_s^2(\bar{q}^\sigma_i \gamma^\mu q^\sigma_j)g^a_\mu + \bar{G}^a\partial^2 G^a + g_s f^{abc}\partial_\mu \bar{G}^a G^b g^c_\mu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu -$$
$$M^2 W^+_\mu W^-_\mu - \frac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \frac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu -$$
$$\frac{1}{2}\partial_\mu H \partial_\mu H - \frac{1}{2}m_h^2 H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2\phi^+\phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 -$$
$$\frac{1}{2c_w^2}M\phi^0\phi^0 - \beta_h[\frac{2M^2}{g^2} + \frac{2M}{g}H + \frac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)] + \frac{2M^4}{g^2}\alpha_h -$$
$$igc_w[\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) +$$
$$Z^0_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)] - igs_w[\partial_\nu A_\mu(W^+_\mu W^-_\nu -$$
$$W^+_\nu W^-_\mu) - A_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu(W^+_\nu \partial_\nu W^-_\mu -$$
$$W^-_\nu \partial_\nu W^+_\mu)] - \frac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \frac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu +$$
$$g^2 c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu -$$
$$A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w[A_\mu Z^0_\nu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) -$$
$$2A_\mu Z^0_\mu W^+_\nu W^-_\nu] - g\alpha[H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-] - \frac{1}{8}g^2\alpha_h[H^4 +$$
$$(\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2 H^2] -$$
$$gMW^+_\mu W^-_\mu H - \frac{1}{2}g\frac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \frac{1}{2}ig[W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) -$$
$$W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)] + \frac{1}{2}g[W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) -$$
$$W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)] + \frac{1}{2}g\frac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) -$$
$$ig\frac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_w MA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) -$$
$$ig\frac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + igs_w A_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) -$$
$$\frac{1}{4}g^2 W^+_\mu W^-_\mu[H^2 + (\phi^0)^2 + 2\phi^+\phi^-] - \frac{1}{4}g^2\frac{1}{c_w^2}Z^0_\mu Z^0_\mu[H^2 + (\phi^0)^2 +$$
$$2(2s_w^2 - 1)^2\phi^+\phi^-] - \frac{1}{2}g^2\frac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) -$$

ghosts shouldn't be here

$$\frac{1}{2}ig^2\frac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \frac{1}{2}g^2 s_w A_\mu\phi^0(W^+_\mu\phi^- +$$
$$W^-_\mu\phi^+) + \frac{1}{2}ig^2 s_w A_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\frac{s_w}{c_w}(2c_w^2 -$$
$$1)Z^0_\mu A_\mu\phi^+\phi^- - g^1 s_w^2 A_\mu A_\mu\phi^+\phi^- - \bar{e}^\lambda(\gamma\partial + m_e^\lambda)e^\lambda - \bar{\nu}^\lambda\gamma\partial\nu^\lambda -$$
$$\bar{u}^\lambda_j(\gamma\partial + m_u^\lambda)u^\lambda_j - \bar{d}^\lambda_j(\gamma\partial + m_d^\lambda)d^\lambda_j + igs_w A_\mu[-(\bar{e}^\lambda\gamma^\mu e^\lambda) +$$
$$\frac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \frac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)] + \frac{ig}{4c_w}Z^0_\mu[(\bar{\nu}^\lambda\gamma^\mu(1 + \gamma^5)\nu^\lambda) +$$
$$(\bar{e}^\lambda\gamma^\mu(4s_w^2 - 1 - \gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(\frac{4}{3}s_w^2 - 1 - \gamma^5)u^\lambda_j) + (\bar{d}^\lambda_j\gamma^\mu(1 -$$
$$\frac{8}{3}s_w^2 - \gamma^5)d^\lambda_j)] + \frac{ig}{2\sqrt{2}}W^+_\mu[(\bar{\nu}^\lambda\gamma^\mu(1 + \gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(1 +$$
$$\gamma^5)C_{\lambda\kappa}d^\kappa_j)] + \frac{ig}{2\sqrt{2}}W^-_\mu[(\bar{e}^\lambda\gamma^\mu(1 + \gamma^5)\nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\lambda\kappa}\gamma^\mu(1 +$$
$$\gamma^5)u^\lambda_j)] + \frac{ig}{2\sqrt{2}}\frac{m_e^\lambda}{M}[-\phi^+(\bar{\nu}^\lambda(1 - \gamma^5)e^\lambda) + \phi^-(\bar{e}^\lambda(1 + \gamma^5)\nu^\lambda)] -$$
$$\frac{g}{2}\frac{m_e^\lambda}{M}[H(\bar{e}^\lambda e^\lambda) + i\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda)] + \frac{ig}{2M\sqrt{2}}\phi^+[-m_d^\kappa(\bar{u}^\lambda_j C_{\lambda\kappa}(1 -$$
$$\gamma^5)d^\kappa_j) + m_u^\lambda(\bar{u}^\lambda_j C_{\lambda\kappa}(1 + \gamma^5)d^\kappa_j] + \frac{ig}{2M\sqrt{2}}\phi^-[m_d^\lambda(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1 +$$
$$\gamma^5)u^\kappa_j) - m_u^\kappa(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1 - \gamma^5)u^\kappa_j] - \frac{g}{2}\frac{m_u^\lambda}{M}H(\bar{u}^\lambda_j u^\lambda_j) -$$
$$\frac{g}{2}\frac{m_d^\lambda}{M}H(\bar{d}^\lambda_j d^\lambda_j) + \frac{ig}{2}\frac{m_u^\lambda}{M}\phi^0(\bar{u}^\lambda_j\gamma^5 u^\lambda_j) - \frac{ig}{2}\frac{m_d^\lambda}{M}\phi^0(\bar{d}^\lambda_j\gamma^5 d^\lambda_j)$$

...this 'fits' in

$$\mathrm{Tr}(f(D/\Lambda)) + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle$$

Num. of generations and $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \longrightarrow$ **NCG** $\longrightarrow$ *Classical* Lagrangian of the Standard Model

[Chamseddine-Connes-Marcolli *ATMP* '07 (Euclidean); J. Barrett *J. Math. Phys.* '07 (Lorenzian)]

Motivation

- From physics to NCG: The Standard Model from the Spectral Action

$$D_F = \left(\begin{array}{cccccccccccccccccc}
0 & 0 & \Upsilon_\nu^* & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_R^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Upsilon_e^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Upsilon_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \Upsilon_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^* & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^* & \otimes 1_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & \Upsilon_u & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \Upsilon_d & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\Upsilon_R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_\nu^T & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_e^T & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar\Upsilon_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar\Upsilon_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^T & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^T & \otimes 1_3 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar\Upsilon_u & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar\Upsilon_d & 0 & 0 & &
\end{array}\right) \in M_{96}(\mathbb{C})_{\text{s.a.}}$$

$\Upsilon_\nu \in M_3(\mathbb{C})$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

fermionic Sp. Action $\langle J\psi, D_F \psi \rangle$

so one zero less among $\sim 10^4$ entries $\Rightarrow$ unseen particle interaction

but here, the zeroes come from geometry!

Num. of generations and $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \longrightarrow$ NCG $\longrightarrow$ *Classical* Lagrangian of the Standard Model

[Chamseddine-Connes-Marcolli *ATMP* '07 (Euclidean); J. Barrett *J. Math. Phys.* '07 (Lorenzian)]

Towards a quantum theory of noncommutative spaces

« *The far distant goal is to set up a functional integral evaluating spectral observables* $\mathscr{S}$

$$\langle \mathscr{S} \rangle = \int \mathscr{S}\, e^{-\operatorname{Tr} f(D/\Lambda) - \frac{1}{2}\langle J\psi, D\psi\rangle + \rho(e,D)}\, de\, d\psi\, dD$$ »

[Eq. 1.892, Connes Marcolli, *NCG, QFT and motives*, 2007]

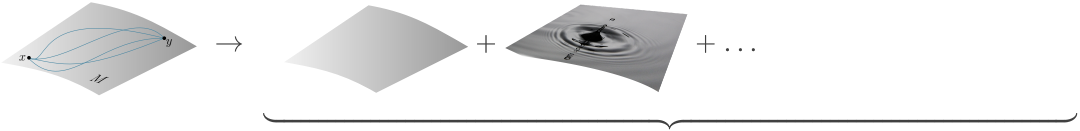
functional integral $\xrightarrow[\text{paradigm shift}]{}$ operator integral

$$\int_{\text{metric}} e^{-\frac{1}{\hbar} S_{\text{EH}}[g]}\, dg \xrightarrow{\text{Einstein-Hilbert} \,\to\, \text{spectral}} \int_{\text{Dirac}} e^{-\frac{1}{\hbar}\operatorname{Tr} f(D)}\, dD$$

(hard to define for manifolds)

$f : \mathbb{R} \to \mathbb{R}$ with $f(D) \to \infty$ at large argument

- Possible application to (Euclidean) quantum gravity



Quantum superposition of geometries

Towards a quantum theory of noncommutative spaces

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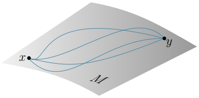
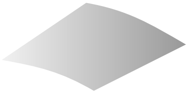
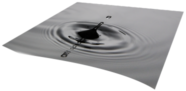


$\to \Big\{$  $+$  $+ \ldots \Big\} \times$  [Hokusai]

Quantum superposition of geometries (small perturbations + instantons)

- ## Origin of noncommutative topology

Connes' *noncommutative (nc) geometry* = nc topology

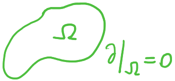
[Gelfand, Najmark *Mat. Sbornik* '43] + metric [A. Connes, *NCG* '94]

$$\{\text{compact Hausdorff topological spaces}\} \simeq \{\text{unital } \textit{commutative } C^*\text{-algebras}\}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\{\text{'noncommutative topological spaces'}\} \simeq \{\text{unital } \sout{\textit{commutative}}\ C^*\text{-algebras}\}$$

- arguably, the 1st NCG-theorem is *Weyl's law* (1911) on the rate of growth of the Laplace spectrum (ordered $\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \ldots$) of $\Omega \subset \mathbb{R}^d$



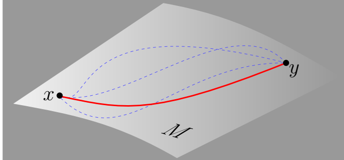
$$\#\{i : \lambda_i \leqslant \Lambda\} = \frac{\mathrm{vol}(\text{unit ball})}{(2\pi)^d} \mathrm{vol}\,\Omega \cdot \Lambda^{d/2} + \mathrm{o}(\Lambda^{d})$$

$d/2$

From this, you cannot answer positively Marek Kac' 1966-question.
But you can 'hear the shape of Ω' knowing a *spectral triple*.
[A, Connes, *JNCG* 2013] (and from it [Connes-van Suijlekom, *CMP* 2021] can hear an MP3;
our story today is not entirely unrelated.)

Replace spin manifold (M, g) by $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

Connes' geodesic distance



$\gamma : \mathbb{R} \to M$

$\inf_{\gamma \text{ as above}} \{ \int_\gamma \mathrm{d}s \} = d(x, y)$

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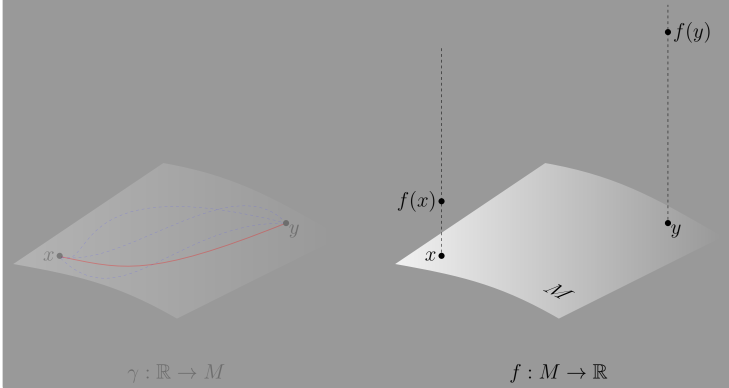

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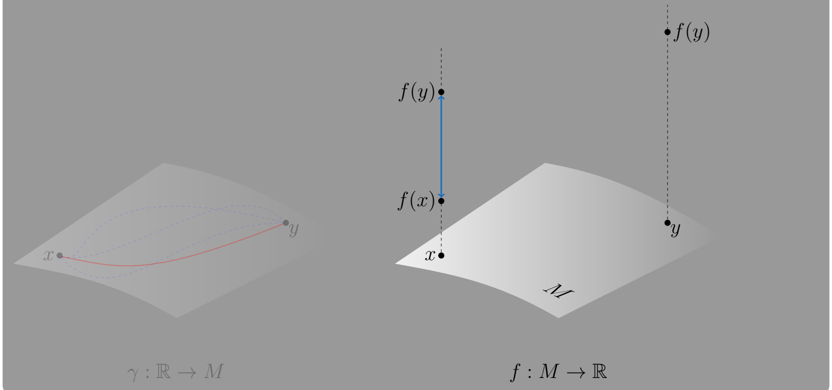

$$|f(x) - f(y)|$$

Replace spin manifold (M, g) by $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

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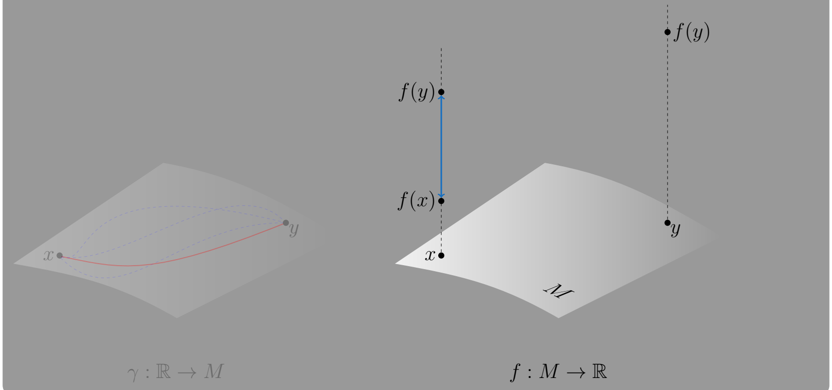
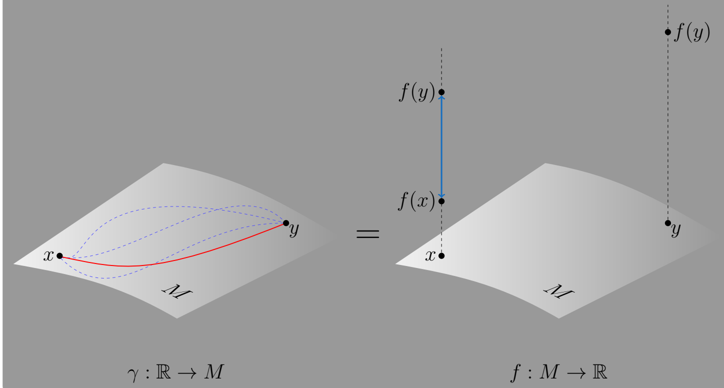


$$\sup_{f \in C^\infty(M)} \{\, |f(x) - f(y)| \;:\; ||D_M f - f D_M|| \leqslant 1 \}$$

Replace spin manifold (M, g) by $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

Connes' geodesic distance

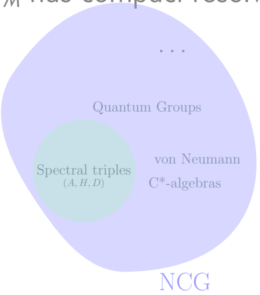


$$\inf_{\gamma \text{ as above}} \{ \textstyle\int_\gamma \mathrm{d}s \} = d(x, y) \;=\; \sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| \;:\; ||D_M f - f D_M|| \leqslant 1 \}$$

Commutative spectral triples

A spin manifold M yields (A_M, H_M, D_M)

- $A_M = C^\infty(M)$ is a comm. $*$-algebra
- $H_M := L^2(M, \mathbb{S})$ a repr. of A_M
- $D_M = -\mathrm{i}\gamma^\mu(\partial_\mu + \omega_\mu)$ is self-adjoint
- for each $a \in A_M$, $[D_M, a]$ is bounded, and in fact $[D_M, x^\mu] = -\mathrm{i}\gamma^\mu$
- D_M has compact resolvent ...



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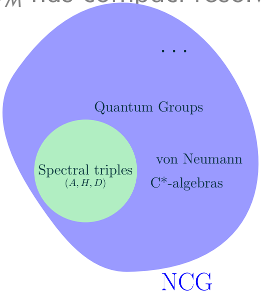

A *spectral triple* (A, H, D) consists of

- a $*$-algebra A
- a representation H of A
- a self-adjoint operator D on H with compact resolvent and such that $[D, a]$ is bounded for each $a \in A$

Commutative spectral triples

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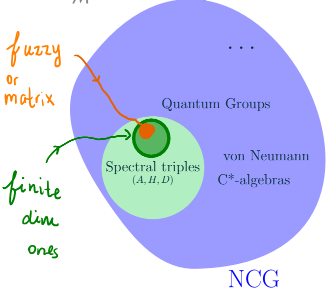

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II. Fuzzy Geometries and Multimatrix Models

A *fuzzy geometry* of signature (p, q), so $\eta = \mathrm{diag}(+_p, -_q)$, consists of

- $A = M_N(\mathbb{C})$
- $H = \mathbb{S} \otimes M_N(\mathbb{C})$, with $\mathbb{S}$ a $\mathbb{C}\ell(p, q)$-module

  ... (axioms for D omitted, `go to axioms` ▽) ...
- Fixing conventions for γ's, characterisation of D in even dimensions:

$$D = \sum_J \Gamma^J_{\text{s.a.}} \otimes \{H_J, \ \cdot\ \} + \sum_J \Gamma^J_{\text{anti.}} \otimes [L_J, \ \cdot\ ]$$

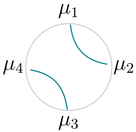
  multi-index J monot. increasing, $|J|$ odd [J. Barrett, *J. Math. Phys.* 2015], $H_J^* = H_J$, $L_J^* = -L_J$
- Examples: [J. Barrett, L. Glaser, *J. Phys. A* 2016]
  - $D_{(1,1)} = \gamma^1 \otimes [L, \ \cdot\ ] + \gamma^2 \otimes \{H, \ \cdot\ \}$
  - $D_{(0,4)} = \sum_\mu \gamma^\mu \otimes [L_\mu, \ \cdot\ ] + \gamma^{\hat\mu} \otimes \{H_{\hat\mu}, \ \cdot\ \}$

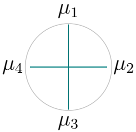
  so we will get double traces from $\mathrm{Tr}_H = \mathrm{Tr}_{\mathbb{S}} \otimes \mathrm{Tr}_{M_N(\mathbb{C})} = \mathrm{Tr}_{\mathbb{S}} \otimes \mathrm{Tr}_N^{\otimes 2}$

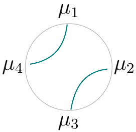
Notation: $\mathrm{Tr}_V\, X$ is the trace on operators $X : V \to V$, $\mathrm{Tr}_V\, 1 = \dim V$. So $\mathrm{Tr}_N\, 1 = N$ but $\mathrm{Tr}_{M_N(\mathbb{C})}(1) = N^2$.

- A tool to organize the fuzzy spectral action is chord diagrams:

$$\mathrm{Tr}_{\mathbb{S}}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}) = \dim\mathbb{S}\big(\overbrace{\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}}^{\theta^{\mu_1\mu_2\mu_3\mu_4}} + \overbrace{(-)\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}}^{\xi^{\mu_1\mu_2\mu_3\mu_4}} + \overbrace{\eta^{\mu_2\mu_3}\eta^{\mu_1\mu_4}}^{\zeta^{\mu_1\mu_2\mu_3\mu_4}}\big)$$

$\theta^{\mu_1\mu_2\mu_3\mu_4} =$ 

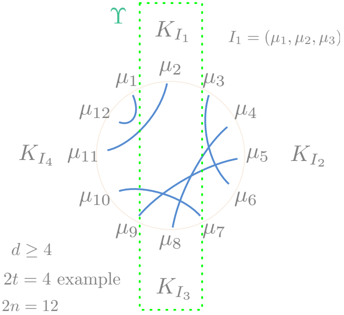
$\xi^{\mu_1\mu_2\mu_3\mu_4} =$ 

$\zeta^{\mu_1\mu_2\mu_3\mu_4} =$ 

- for dimension-d geometries, the combinatorial formula [CP '19] reads

$$\frac{1}{\dim\mathbb{S}}\mathrm{Tr}(D^{2t}) = \overbrace{\sum_{I_1,\ldots,I_{2t}\in\Lambda_d^-}}^{\text{if } J\in\Lambda_d,\ \mathrm{d}x^J\neq 0 \text{ on } \mathbb{R}^d} \Bigg\{ \overbrace{\sum_{\substack{\chi\in\mathrm{CD}_{2n}\\ 2n=\sum_i |I_i|}} \chi^{I_1\ldots I_{2t}}}^{\text{decorated chord diags}}$$

$$\times\Big(\sum_{\Upsilon\in\mathscr{P}_{2t}} \mathrm{sgn}(I_\Upsilon)\times \mathrm{Tr}_N(K_{I_{\Upsilon^c}})\times\mathrm{Tr}_N\big[(K^T)_{I_\Upsilon}\big]\Big)\Bigg\}$$



- A tool to organize the fuzzy spectral action is chord diagrams:

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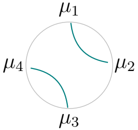
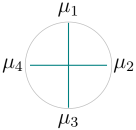
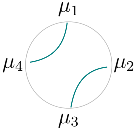
$\theta^{\mu_1\mu_2\mu_3\mu_4} =$ 

$\xi^{\mu_1\mu_2\mu_3\mu_4} =$ 

$\zeta^{\mu_1\mu_2\mu_3\mu_4} =$ 

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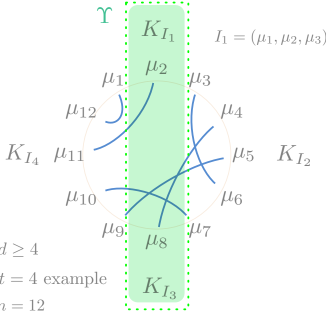
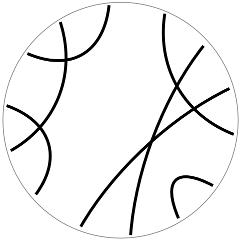


$d \geq 4$
$2t = 4$ example
$2n = 12$

trivial partitions yield $N \cdot \mathrm{Tr}_N(P)$ we all know and love...

Warning: These chord diagrams are not Feynman diagrams, they just determine the classical action.



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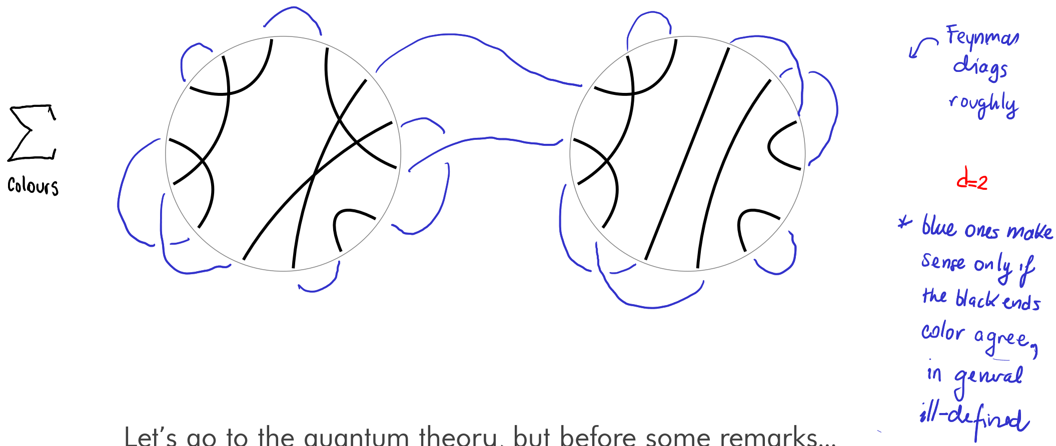


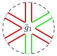
Let's go to the quantum theory, but before some remarks...

Multimatrix models with multitraces & ribbon graphs

- The chord-diagram description holds in general dimension and signature [CP '19]

$$\mathcal{Z} = \int_{\text{Dirac}} \mathrm{e}^{-\operatorname{Tr}_H f(D)} \mathrm{d}D \quad (\hbar = 1)$$
$$= \int_{M_{p,q}} \mathrm{e}^{-N \operatorname{Tr}_N P - \operatorname{Tr}_N^{\otimes 2}(Q_{(1)} \otimes Q_{(2)})} \mathrm{d}\mathbb{X}_{\text{Leb}}$$

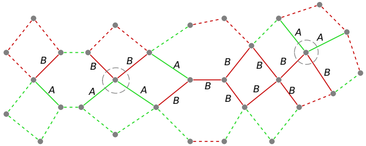
- $\mathbb{X} \in M_{p,q}$ = products of $\mathfrak{su}(N)$ and $\mathcal{H}_N$
- $\mathrm{d}\mathbb{X}_{\text{Leb}}$ is the Lebesgue measure on $M_{p,q}$
- P, $Q_{(i)}$ are certain nc-polynomials
- $\mathcal{Z}_{\text{formal}}$ is the gen. func. of colored ribbon graphs (maps)

$\bar{g}_1 \operatorname{Tr}_N(ABBBAB) \quad \leftrightarrow$ 

$\bar{g}_2 \operatorname{Tr}_N^{\otimes 2}(AABABA \otimes AA) \quad \leftrightarrow$ 

- Multitrace random matrices:
  - 'touching interactions' [Klebanov, *PRD* '95]
  - wormholes [Ambjørn-Jurkiewicz-Loll-Vernizzi, *JHEP* '01]
  - stuffed maps [G. Borot *AHP-D* '14]
  - trace polynomials [G. Cébron, *J.Funct.Anal.* '13]
    [D. Jekel-W. Li-D. Shlyakhtenko, 2021; A. Guionnet...,]
  - AdS/CFT [Witten, `hep-th/0112258`]

- Ribbon graphs: Enumeration of maps [Brezin, Itzykson, Parisi, Zuber, *CMP* '78], here 'face-worded'



III. Functional renormalisation in random matrices

Motivation from '2D-Quantum Gravity'

discrete surfaces ↔ matrix integrals $\mathcal{Z}(\lambda)$

[B. Eynard, *Counting Surfaces* '16]

smooth surface ↔ $\langle \text{area} \rangle$ finite
& infinitesimal mesh $\mathfrak{a}$
$\langle \text{area} \rangle_g \sim \frac{\mathfrak{a}^2(2-2g)}{\lambda/\lambda_c - 1}$

all topologies ↔ $\mathcal{Z}(\lambda) = \sum_g N^{2-2g} \mathcal{Z}_g(\lambda)$

$\wr$

$(\lambda_c - \lambda)^{(2-2g)/\theta}$

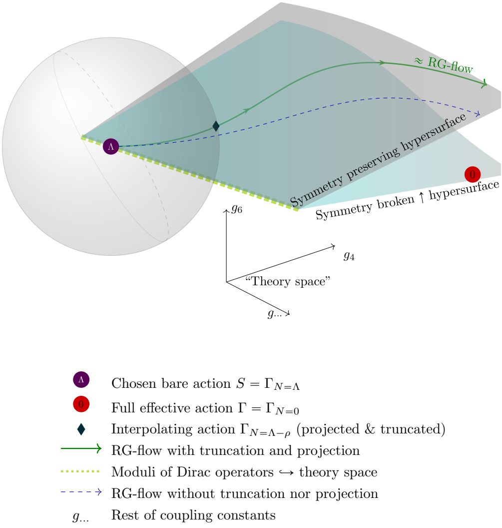
⇑

double-scaling limit $N(\lambda_c - \lambda)^{1/\theta} = C$

lin. RG-flow near a fixed point ↔ $\lambda(N) = \lambda_c + (N/C)^{-\theta}$
$\theta = -(\partial\beta/\partial\lambda)|_{\lambda_c}$

[Eichhorn-Koslowski, PRD, '13]

[CP 20] →



Homogeneisation of notation

- Let's write $\mathbb{E}[\bullet]$ for $\langle \bullet \rangle$. Wick's theorem [L. Isserlis *Biometrika* 1918]: for zero-mean x_i's, . . .

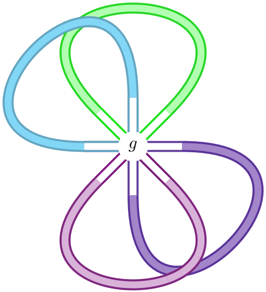
$$\mathbb{E}[x_{j_1} \cdots x_{j_{2n}}] = \sum_{\substack{\pi \in P_2(2n) \\ \text{(pairings)}}} \prod_{(p,q)\in\pi} \mathbb{E}[x_{j_p} x_{j_q}]$$

- k the number of Hermitian matrices of size N, $X_1^{(N)}, \ldots, X_k^{(N)}$ [Piotr Śniady's notation]

- Ribbon graphs. For $\mathbb{E}[(X_\mu^{(N)})_{i,j}(X_\rho^{(N)})_{l,m}] = \delta_{\mu\rho}\delta_{im}\delta_{jl}$ $\qquad \mu,\rho = 1,\ldots,k;\ i,\ldots,m = 1,\ldots,N$

$1/N$

$$(gN) \cdot \mathbb{E}\Big[\, \mathrm{Tr}_N \left(X_1^{(N)} X_2^{(N)} X_1^{(N)} X_2^{(N)} X_3^{(N)} X_4^{(N)} X_3^{(N)} X_4^{(N)} \right) \Big] =$$

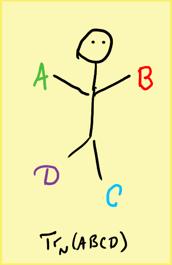


$\sim N^{-2}$

- Some Feynman graphs of multimatrix ϕ^4-theory...

Several-loop graph

[pic by 'Princi19skydiver', Wikipedia]



One-loop graph

[pic by 'Wojciech Kielar', Wikipedia]

foot-foot, foot-hand, ok.

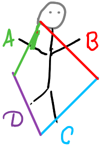
but letters must coincide!

(ignore the head)

- Some Feynman graphs of multimatrix ϕ^4-theory...

Several-loop graph

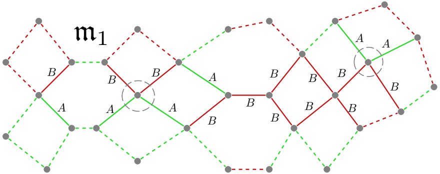
[pic by 'Princi19skydiver', Wikipedia]

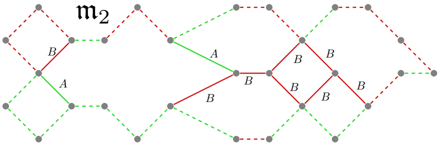


One-loop graph

[pic by 'Wojciech Kielar', Wikipedia]

- advantage of functional renormalisation: 1-loops only





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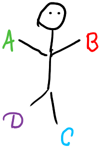
[pic by 'Princi19skydiver', Wikipedia]



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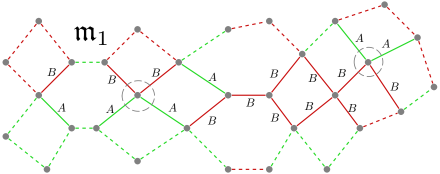
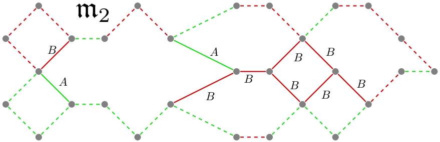


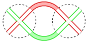


- *Question (sloppy version):* Given an operator $\mathrm{Tr}_N\, w$, in $w \in \mathbb{C}^{(N)}_{\langle k \rangle}$, find (up to given degree in the couplings) all 1-loops it might come from, in the above example sense

Functional Renormalisation for k-matrix models (w/multitrace-measures)

Quantum theories 'flow' with energy, here in RG-time $t = \log N$, $1 \ll N < \mathcal{N}$. E.g. for $k = 2$ and with bare action

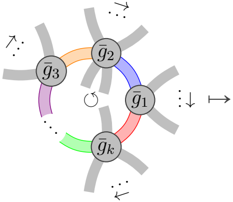
$$S[A, B] = \mathcal{N}\, \mathrm{Tr}_{\mathcal{N}} \left\{ \frac{1}{2} A^2 + \frac{1}{2} B^2 + g_{A^4} \frac{1}{4} A^4 + g_{B^4} \frac{1}{4} B^4 + \frac{1}{2} g_{ABAB} ABAB \right\}$$

radiative corrections 'generate' *effective vertices*. For instance  generates $\mathcal{N}\, \mathrm{Tr}_{\mathcal{N}}(ABBA)$.

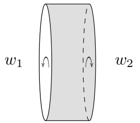
$$\Gamma_N[A, B] = \mathrm{Tr}_N \Big\{ \overbrace{\frac{Z_A}{2} A^2 + \frac{Z_B}{2} B^2 + \bar{g}_{A^4} \frac{1}{4} A^4 + \bar{g}_{B^4} \frac{1}{4} B^4 + \frac{1}{2} \bar{g}_{ABAB} ABAB}^{\text{operators from the bare action (but with 'running couplings')}} + \overbrace{\frac{1}{2} \bar{g}_{ABBA} ABBA + \frac{1}{2} \bar{g}_{A|A}\, \mathrm{Tr}_N(A) \times A + \dots}^{\text{radiative corrections}} \Big\}$$

We are interested in *one-loop graphs*. The *effective vertex* O_G^{eff} of such a graph is formed by reading off each word w_i traveling around all ribbon edges (propagators) by both sides:

$$O_G^{\mathrm{eff}} = \overbrace{\mathrm{Tr}_N(w_1) \times \mathrm{Tr}_N(w_2) \times \cdots \times \mathrm{Tr}_N(w_s)}^{\text{from vertices contracted with propagators}}$$
$$\underbrace{\times\, \mathrm{Tr}_N(U_1) \times \mathrm{Tr}_N(U_2) \dots \times \mathrm{Tr}_N(U_r)}_{\text{from vertices uncontracted with propagators}}$$

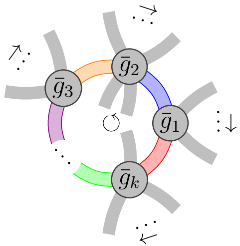


gral. situation (yet, without multitraces)

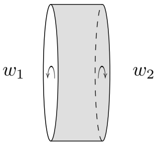


inner word $w_1 \otimes$ outer word w_2

So the actual question is: find the pre-image of the map

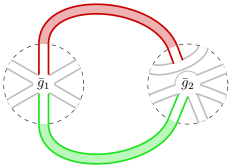


$\longmapsto$



gral. situation (yet, without multitraces) — inner word $w_1 \otimes$ outer word w_2

... actually, for multitrace operators: pre-image of any $O = \prod_\alpha \mathrm{Tr}_N\, w_\alpha$:



One then sums over the product of all g_i's appearing in such 1-loops.
These polynomials span the β-function for O.

Two steps

1. Understanding the Func. Renormalisation Eq.

[CP 2007.10914, *Ann. Henri Poincaré* 2021]

- prove Wetterich Equation, or FRGE; its proof determines the algebra that governs geometric series in the Hessian of Γ
- [A. Eichhorn, T. Koslowski, *Phys. Rev. D* '13] oriented us, but the proof of the FRGE dictates an algebra not reported there
- β-equations found for a sextic truncation (48 running operators). For the unique real solution g^* leading to a single relevant direction (positive e.v. of $-(\partial\beta_i/\partial g_j)_{i,j}|_{g^*}$) yields an R_N-dependent

$$g^*_{A^4} = 1.002\times\left(g^*_{A^4}\big|_{\text{[Kazakov-Zinn-Justin, Nucl. Phys. B '99]}}\right)$$

2. Unicity (using a ribbon graph argument)

[CP 2111.02858 *Lett. Math. Phys.* 2022]

- write down Wetterich Equation
  "$\dot\Gamma = \frac{1}{2}\,\mathrm{Tr}_{M_k(\mathcal{A})}\left\{\dot R_N/[\Gamma^{(2)}+R_N]\right\}$"
- assume an expansion of its rhs in unitary-invariant operators ($\neq$ exact RG)
- impose the one-loop structure and solve for the algebra $\mathcal{A}=\mathcal{A}_{k,N}$
- determine from it the algebra that computes Wetterich equation; it is unique and the one reported in [CP 2007.10914]

- *nc-derivative* $\partial_A : \mathbb{C}_{\langle k \rangle} \to \mathbb{C}_{\langle k \rangle}^{\otimes 2}$ sums over 'replacements of A by $\otimes$' [Turnbull+Rota-Sagan-Stein+Voiculescu]:

$$\partial_A(PAAR) = P \otimes AR + PA \otimes R\,, \text{ but}$$
$$\partial_A(ALGEBRA) = 1 \otimes LGEBRA + ALGEBR \otimes 1$$

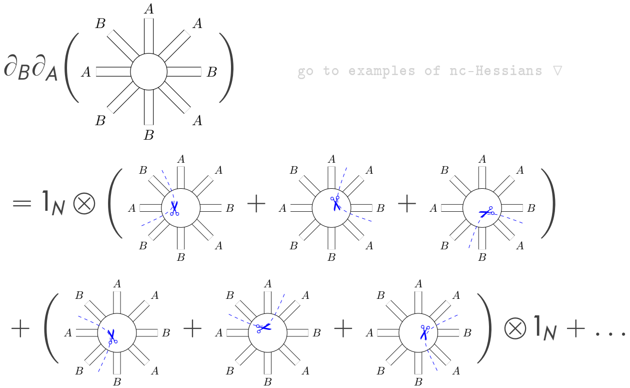
- parenthetically, nc-derivatives allow to compactly write loop-equations (or Dyson-Schwinger eqs.) in random matrix theory

$$\mathbb{E}\Big[\Big(\frac{1}{N}\operatorname{Tr}_N \otimes \frac{1}{N}\operatorname{Tr}_N\Big)(\partial_X P)\Big] = \mathbb{E}[P \mathscr{D}_X V] \qquad P \in \mathbb{C}_{\langle k \rangle}$$

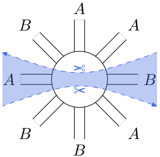
[J. Mingo, R. Speicher *Probab. Math. Stat.* 2013] [A. Guionnet *Jpn. J. Math.* 2016] (cf. S. Azarfar and N. Pagliaroli's talks)

$$\mathscr{D}_X P = \partial_X \operatorname{Tr}_N P \qquad P \in \mathbb{C}_{\langle k \rangle}^{(N)}$$

- $W \in \mathbb{C}_{\langle k \rangle}$, the *nc-Hessian* $\operatorname{Hess} \operatorname{Tr}_N W \in M_k(\mathbb{C}_{\langle k \rangle} \otimes \mathbb{C}_{\langle k \rangle})$ has entries are $\operatorname{Hess}_{b,a} \operatorname{Tr} W = (\partial_{X_b} \circ \partial_{X_a}) \operatorname{Tr}_N W$. Are computed by 'cuts': e.g. $W = ABAABABB$ [CP '21]



go to examples of nc-Hessians ▽

in ellipsis $\sum_{\text{cuts}}$ like  $\rightarrow BAA \otimes ABB$

- products of traces $\Rightarrow$ extend by $\boxtimes$, $\mathcal{A}_k = (\mathbb{C}_{\langle k \rangle} \otimes \mathbb{C}_{\langle k \rangle}) \oplus (\mathbb{C}_{\langle k \rangle} \boxtimes \mathbb{C}_{\langle k \rangle})$

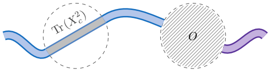
✓ in $\underline{\text{Vect}}_{\mathbb{C}}$, just $\mathbb{C}_{\langle k \rangle}^{\otimes 2}$, but as algebra ...

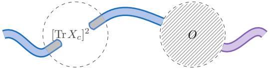
yields an ordinary unit

generates the constants in the real vect. space

$$\mathrm{Hess}_{a,b}(\mathrm{Tr}\, P \cdot \mathrm{Tr}\, Q) = \mathrm{Tr}\, P \cdot \mathrm{Hess}_{a,b}[\mathrm{Tr}\, Q] + (\partial_{X_a} \mathrm{Tr}\, P) \boxtimes (\partial_{X_b} \mathrm{Tr}\, Q) + (P \leftrightarrow Q)$$

- products of traces $\Rightarrow$ extend by $\boxtimes$, $\mathcal{A}_k = (\mathbb{C}_{\langle k \rangle} \otimes \mathbb{C}_{\langle k \rangle}) \oplus (\mathbb{C}_{\langle k \rangle} \boxtimes \mathbb{C}_{\langle k \rangle})$

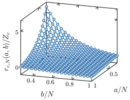




$$\mathrm{Hess}_{a,b}(\mathrm{Tr}\, P \cdot \mathrm{Tr}\, Q) = \mathrm{Tr}\, P \cdot \mathrm{Hess}_{a,b}[\mathrm{Tr}\, Q] + (\partial_{X_a} \mathrm{Tr}\, P) \boxtimes (\partial_{X_b} \mathrm{Tr}\, Q) + (P \leftrightarrow Q)$$

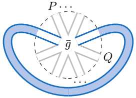
- Wetterich Eq. governs the functional RG $t = \log N$ $\quad \eta_j = -\partial_t \log(Z_j)$

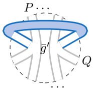
$$\partial_t \Gamma_N[\mathbb{X}] = \frac{1}{2} \mathrm{STr} \left\{ \frac{\partial_t R_N}{\mathrm{Hess}\, \Gamma_N[\mathbb{X}] + R_N} \right\}$$



$$\overset{\text{assume}}{=} \sum_{k=0}^{\infty} \bar{h}_k(N, \eta_1, \ldots, \eta_n) \times \underbrace{\frac{1}{2}(-1)^k \mathrm{Tr}_{M_k(\mathcal{A})} \left\{ (\mathrm{Hess}\, \Gamma_N^{\mathrm{Int}}[\mathbb{X}])^{\star k} \right\}}_{\text{regulator-independent part}}$$

- $\mathrm{STr} = \mathrm{Tr}_k \otimes \mathrm{Tr}_{\mathcal{A}_k}$. Tadpoles

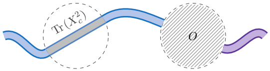


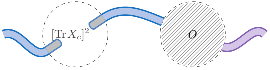


imply

$$\mathrm{Tr}_{\mathcal{A}_k}(P \otimes Q) = \mathrm{Tr}_N\, P \cdot \mathrm{Tr}_N\, Q, \qquad \mathrm{Tr}_{\mathcal{A}_k}(P \boxtimes Q) = \mathrm{Tr}_N(PQ)$$

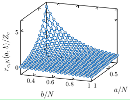
- products of traces $\Rightarrow$ extend by $\boxtimes$, $\mathcal{A}_k = (\mathbb{C}_{\langle k\rangle} \otimes \mathbb{C}_{\langle k\rangle}) \oplus (\mathbb{C}_{\langle k\rangle} \boxtimes \mathbb{C}_{\langle k\rangle})$





$$\mathrm{Hess}_{a,b}(\mathrm{Tr}\, P \cdot \mathrm{Tr}\, Q) = \mathrm{Tr}\, P \cdot \mathrm{Hess}_{a,b}[\mathrm{Tr}\, Q] + (\partial_{X_a} \mathrm{Tr}\, P) \boxtimes (\partial_{X_b} \mathrm{Tr}\, Q) + (P \leftrightarrow Q)$$

- Wetterich Eq. governs the functional RG $t = \log N$



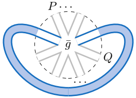
$$\partial_t \Gamma_N[\mathbb{X}] = \frac{1}{2}\mathrm{STr}\left\{\frac{\partial_t R_N}{\mathrm{Hess}\,\Gamma_N[\mathbb{X}] + R_N}\right\}$$

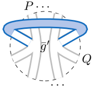
U(N)-invariants

$$\overset{\text{assume}}{=} \sum_{k=0}^{\infty} \bar{h}_k(N, \eta_1, \ldots, \eta_n) \times \underbrace{\frac{1}{2}(-1)^k \mathrm{Tr}_{M_k(\mathcal{A})}\left\{(\mathrm{Hess}\,\Gamma_N^{\mathrm{Int}}[\mathbb{X}])^{\star k}\right\}}_{\text{regulator-independent part}}$$

Projecting to U(N)-invariants

- $\mathrm{STr} = \mathrm{Tr}_k \otimes \mathrm{Tr}_{\mathcal{A}_k}$. Tadpoles*





imply

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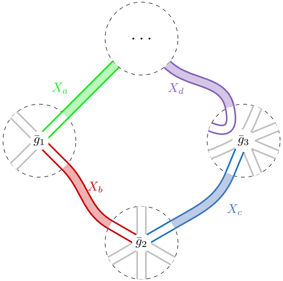
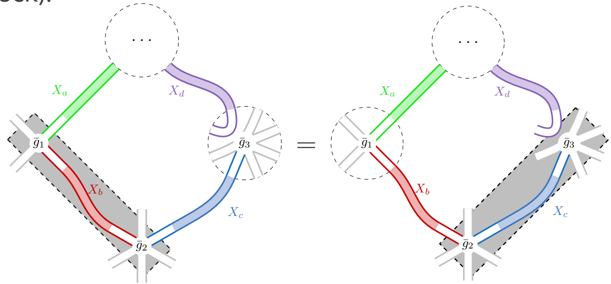
called * tadpoles, since usually look like

Finding ⋆

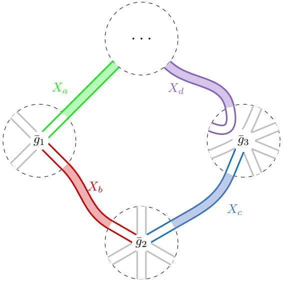
Want: 

$\subset \mathrm{Hess}_{a,b}\, O_1 \star \mathrm{Hess}_{b,c}\, O_2 \star \mathrm{Hess}_{c,d}\, O_3 \star \ldots \star \mathrm{Hess}_{*,a}\, O_\ell$

Associativity (trivial check):



Finding ⋆

Want: 

$\subset \mathrm{Hess}_{a,b}\, O_1 \star \mathrm{Hess}_{b,c}\, O_2 \star \mathrm{Hess}_{c,d}\, O_3 \star \ldots \star \mathrm{Hess}_{*,a}\, O_\ell$

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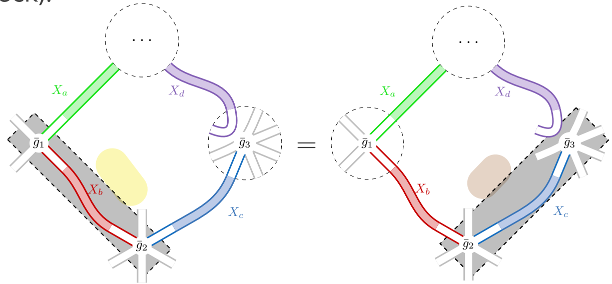
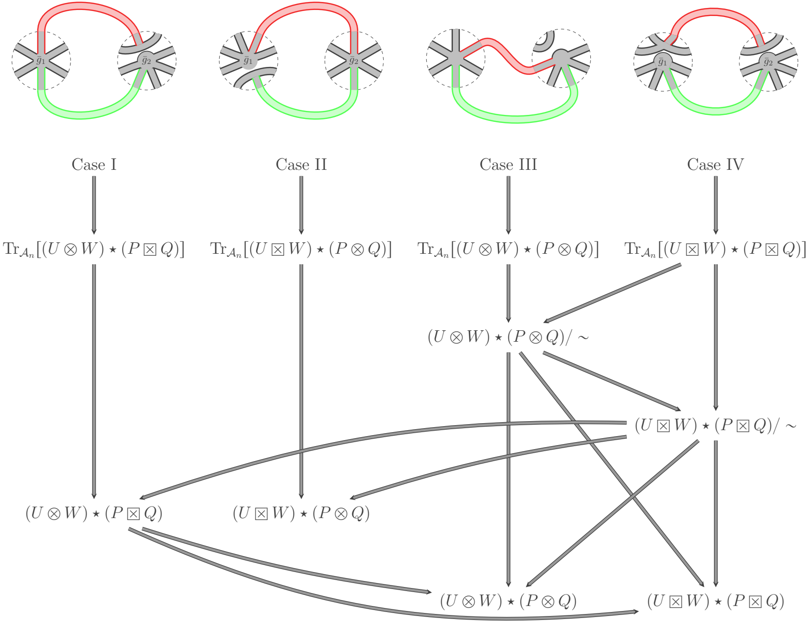
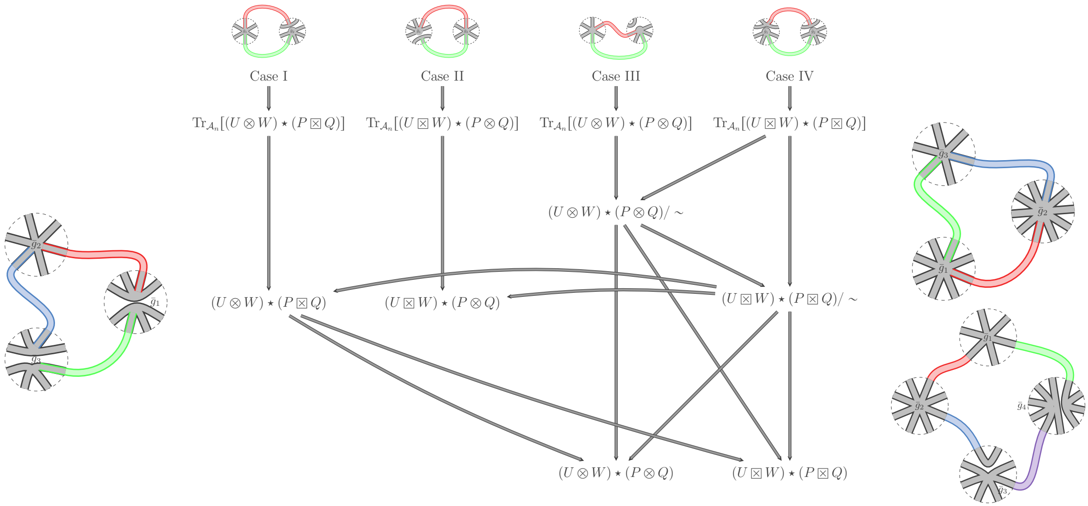




$\bar{g}_1$
$\bar{g}_2$
Case I
Case II
Case III
Case IV
$\mathrm{Tr}_{\mathcal{A}_n}[(U \otimes W) \star (P \boxtimes Q)]$
$\mathrm{Tr}_{\mathcal{A}_n}[(U \boxtimes W) \star (P \otimes Q)]$
$\mathrm{Tr}_{\mathcal{A}_n}[(U \otimes W) \star (P \otimes Q)]$
$\mathrm{Tr}_{\mathcal{A}_n}[(U \boxtimes W) \star (P \boxtimes Q)]$
$(U \otimes W) \star (P \otimes Q)/\sim$
$(U \boxtimes W) \star (P \boxtimes Q)/\sim$
$(U \otimes W) \star (P \boxtimes Q)$
$(U \boxtimes W) \star (P \otimes Q)$
$(U \otimes W) \star (P \otimes Q)$
$(U \boxtimes W) \star (P \boxtimes Q)$



Case I
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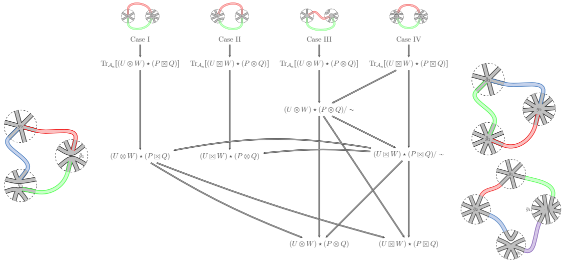
$(U \otimes W) \star (P \otimes Q)$
$(U \boxtimes W) \star (P \boxtimes Q)$
$\bar{g}_1$
$\bar{g}_2$
$\bar{g}_3$
$\bar{g}_4$



Thm. [CP '22] If the RG-flow is computable in terms of $\mathrm{U}(N)$-invariants, the algebra of Functional Renormalisation is $M_k(\mathcal{A}_{k,N},\star)$ where

$$\mathcal{A}_{k,N} = (\mathbb{C}^{(N)}_{\langle k\rangle}\otimes\mathbb{C}^{(N)}_{\langle k\rangle})\oplus(\mathbb{C}^{(N)}_{\langle k\rangle}\boxtimes\mathbb{C}^{(N)}_{\langle k\rangle})$$

whose product in hom. elements reads:

$$(U\otimes W)\star(P\otimes Q) = PU\otimes WQ\,,$$
$$(U\boxtimes W)\star(P\otimes Q) = U\boxtimes PWQ\,,$$
$$(U\otimes W)\star(P\boxtimes Q) = WPU\boxtimes Q\,,$$
$$(U\boxtimes W)\star(P\boxtimes Q) = \mathrm{Tr}_N(WP)U\boxtimes Q\,,$$

and traces $\mathrm{Tr}_k\otimes\mathrm{Tr}_{\mathcal{A}_k}$

$$\mathrm{Tr}_{\mathcal{A}_k}(P\otimes Q) = \mathrm{Tr}_N P\cdot\mathrm{Tr}_N Q\,,$$
$$\mathrm{Tr}_{\mathcal{A}_k}(P\boxtimes Q) = \mathrm{Tr}_N(PQ)\,.$$

Remark: To be more precise, any occurrence of the free algebra in $\mathcal{A}_{k,N}$ should be replaced by the algebra of 'trace polynomials' (e.g. $\mathrm{Tr}_N(X_1X_3)X_2 + N\,\mathrm{Tr}_N(X_2^2)$) [D. Jekel-W. Li-D. Shlyakhtenko, '21]

Example: A Hermitian 3-matrix model. Consider $O_1 = \frac{\bar{g}_1}{2}[\mathrm{Tr}_N(\frac{A^2}{2})]^2$ and $O_2 = \bar{g}_2 \, \mathrm{Tr}_N(ABC)$.

$$\mathrm{Hess}_{I,J}\, O_1 = \delta_I^J \delta_I^A \bar{g}_1 \{ \underbrace{\mathrm{Tr}_N(A^2/2)\cdot[1_N \otimes 1_N]} + \underbrace{A \boxtimes A} \} \,,$$


where a ‘filled ribbon’ means contracted in the one-loop graph, and ‘white ribbon’ uncontracted.

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A B C

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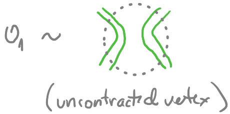

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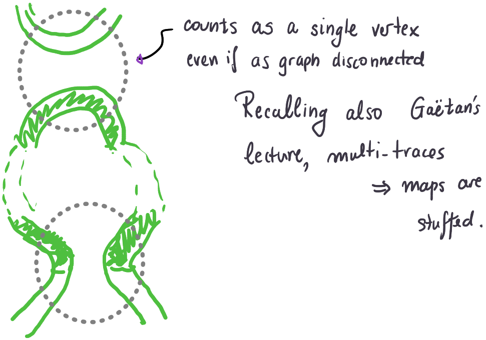

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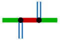
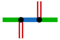

$$\mathrm{Hess}\, O_2 = \bar{g}_2 \begin{bmatrix} 0 & C\otimes 1_N & B\otimes 1_N \\ 1_N\otimes C & 0 & A\otimes 1_N \\ 1_N \otimes B & 1_N\otimes A & 0 \end{bmatrix} \Rightarrow [\mathrm{Hess}\, O_2]^{\star 2} = \bar{g}_2^2 \begin{bmatrix} \overbrace{C\otimes C} + \overbrace{B\otimes B} & B\otimes A & C\otimes A \\ A\otimes B & A\otimes A + C\otimes C & C\otimes B \\ A\otimes C & B\otimes C & B\otimes B + A\otimes A \end{bmatrix}.$$

Extracting coefficients

$$[\bar{g}_1\bar{g}_2^2]\,\mathrm{Tr}_{M_3(\mathcal{A})}\{\mathrm{Hess}\, O_1 \star [\mathrm{Hess}\, O_2]^{\star 2}\} = \mathrm{Tr}_N(A^2/2)\times\left[(\mathrm{Tr}_N\, C)^2 + (\mathrm{Tr}_N\, B)^2\right] + \mathrm{Tr}_N(ACAC + ABAB)\,,$$

which are effective vertices of the four one-loop graphs that can be formed with the contractions of (the filled ribbon half-edges of) any of {, } with any of {, }.

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Why not using graphs? Soon, nc-Hessians get bulky: In [CP '20] 48 such operators run

Operator	Its nc-Hessian
$\operatorname{Tr}(A)\operatorname{Tr}(A^3)$	$3\cdot\begin{pmatrix} \operatorname{Tr}A\cdot(A\otimes 1+1\otimes A) + 1\boxtimes A^2 + A^2\boxtimes 1 & 0 \\ 0 & 0 \end{pmatrix}$
$\operatorname{Tr}A\operatorname{Tr}(AAABB)$	$\begin{pmatrix} (\operatorname{Tr}(A)1\otimes(ABB) + \operatorname{Tr}(A)1\otimes(BBA) + \operatorname{Tr}(A)(ABB)\otimes 1 + \operatorname{Tr}(A)(BBA)\otimes 1 + 1\boxtimes(AABB) + 1\boxtimes(ABBA) + 1\boxtimes(BBAA) + (AABB)\boxtimes 1 + (ABBA)\boxtimes 1 + (BBAA)\boxtimes 1 + \operatorname{Tr}(A)A\otimes B^2 + \operatorname{Tr}(A)B^2\otimes A) & \operatorname{Tr}(A)(1\otimes(AAB) + (BAA)\otimes 1 + A\otimes AB + B\otimes A^2 + A^2\otimes B + BA\otimes A) + 1\boxtimes(A^3B) + 1\boxtimes(BA^3) \\ \operatorname{Tr}(A)(1\otimes(BAA) + (AAB)\otimes 1 + A\otimes BA + B\otimes A^2 + A^2\otimes B + AB\otimes A) + (A^3B)\boxtimes 1 + (BA^3)\boxtimes 1 & \operatorname{Tr}(A)(1\otimes(A^3) + (A^3)\otimes 1) \end{pmatrix}$

these entries are not in $\mathbb{C}\langle 2\rangle$ / cyclicity

empty ⇒ alive

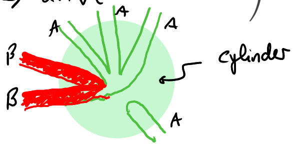
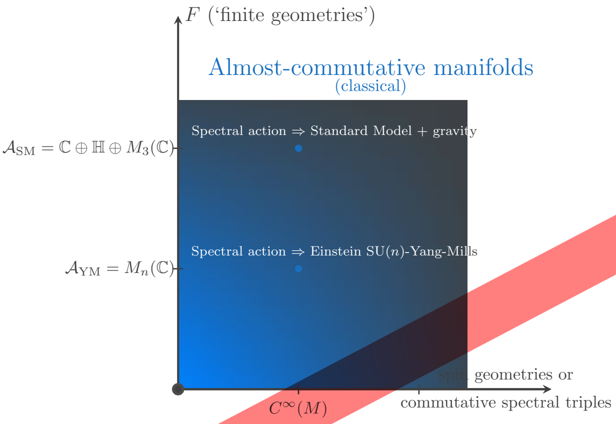


Table: Some Hessians. Here $\operatorname{Tr} = \operatorname{Tr}_N$.

III. Matrix gauge theory



$$\mathcal{Z}_{\mathrm{AC}} \overset{?}{=} \int_{\mathrm{Dirac}} e^{-\frac{1}{\hbar}\operatorname{Tr} f(D)} \mathrm{d}D$$

(hard for almost-commutative manifolds)

True, but I didn't mention this in the actual talk

Definition [CP 2105.01025] We define a *gauge matrix spectral triple* $G_\ell \times F$ as the spectral triple product of a fuzzy geometry G_ℓ with a finite geometry $F = (A_F, H_F, D_F)$, $\dim A_F < \infty$.

Lemma-Definition [CP 2105.01025] Consider a gauge matrix spectral triple $G_\ell \times F$ with

$$F = (M_n(\mathbb{C}), M_n(\mathbb{C}), D_F)$$

and G_ℓ Riemannian ($d = 4$) fuzzy geometry on $M_N(\mathbb{C})$, whose fluctuated Dirac op. is

$$D_\omega = \sum_{\mu=0}^{3} \overbrace{\gamma^\mu \otimes (\ell_\mu + a_\mu) + \gamma^{\hat\mu} \otimes (x_\mu + s_\mu)}^{D_{\text{gauge}}} + \overbrace{\gamma \otimes \Phi}^{D_{\text{Higgs}}}, \qquad a_\mu = \text{'gauge potential'},\ x_\mu = \text{spin connection?}$$

The *field strength* is given by

$$\mathscr{F}_{\mu\nu} := [\overbrace{\ell_\mu + a_\mu}^{d_\mu}, \ell_\nu + a_\nu] =: [\mathsf{F}_{\mu\nu}, \ \cdot\ ]$$

Lemma The gauge group $\mathsf{G}(A) \cong \mathrm{PU}(N) \times \mathrm{PU}(n)$ acts as follows

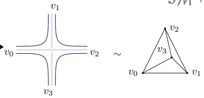
$$\mathsf{F}_{\mu\nu} \mapsto \mathsf{F}^u_{\mu\nu} = u\mathsf{F}_{\mu\nu}u^* \text{ for all } u \in \mathsf{G}(A)$$

The content of the Spectral Action ...

Meaning	Random matrix case, flat $d = 4$ Riem.	Smooth operator
	Tr = trace of ops. $M_N \otimes M_n \to M_N \otimes M_n$	
Derivation	$\ell_\mu = [L_\mu \otimes 1_n, \ \cdot \ ]$	∂_i
Gauge potential	$\mathscr{a}_\mu = [A_\mu, \ \cdot \ ]$	$\mathbb{A}_i$
Covariant derivative	$\mathscr{d}_\mu = \ell_\mu + \mathscr{a}_\mu$	$\mathbb{D}_i = \partial_i + \mathbb{A}_i$
Field strength	$[\mathscr{d}_\mu, \mathscr{d}_\nu] = \overbrace{[\ell_\mu, \ell_\nu]}^{\not\equiv 0} + [\ell_\mu, \mathscr{a}_\nu] - [\ell_\nu, \mathscr{a}_\mu] + [\mathscr{a}_\mu, \mathscr{a}_\nu]$	$[\mathbb{D}_i, \mathbb{D}_j] = \overbrace{[\partial_i, \partial_j]}^{\equiv 0} + \partial_i \mathbb{A}_j - \partial_j \mathbb{A}_i + [\mathbb{A}_i, \mathbb{A}_j]$
Yang-Mills action	$-\frac{1}{4} \mathrm{Tr}(\mathscr{F}_{\mu\nu} \mathscr{F}^{\mu\nu})$	$-\frac{1}{4} \int_M \mathrm{Tr}_{\mathfrak{su}(n)} (\mathbb{F}_{ij} \mathbb{F}^{ij}) \mathrm{vol}$
Higgs field	Φ	h
Higgs potential	$\mathrm{Tr}(f_2 \Phi^2 + f_4 \Phi^4)$	$\int_M (f_2 \|h\|^2 + f_4 \|h\|^4) \mathrm{vol}$
Gauge-Higgs coupling	$-\mathrm{Tr}(\mathscr{d}_\mu \Phi \mathscr{d}^\mu \Phi)$	$-\int_M \|\mathbb{D}_i h\|^2 \mathrm{vol}$

+

Propagators and $\sim (\ell_\mu)_{ij} (\ell_\nu)_{jm} (\ell^\mu)_{ml} (\ell^\nu)_{li} \leftrightarrow$



True but I didn't talk about this (no time)

Conclusion

- spectral triple $\equiv$ spin manifold mod. commutativity of the 'algebra of functions'
- spin $M \times \{$finite spectral triple$\} \equiv$ almost-commutative
  (reproduces classical Standard Model, but hard to quantize)
- $G_\ell \times F$= fuzzy $\times$ finite $=$ gauge matrix spectra triple
  is $\mathrm{PU}(n)$-Yang-Mills-Higgs-like if F is over $M_n(\mathbb{C})$; Small step towards
  [Connes Marcolli, *NCG, QFT and motives*, '07, next sceenshot]

  « The far distant goal is to set up a functional integral evaluating spectral observables $\mathcal{S}$ as

  $$\langle \mathcal{S} \rangle = \mathcal{N} \int \mathcal{S}\, e^{-\mathrm{Tr}(f(D/\Lambda)) - \frac{1}{2}\langle J\psi, D\,\psi\rangle - \rho(e,D)}\, \mathcal{D}[\psi]\mathcal{D}[D]\mathcal{D}[e], \quad (1.892)$$ »

- The matrix algebra $M_k(\mathcal{A}_{k,N})$ where functional renormalisation for random matrices (k-martix model) takes place was provided. $\mathcal{A}_{k,N}$ is a bigger relative of $\mathbb{C}_{\langle k \rangle}^{(N)}$

Conclusion

- spectral triple $\equiv$ spin manifold mod. commutativity of the 'algebra of functions'
- spin $M \times$ {finite spectral triple} $\equiv$ almost-commutative (reproduces classical Standard Model, but hard to quantize)
- $G_\ell \times F$= fuzzy $\times$ finite = gauge matrix spectra triple True but I didn't talk about this (no time) is $\mathrm{PU}(n)$-Yang-Mills-Higgs-like if F is over $M_n(\mathbb{C})$; Small step towards

[Connes Marcolli, *NCG, QFT and motives*, '07, next sceenshot]

« The far distant goal is to set up a functional integral evaluating spectral observables $\mathcal{S}$ as

$$\langle \mathcal{S} \rangle = \mathcal{N} \int \mathcal{S}\, e^{-\mathrm{Tr}(f(D/\Lambda)) - \frac{1}{2}\langle J\psi, D\psi\rangle - \rho(e,D)} \mathcal{D}[\psi]\mathcal{D}[D]\mathcal{D}[e], \quad (1.892)$$

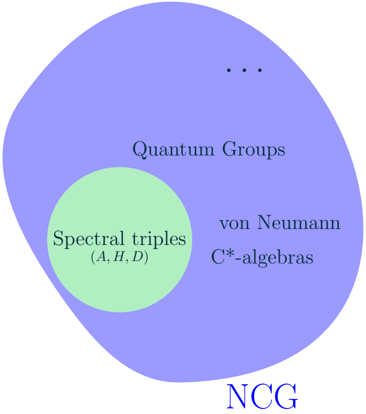
»

- The matrix algebra $M_k(\mathcal{A}_{k,N})$ where functional renormalisation for random matrices (k-martix model) takes place was provided. $\mathcal{A}_{k,N}$ is a bigger relative of $\mathbb{C}^{(N)}_{\langle k \rangle}$

thank you!

References: [CP 1912.13288] [CP 2007.10914] [CP 2102.06999] [CP 2105.01025] [CP 2111.02858]

Sort of appendix,
which wasn't needed



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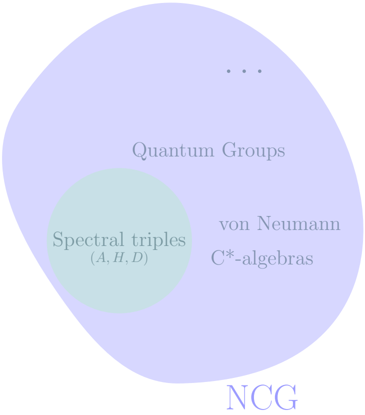
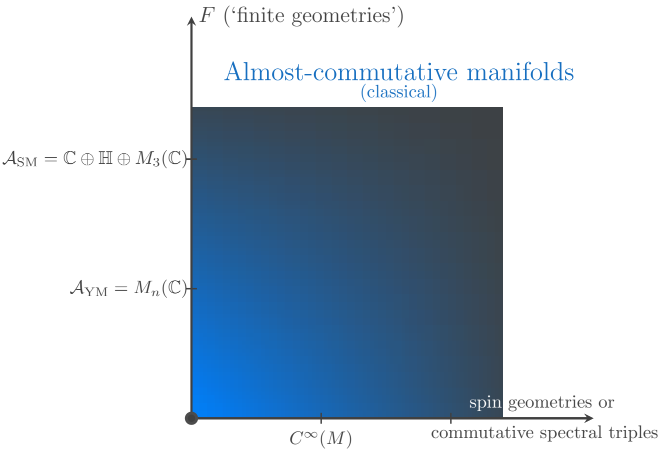
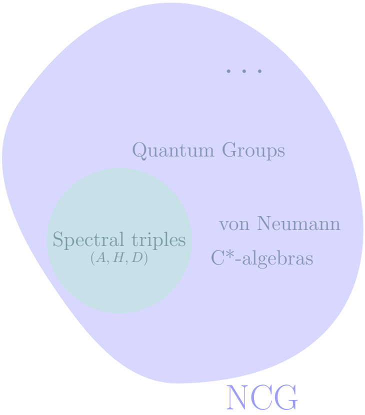
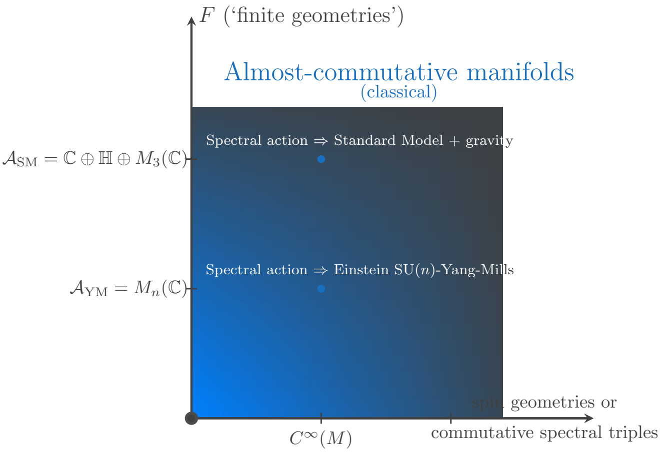





. . .
Quantum Groups
von Neumann
Spectral triples
(A, H, D)
C*-algebras
NCG


F ('finite geometries')
Almost-commutative manifolds
(classical)
Spectral action ⇒ Standard Model + gravity
$\mathcal{A}_{\mathrm{SM}} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$
Spectral action ⇒ Einstein SU(n)-Yang-Mills
$\mathcal{A}_{\mathrm{YM}} = M_n(\mathbb{C})$
spin geometries or
commutative spectral triples
$C^\infty(M)$

NCG toolkit in high energy physics

- On a spectral triple (A, H, D) the (bosonic) classical action is given by

$$S(D) = \mathrm{Tr}_H\, f(D/\Lambda)$$ [Chamseddine-Connes *CMP* '97]

for a bump function f, Λ a scale. It's computed with heat kernel expansion

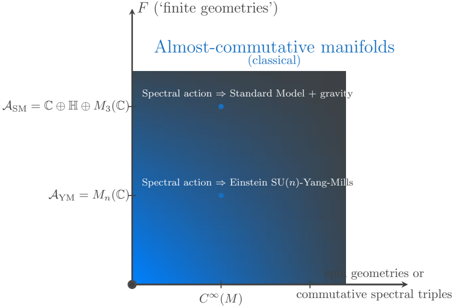
[P. Gilkey, *J. Diff. Geom.* '75]

- Realistic, classical models come from *almost-commutative manifolds* $M \times F$, where F is a finite-dim. spectral triple

$$(C^\infty(A_F), H_M \otimes H_F, D_M \otimes 1_F + \gamma_5 \otimes D_F)$$

- applications require (A, H, D) to have a *reality* $J : H \to H$ antiunitary$^{\text{axioms}}$, implementing a right A-action on H

Sort of appendix, which wasn't needed



NCG toolkit in high energy physics

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$$S^G \rightsquigarrow S^{\mathrm{Maps}(M,G)}$$

$$\mathrm{d} \rightsquigarrow \mathrm{d} + \mathbb{A} \qquad \mathbb{A} \in \Omega^1(M) \otimes \mathfrak{g}$$

$$\mathbb{A}' = u\mathbb{A}u^{-1} + u\mathrm{d}u^{-1} \qquad u \in \mathrm{Maps}(M, G)$$

NCG toolkit in high energy physics

Sort of appendix, which wasn't needed

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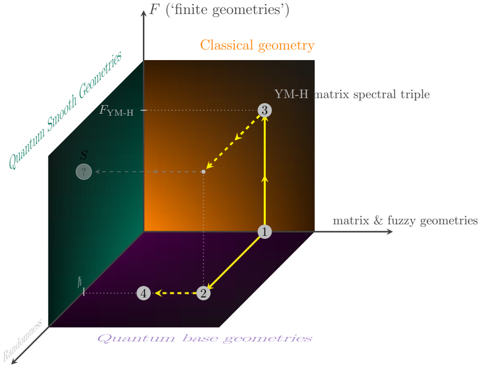
- given (A, H, D) and a Morita equivent algebra B (i.e. $\mathrm{End}_A(E) \cong B$) yields new $(B, E \otimes_A H, D')$. For $A = B$, in fact a tower

$$\left\{(A, H, D + \omega \pm J\omega J^{-1})\right\}_{\omega \in \Omega^1_D(A)}$$

$$D_\omega \mapsto \mathrm{Ad}(u) D_\omega \mathrm{Ad}(u)^* = D_{\omega_u}$$

$$\omega \mapsto \omega_u = u\omega u^* + u[D, u^*] \quad u \in \mathcal{U}(A)$$ skip cube

Organisation



1. Matrix Geometries
   [J. Barrett, *J. Math. Phys.* 2015]
2. Dirac ensembles [J. Barrett, L. Glaser, *J. Phys. A* 2016] and how to compute the spectral action
   [CP 1912.13288]
3. Gauge matrix spectral triples *(this talk)*
   [CP 2105.01025]
4. Functional Renormalisation [CP 2007.10914] and
   [CP 2111.02858]

Sort of appendix, which wasn't needed

β-functions of NCG two-matrix models, signature $\eta = \mathrm{diag}(e_1, e_2)$

Sort of appendix, which wasn't needed

$$2h_1(a_4 + c_{22} + 2d_{2|02} + 6d_{2|2}) = \eta_a$$
$$2h_1(b_4 + c_{22} + 6d_{02|02} + 2d_{2|02}) = \eta_b$$
$$-h_1[e_a(a_4 - c_{1111}) + 2d_{1|12} + 6d_{1|3}] + d_{1|1}(\eta + 1) = \beta(d_{1|1})$$
$$-h_1[e_b(b_4 - c_{1111}) + 6d_{01|03} + 2d_{01|21}] + d_{01|01}(\eta + 1) = \beta(d_{01|01})$$

The next block encompasses the connected quartic couplings:

$$h_2(4a_4^2 + 4c_{22}^2) + a_4(2\eta + 1)$$
$$-h_1(24a_6 e_a + 4c_{42} e_b + 4d_{02|4} e_b + 4d_{2|4} e_a) = \beta(a_4)$$

$$h_2(4b_4^2 + 4c_{22}^2) + b_4(2\eta + 1)$$
$$-h_1(24b_6 e_b + 4c_{24} e_a + 4d_{02|04} e_b + 4d_{2|04} e_a) = \beta(b_4)$$

$$-h_1(2e_a c_{1212} + e_b 2c_{2121} + 3e_a c_{24} + 3e_b c_{42} + e_a d_{02|22} + e_b d_{2|22})$$
$$+h_2(2a_4 c_{22} + 2b_4 c_{22} + 2e_a e_b c_{1111}^2 + 2e_a e_b c_{22}^2) + c_{22}(2\eta + 1) = \beta(c_{22})$$

$$8e_a e_b c_{1111} c_{22} h_2 + c_{1111}(2\eta + 1)$$
$$+h_1(4e_a c_{1311} + 4e_b c_{3111} + 2e_a d_{02|1111} + 2e_b d_{2|1111}) = \beta(c_{1111})$$

Matrix or Fuzzy Geometries GO TO characterization ⋘

Sort of appendix, which wasn't needed

Definition ("condensed" from [J. Barrett, *J. Math. Phys.* 2015]).

A *fuzzy geometry* of *signature* $(p,q)\in\mathbb{Z}^2_{\geqslant 0}$ is given by

- a simple matrix algebra $\mathcal{A}$ – we take always $\mathcal{A}=M_N(\mathbb{C})$
- a Hermitian $\mathbb{C}\ell(p,q)$-module $\mathbb{S}$ with a *chirality* γ. That is a linear map $\gamma:\mathbb{S}\to\mathbb{S}$ satisfying $\gamma^*=\gamma$ and $\gamma^2=1$
- a Hilbert space $\mathcal{H}=\mathbb{S}\otimes M_N(\mathbb{C})$ with inner product $\langle v\otimes R, w\otimes S\rangle=(v,w)\,\mathrm{Tr}_N(R^*S)$ for each $R,S\in M_N(\mathbb{C})$, being $(\cdot\,,\cdot)$ the inner product of $\mathbb{S}$
- a left-$\mathcal{A}$ representation $\rho(a)(v\otimes R)=v\otimes(aR)$ on $\mathcal{H}$, $a\in\mathcal{A}$ and $v\otimes R\in\mathcal{H}$
- three signs $\epsilon,\epsilon',\epsilon''\in\{-1,+1\}$ determined through $s:=q-p$ by the following table:

$s\equiv q-p \pmod 8$	0	1	2	3	4	5	6	7
ϵ	+	+	−	−	−	−	+	+
ϵ'	+	−	+	+	+	−	+	+
ϵ''	+	+	−	+	+	+	−	+

- a real structure $J=C\otimes *$, where $*$ is complex conjugation and C is an anti-unitarity on $\mathbb{S}$ satisfying $C^2=\epsilon$ and $C\gamma^\mu=\epsilon'\gamma^\mu C$ for all the gamma matrices $\mu=1,\dots,p+q$.
- a self-adjoint operator D on $\mathcal{H}$ satisfying the *order-one condition*

$$[\,[D,\rho(a)]\,,J\rho(b)J^{-1}]=0 \qquad \text{for all } a,b\in\mathcal{A}$$

- a chirality $\Gamma=\gamma\otimes 1_{\mathcal{A}}$ for $\mathcal{H}$, where γ is the chirality of $\mathbb{S}$. The signs above impose: