# On random noncommutative geometry, multi-matrix models and free algebra

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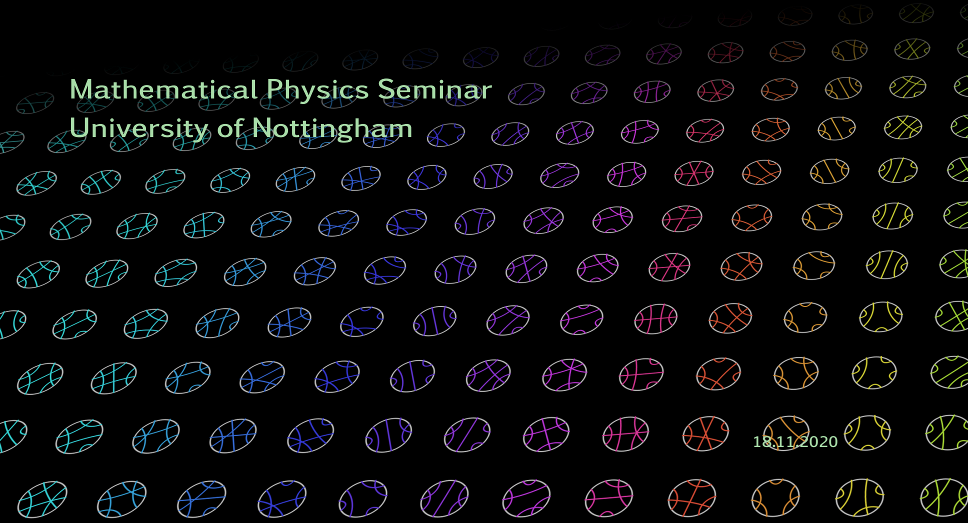

Mathematical Physics Seminar
University of Nottingham

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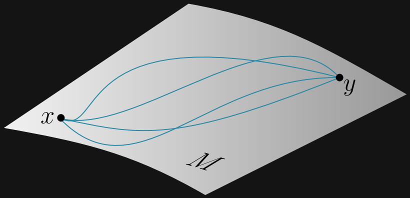
# Quantum Space & Quantisation

- Path integrals on $M$



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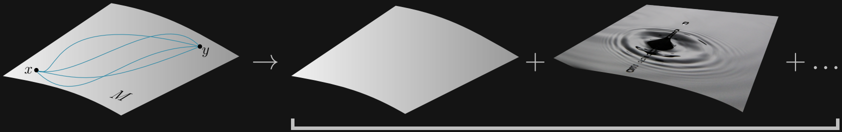
- Path integrals on $M \rightarrow$ Path integral of spacetime (Quantum Gravity)




Quantum superposition of geometries

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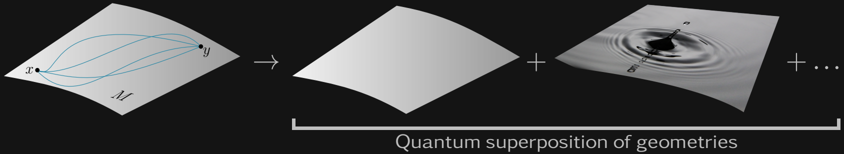


- Quantum Gravity $\to$ Random Geometry (Euclidean QFT)

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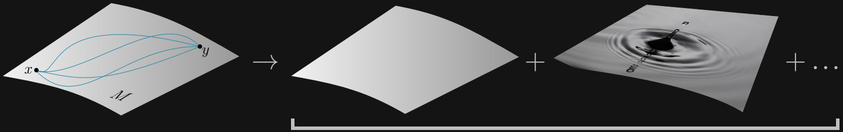


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- Quantum Gravity $\to$ Random Geometry (Euclidean QFT)

- Discretisation approaches: Causal Dynamical Triangulations, Matrix Models, Group Field Theory, Tensor Models (TM),

(Wikipedia) 

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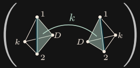

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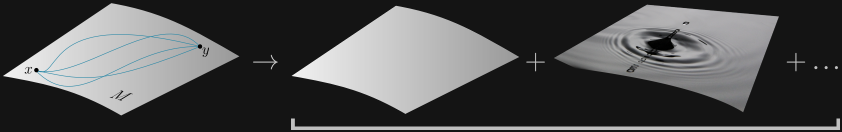


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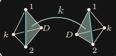
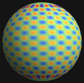



- Algebraisation approach: Noncommutative Geometry (NCG)



$$\mathcal{Z}_{\text{NCG}} = \int_{\text{Dirac}} e^{-\operatorname{Tr} f(D)} \mathrm{d}D$$

# Noncommutative Geometry (Motivation)

$$
\begin{gathered}
-\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4}g^2_s f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu + \\
\tfrac{1}{2}ig^2_s(\bar{q}^\sigma_i \gamma^\mu q^\sigma_j)g^a_\mu + \bar{G}^a\partial^2 G^a + g_s f^{abc}\partial_\mu \bar{G}^a G^b g^c_\mu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - \\
M^2 W^+_\mu W^-_\mu - \tfrac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c^2_w}M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \\
\tfrac{1}{2}\partial_\mu H \partial_\mu H - \tfrac{1}{2}m^2_h H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2\phi^+\phi^- - \tfrac{1}{2}\partial_\mu\phi^0\partial_\mu\phi^0 - \\
\tfrac{1}{2c^2_w}M\phi^0\phi^0 - \beta_h[\tfrac{2M^2}{g^2} + \tfrac{2M}{g}H + \tfrac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)] + \tfrac{2M^4}{g^2}\alpha_h - \\
igc_w[\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + \\
Z^0_\mu(W^+_\nu\partial_\nu W^-_\mu - W^-_\nu\partial_\nu W^+_\mu)] - igs_w[\partial_\nu A_\mu(W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - A_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + A_\mu(W^+_\nu\partial_\nu W^-_\mu - \\
W^-_\nu\partial_\nu W^+_\mu)] - \tfrac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \tfrac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + \\
g^2 c^2_w(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s^2_w(A_\mu W^+_\mu A_\nu W^-_\nu - \\
A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w[A_\mu Z^0_\nu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - \\
2A_\mu Z^0_\mu W^+_\nu W^-_\nu] - g\alpha[H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-] - \tfrac{1}{8}g^2\alpha_h[H^4 + \\
(\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2] - \\
gMW^+_\mu W^-_\mu H - \tfrac{1}{2}g\tfrac{M}{c^2_w}Z^0_\mu Z^0_\mu H - \tfrac{1}{2}ig[W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - \\
W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)] + \tfrac{1}{2}g[W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) - \\
W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)] + \tfrac{1}{2}g\tfrac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) - \\
ig\tfrac{s^2_w}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_w MA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) - \\
ig\tfrac{1-2c^2_w}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + igs_w A_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \\
\tfrac{1}{4}g^2 W^+_\mu W^-_\mu[H^2 + (\phi^0)^2 + 2\phi^+\phi^-] - \tfrac{1}{4}g^2\tfrac{1}{c^2_w}Z^0_\mu Z^0_\mu[H^2 + (\phi^0)^2 + \\
2(2s^2_w - 1)^2\phi^+\phi^-] - \tfrac{1}{2}g^2\tfrac{s^2_w}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) -
\end{gathered}
$$

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\begin{gathered}
\tfrac{1}{2}ig^2\tfrac{s^2_w}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \tfrac{1}{2}g^2 s_w A_\mu\phi^0(W^+_\mu\phi^- + \\
W^-_\mu\phi^+) + \tfrac{1}{2}ig^2 s_w A_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\tfrac{s_w}{c_w}(2c^2_w - \\
1)Z^0_\mu A_\mu\phi^+\phi^- - g^1 s^2_w A_\mu A_\mu\phi^+\phi^- - \bar{e}^\lambda(\gamma\partial + m^\lambda_e)e^\lambda - \bar{\nu}^\lambda\gamma\partial\nu^\lambda - \\
\bar{u}^\lambda_j(\gamma\partial + m^\lambda_u)u^\lambda_j - \bar{d}^\lambda_j(\gamma\partial + m^\lambda_d)d^\lambda_j + igs_w A_\mu[-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \\
\tfrac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \tfrac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)] + \tfrac{ig}{4c_w}Z^0_\mu[(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + \\
(\bar{e}^\lambda\gamma^\mu(4s^2_w - 1 - \gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(\tfrac{4}{3}s^2_w - 1 - \gamma^5)u^\lambda_j) + (\bar{d}^\lambda_j\gamma^\mu(1 - \\
\tfrac{8}{3}s^2_w - \gamma^5)d^\lambda_j)] + \tfrac{ig}{2\sqrt{2}}W^+_\mu[(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(1 + \\
\gamma^5)C_{\lambda\kappa}d^\kappa_j)] + \tfrac{ig}{2\sqrt{2}}W^-_\mu[(\bar{e}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\lambda\kappa}\gamma^\mu(1 + \\
\gamma^5)u^\lambda_j)] + \tfrac{ig}{2\sqrt{2}}\tfrac{m^\lambda_e}{M}[-\phi^+(\bar{\nu}^\lambda(1-\gamma^5)e^\lambda) + \phi^-(\bar{e}^\lambda(1+\gamma^5)\nu^\lambda)] - \\
\tfrac{g}{2}\tfrac{m^\lambda_e}{M}[H(\bar{e}^\lambda e^\lambda) + i\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda)] + \tfrac{ig}{2M\sqrt{2}}\phi^+[-m^\kappa_d(\bar{u}^\lambda_j C_{\lambda\kappa}(1 - \\
\gamma^5)d^\kappa_j) + m^\lambda_u(\bar{u}^\lambda_j C_{\lambda\kappa}(1+\gamma^5)d^\kappa_j] + \tfrac{ig}{2M\sqrt{2}}\phi^-[m^\lambda_d(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1 + \\
\gamma^5)u^\kappa_j) - m^\kappa_u(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1-\gamma^5)u^\kappa_j] - \tfrac{g}{2}\tfrac{m^\lambda_u}{M}H(\bar{u}^\lambda_j u^\lambda_j) - \\
\tfrac{g}{2}\tfrac{m^\lambda_d}{M}H(\bar{d}^\lambda_j d^\lambda_j) + \tfrac{ig}{2}\tfrac{m^\lambda_u}{M}\phi^0(\bar{u}^\lambda_j\gamma^5 u^\lambda_j) - \tfrac{ig}{2}\tfrac{m^\lambda_d}{M}\phi^0(\bar{d}^\lambda_j\gamma^5 d^\lambda_j)
\end{gathered}
$$

$\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \rightarrowtail$ NCG $\rightarrowtail$ Classical SM

[Chamseddine-Connes-Marcolli *ATMP* (Euclidean); J. Barrett *J. Math. Phys.* 2007 (Lorenzian)]

# On random NCG

- Sketchy: Spectral Action on fuzzy () geometries [C.P. arXiv:1912.13288] in terms multi-matrix models: noncommutative polynomials, e.g. for 2D

$$A^2, B^2, \cancel{AB}, \ldots, ABAB, A^2B^2, \cancel{ABABAB}, \ldots$$

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- More in detail: Functional Renormalisation Group Equation (FRGE) [C.P. arXiv:2007.10914] described in terms of a "noncommutative calculus" [Turnbull '28; Rota-Sagan-Stein '80; Voiculescu '00]

- we motivate the FRGE, before going to classical NCG (spectral formalism)

# Motivation for the FRGE

- First motivation (the talk won't rely on)

| | | |
|---|---|---|
| discrete surfaces | $\leftrightarrow$ | matrix integrals $\mathcal{Z}(\lambda)$<br>[B. Eynard, *Counting Surfaces* '16] |
| smooth surface | $\leftrightarrow$ | $\langle \text{area} \rangle\|_{\lambda=\lambda_c} \to \infty$<br>& infinitesimal mesh |

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| all topologies $\Uparrow$ | $\leftrightarrow$ | $\mathcal{Z}(\lambda) = \sum_g N^{2-2g} \mathcal{Z}_g(\lambda)$ $\uparrow \sim (\lambda_c - \lambda)^{(2-2g)/\theta}$ |
| double-scaling limit | | $N(\lambda_c - \lambda)^{1/\theta} = C$ |

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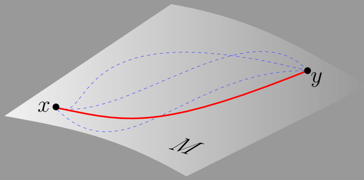
- Second motivation of the Renormalisation Group (not this talk):
  - gravity loops  influence matter, and vice versa 

# Connes′ geodesic distance formula ($M$ spin)




$\gamma : \mathbb{R} \to M$

$$\inf \int_\gamma \mathrm{ds} = d(x,y)$$

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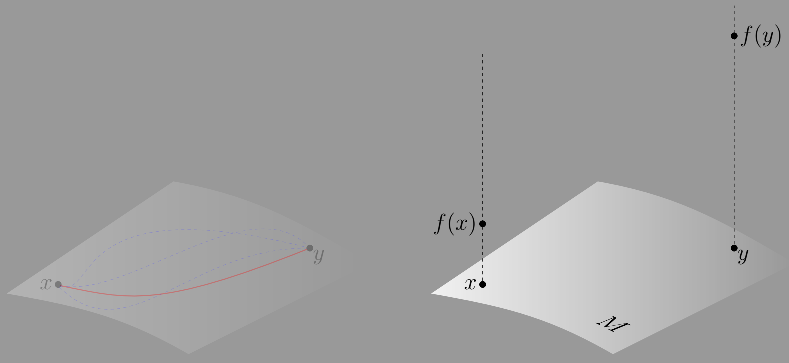


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$f : M \to \mathbb{R}$

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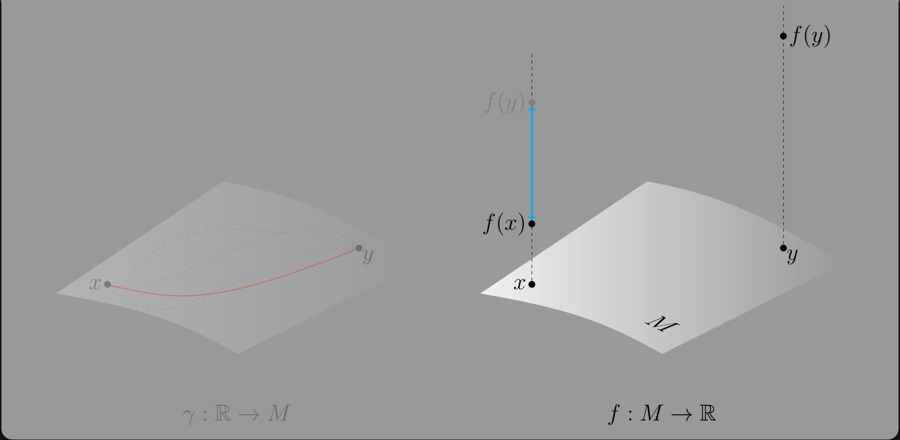


$$|f(x) - f(y)|$$

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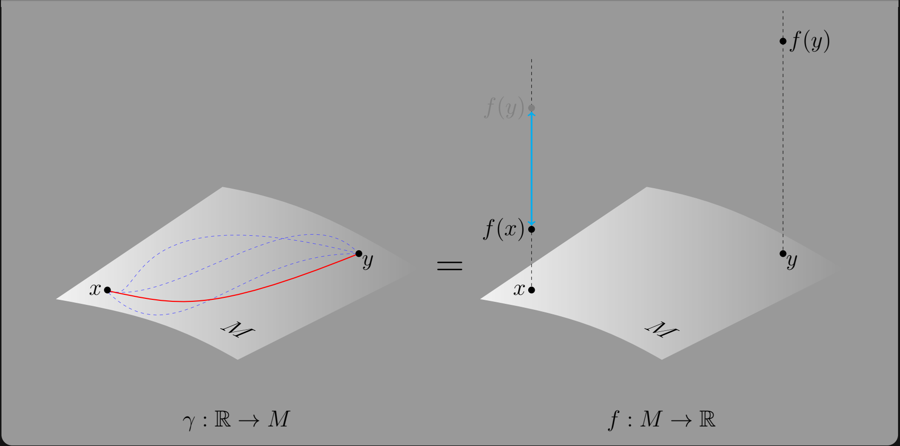


$$\sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| \, : \, \|Df - fD\| \leq 1 \}$$

- with $f$ as multiplication operator on $\mathcal{H} = L^2(M, S)$

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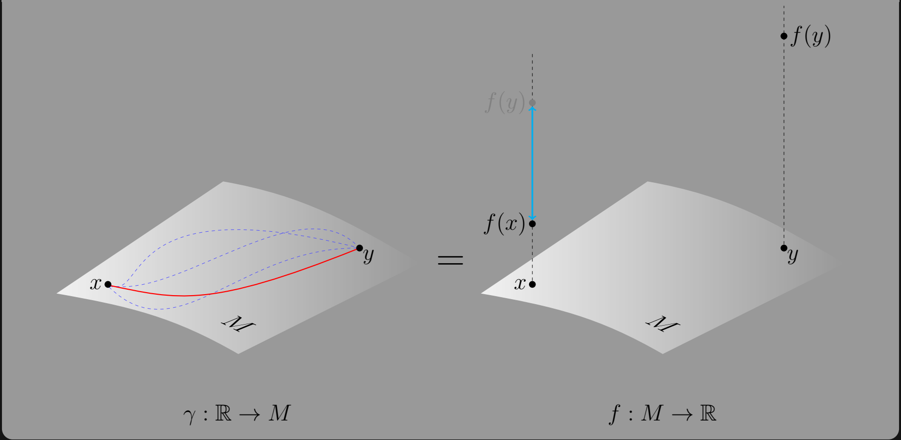


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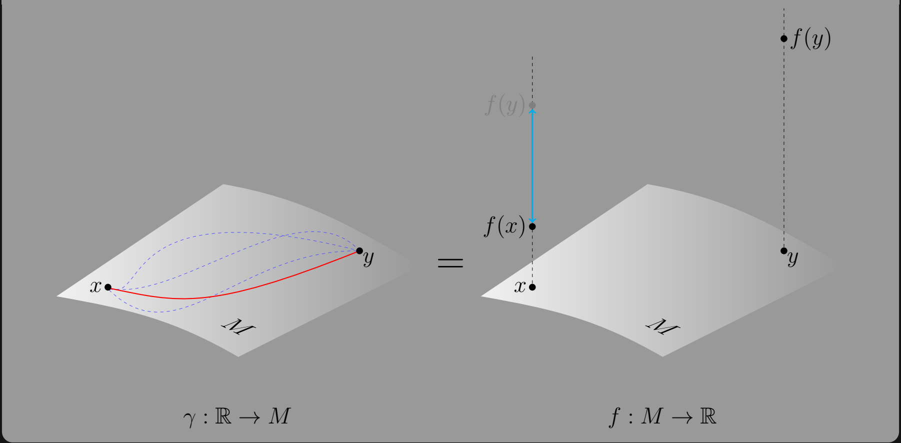


$$\inf \int_\gamma \mathrm{d}s = d(x,y) = \sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| : \|Df - fD\| \leq 1 \}$$

- with $f$ as multiplication operator on $\mathcal{H} = L^2(M,S)$
- $\big(C^\infty(M), L^2(M,S), D = \gamma^\mu[\partial_\mu + \omega_\mu]\big)$ is a spectral triple!

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- Spin geometries $M$ are commutative spectral triples:
  - $\mathcal{A} = C(M)$, a commutative $*$-algebra
  - $\mathcal{H} = L^2(M, S)$ is a representation of $\mathcal{A}$
  - $\not{D}_M : \mathcal{H} \to \mathcal{H}$, a self-adjoint Dirac

  the converse$^{+\text{axioms}}$ is also true
  [A. Connes *Commun. Math. Phys.* 1996; J. Várilly A. Rennie arXiv:0610418; A. Connes *JNCG* 2013]

# Fuzzy geometries [J. Barrett, *J. Math. Phys.* 2015]

A fuzzy geometry of signature $(p,q)$ (thus of dim. $p+q$ and KO-dim $q-p$) consists of

- $\mathcal{A} = M_N(\mathbb{C})$
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  … (axioms for $D$ omitted ▹▹) …
- Characterization of Dirac operators for even $p+q$
  - $(\gamma^\mu)^2 = +1, \quad \mu = 1,\ldots,p,$ $\qquad \gamma^\mu$ Hermitian,
  - $(\gamma^\mu)^2 = -1, \quad \mu = p+1,\ldots,p+q,$ $\qquad \gamma^\mu$ anti-Hermitian,
  - $\Gamma^I := \gamma^{\mu_1}\cdots\gamma^{\mu_r}$ for $\mu_i = 1,\ldots,p+q$, $I = (\mu_1,\ldots,\mu_r)$

  then [*Op. cit.*] in terms of Hermitian $H_I$ and anti-Hermitian $L_I$ in $M_N(\mathbb{C})$

$$D = \sum_I \Gamma^I_{\text{s.a.}} \otimes \{H_I, \cdot\} + \sum_I \Gamma^I_{\text{anti.}} \otimes [L_I, \cdot] \qquad |I| \text{ odd, } I \text{ monot. incr.}$$

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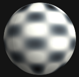

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  - $(\gamma^\mu)^2 = -1, \quad \mu = p+1, \dots, p+q,$ $\qquad$ $\gamma^\mu$ anti-Hermitian,
  - $\Gamma^I := \gamma^{\mu_1} \cdots \gamma^{\mu_r}$ for $\mu_i = 1, \dots, p+q$, $I = (\mu_1, \dots, \mu_r)$

  then [*Op. cit.*] in terms of Hermitian $H_I$ and anti-Hermitian $L_I$ in $M_N(\mathbb{C})$

  $$D = \sum_I \Gamma^I_{\text{s.a.}} \otimes \{H_I, \cdot\,\} + \sum_I \Gamma^I_{\text{anti.}} \otimes [L_I, \cdot\,] \qquad |I| \text{ odd, } I \text{ monot. incr.}$$

-
  $$D^{(1,3)} = \gamma^0 \otimes \{H_0, \cdot\,\} + \sum_c \gamma^c \otimes [L_c, \cdot\,] \qquad \text{‘matrix } \partial_\mu\text{’s’}$$

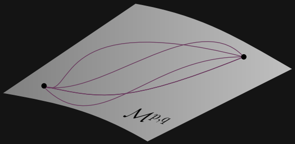
  $$+ \underbrace{\gamma^1\gamma^2\gamma^3}_{\equiv \Gamma^{\hat{0}}} \otimes \{H_{\hat{0}}, \cdot\,\} + \sum_a \Gamma^{\hat{a}} \otimes [L_{\hat{a}}, \cdot\,] \qquad \text{‘matrix spin connection’}$$

# Path integral picture

- ...or for fixed fixed $\xi = (\mathcal{A}, \mathcal{H})$, and $D$ of the form

$$D = \sum_I \Gamma^I_{\text{s.a.}} \otimes \{H_I, \bullet\} + \sum_I \Gamma^I_{\text{anti.}} \otimes [L_I, \cdot\,]$$

$$\mathcal{M}^{p,q} = \{D : \xi, \textit{adding } D \textit{ to } \xi \textit{ is a (p,q)-fuzzy geometry}\}$$




$$\mathcal{M}^{p,q} = (\mathbb{H}_N)^{\times p} \times \mathfrak{su}(N)^{\times q} \qquad (d=2)$$

$$\mathcal{M}^{p,q} = \begin{cases} \mathbb{H}_N^{\times 4} \times \mathfrak{su}(N)^{\times 4} & \text{(Riemannian)} \\ \mathbb{H}_N^{\times 2} \times \mathfrak{su}(N)^{\times 6} & \text{(Lorentzian)} \end{cases}$$

# Computing the spectral action [C.P. arXiv:1912.13288]

Aim: for polynomial $f$, systematically restate

$$\mathcal{Z}_{\text{NCG}}^{\text{fuzzy}} = \int_{\text{Dirac}} \mathrm{e}^{-\operatorname{Tr}_{\mathcal{H}} f(D)} \mathrm{d}D$$

for even dimension and $(p,q)$ signature $\eta = \operatorname{diag}(+1,\ldots,+1,-1,\ldots,-1)$ as multi-matrix model

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Strategy: Random fuzzy $\rightarrow$ random matrices [J. Barrett, L. Glaser, *J. Phys. A* 2016]. Since $\mathcal{H} = V \otimes M_N(\mathbb{C})$ we need:

- traces of products of gamma matrices
- traces of products of parametrizing matrices $L_I$ and $H_I$

# Chord Diagrams (CD) for $d=2$ geometries, $\eta=\mathrm{diag}(e_1,e_2)$ [C.P. '19]

$$\mathrm{Tr}_{\mathcal{H}}(D^6)=2N\sum_{\mu}\overbrace{\mathrm{Tr}_V(\gamma^{\mu_1}\cdots\gamma^{\mu_6})}\times$$

$$+(-1)^2\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_4}\eta^{\mu_3\mu_6}\quad+(-1)^1\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_6}\eta^{\mu_3\mu_4}\quad+(-1)^0\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_3}\eta^{\mu_4\mu_5}\quad+(-1)^1\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_4}\eta^{\mu_3\mu_5}\quad+(-1)^0\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_5}\eta^{\mu_3\mu_4}$$

$$+(-1)^1\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_6}\eta^{\mu_4\mu_5}\quad+(-1)^0\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}\eta^{\mu_5\mu_6}\quad+(-1)^3\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_5}\eta^{\mu_3\mu_6}\quad+(-1)^2\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_6}\eta^{\mu_3\mu_5}\quad+(-1)^1\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_3}\eta^{\mu_4\mu_6}$$

$$+(-1)^0\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}\eta^{\mu_5\mu_6}\quad+(-1)^1\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_5}\eta^{\mu_4\mu_6}\quad+(-1)^0\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_6}\eta^{\mu_4\mu_5}\quad+(-1)^1\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}\eta^{\mu_5\mu_6}\quad+(-1)^2\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_5}\eta^{\mu_4\mu_6}$$

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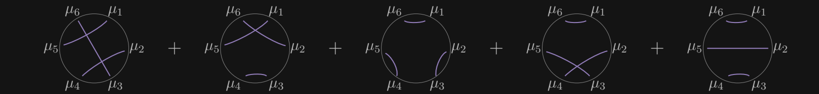
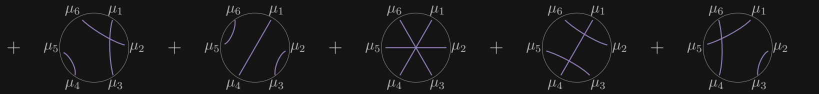


$$+(-1)^2\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_4}\eta^{\mu_3\mu_6} \quad +(-1)^1\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_6}\eta^{\mu_3\mu_4} \quad +(-1)^0\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_3}\eta^{\mu_4\mu_5} \quad +(-1)^1\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_4}\eta^{\mu_3\mu_5} \quad +(-1)^0\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_5}\eta^{\mu_3\mu_4}$$



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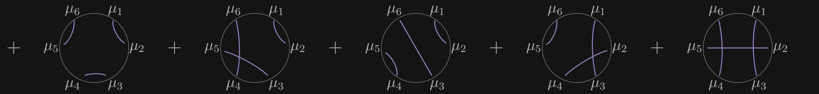

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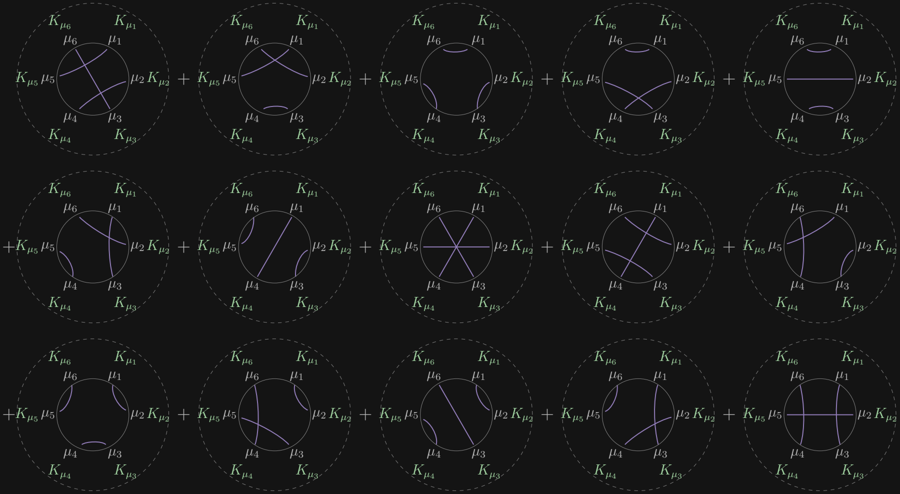

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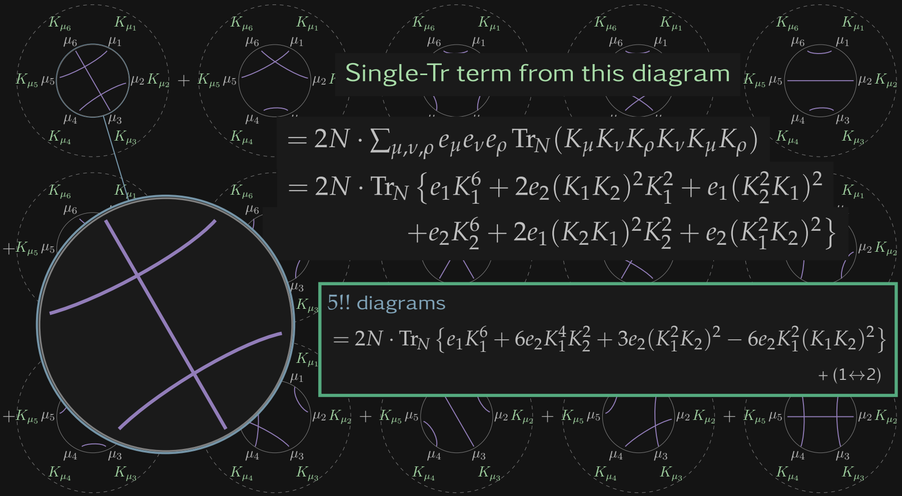


Single-Tr term from this diagram

$$= 2N \cdot \sum_{\mu,\nu,\rho} e_\mu e_\nu e_\rho \,\mathrm{Tr}_N(K_\mu K_\nu K_\rho K_\nu K_\mu K_\rho)$$
$$= 2N \cdot \mathrm{Tr}_N \{ e_1 K_1^6 + 2e_2 (K_1K_2)^2 K_1^2 + e_1 (K_2^2 K_1)^2$$
$$+ e_2 K_2^6 + 2e_1 (K_2K_1)^2 K_2^2 + e_2 (K_1^2 K_2)^2 \}$$

5!! diagrams

$$= 2N \cdot \mathrm{Tr}_N \{ e_1 K_1^6 + 6e_2 K_1^4 K_2^2 + 3e_2 (K_1^2 K_2)^2 - 6e_2 K_1^2 (K_1K_2)^2 \}$$
$$+ (1 \leftrightarrow 2)$$

# Recap

- random noncommutative geometry leads to random multi-matrix models [Barrett '15, Barrett-Glaser '16]

$$\mathcal{Z}_{\text{NCG}}^{\text{fuzzy}} = \int_{\mathcal{M}_N^{p,q}} \mathrm{e}^{-N\cdot \mathrm{Tr}_N P - \mathrm{Tr}_N Q^{(1)} \mathrm{Tr}_N Q^{(2)}} \, \mathrm{d}\mu$$

- systematically computable in terms of cyclically self-adjoint NC polynomials $P$ and 'bipolynomials' in $Q^{(1)} \otimes Q^{(2)}$, in $2^{p+q-1}$ matrices [C.P. '19]

- chord diagram organization; same CD structure also appeared in [Sati-Schreiber arXiv:1912.10425 & ncatlab article for fuzzy sphere]

# Free algebra and differential operators

- Let $\mathbb{C}_{\langle n \rangle} = \mathbb{C}\langle X_1, \ldots X_n \rangle$ be generated by $X_1, \ldots X_n \in M_N^{\pm}(\mathbb{C})$,

  $\mathbb{C}_{\langle n \rangle} = \{$words or noncommutative (NC) polynomials in $X_i^* = \pm X_i\}$

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- NC-derivative [Turnbull '28; Rota-Sagan-Stein '80; Voiculescu '00]

$$\partial^{X_j} : \mathbb{C}_{\langle n\rangle} \to \mathbb{C}_{\langle n\rangle} \otimes \mathbb{C}_{\langle n\rangle}$$

$$X_{\ell_1}\cdots X_{\ell_k} \mapsto \sum_{i=1}^{k} \delta^{j}_{\ell_i} \cdot X_{\ell_1}\cdots X_{\ell_{i-1}} \otimes X_{\ell_{i+1}}\cdots X_{\ell_k}$$

- Example: $\partial^{\mathsf{E}}(\mathsf{FREENESS}) = \mathsf{FR} \otimes \mathsf{ENESS} + \mathsf{FRE} \otimes \mathsf{NESS} + \mathsf{FREEN} \otimes \mathsf{SS}$, on $\mathbb{C}\langle$A,B,...,Z$\rangle$

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- Example: $\partial^{\mathsf{E}}(\mathsf{FREENESS}) = \mathsf{FR} \otimes \mathsf{ENESS} + \mathsf{FRE} \otimes \mathsf{NESS} + \mathsf{FREEN} \otimes \mathsf{SS}$, on $\mathbb{C}\langle$A,B,...,Z$\rangle$, but

$$\partial^{\mathsf{S}}(\mathsf{FREENESS}) = \mathsf{FREENE} \otimes \mathsf{S} + \mathsf{FREENES} \otimes 1$$

- $\partial^X_{ab} = \delta/\delta(X_{ba})$, but with $(U \otimes W)_{ab;cd} = U_{ab}W_{cd}$ for $P \in \mathbb{C}_{\langle n\rangle}$,

$$[(\partial^X P)(X_1, \ldots, X_n)]_{ab;cd} = \partial^X_{cb}[P(X_1, \ldots, X_n)]_{ad}$$

  Important transposition $\tau(ab;cd) = (cb;ad)$ for $\tau = (13) \in \mathrm{Sym}(4)$

- Cyclic derivative: $\mathscr{D}^X = \tilde{m} \circ \partial^X : \mathbb{C}_{\langle n \rangle} \to \mathbb{C}_{\langle n \rangle}$ and $\tilde{m}(U \otimes W) = WU$

$$\mathscr{D}^{\mathsf{E}}(\mathsf{FREENESS}) = \tilde{m}\big(\mathsf{FR} \otimes \mathsf{ENESS} + \mathsf{FRE} \otimes \mathsf{NESS} + \mathsf{FREEN} \otimes \mathsf{SS}\big)$$

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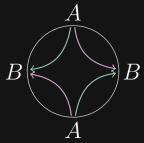
- $\partial^{X_j} \operatorname{Tr} P = \mathscr{D}^{X_j} P$. For instance,

$$\begin{aligned}(\partial^B \circ \partial^A) \operatorname{Tr}(ABAB) &= \partial^B \mathscr{D}^A (ABAB) \\ &= 2\partial^B (BAB) = 2[1 \otimes AB + BA \otimes 1] \sim\end{aligned}$$



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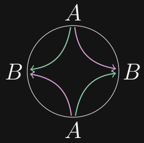
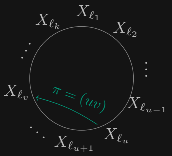



- (Optional:) Double derivatives on traces:

$$(\partial^{X_i} \circ \partial^{X_j}) \operatorname{Tr} Q = \sum_{\pi = (uv)} \delta^j_{\ell_u} \delta^i_{\ell_v} \pi_1(Q) \otimes \pi_2(Q),$$




$\pi_2(Q) X_{\ell_u} \pi_1(Q) X_{\ell_v}$ matches $Q$

# NC-Hessian and NC-Laplacian

[C.P. '20]

- The *noncommutative Hessian* (NC Hessian) is the operator

$$\mathrm{Hess} : \underbrace{\mathrm{im}\,\mathrm{Tr}}_{\text{“cyclic words”} \subset \mathbb{C}_{\langle n \rangle}} \to M_n(\mathbb{C}) \otimes \mathbb{C}_{\langle n \rangle}^{\otimes 2}$$

whose $(ij)$-entry $(1 \leq i,j \leq n)$ is

$$(\mathrm{Hess}\,\mathrm{Tr}_N\, P)_{ij} := (\partial^{X_i} \circ \partial^{X_j}\, \mathrm{Tr}_N\, P) \in \mathbb{C}_{\langle n \rangle}^{\otimes 2}\,.$$

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- Hess is not symmetric. For instance $(n = 2)$

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$$= 2\begin{pmatrix} B \otimes B & AB \otimes 1 + 1 \otimes BA \\ BA \otimes 1 + 1 \otimes AB & A \otimes A \end{pmatrix}$$

- Optional: The NC-Laplacian $\nabla$ of a “cyclic word” is the $M_n(\mathbb{C})$-trace of the NC-Hessian

$$\nabla^2\{\mathrm{Tr}(ABAB)\} = B \otimes B + A \otimes A.$$

# Twisted products $\otimes_\tau$

- The RG-flow (later) generates multi-traces... it thus twists

$$\nabla^2(\operatorname{Tr} P \cdot \operatorname{Tr} Q) = (\nabla^2 \operatorname{Tr} P) \cdot \operatorname{Tr} Q + (\nabla^2 \operatorname{Tr} Q) \cdot \operatorname{Tr} P + \sum_j \{ \mathscr{D}^{X_j} P \otimes_\tau \mathscr{D}^{X_j} Q + \mathscr{D}^{X_j} Q \otimes_\tau \mathscr{D}^{X_j} P \},$$

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- With the twisted product by $\tau = (13) \in \operatorname{Sym}(4)$ of the four indices,

$$(U \otimes_\tau W)_{a_1 a_2; a_3 a_4} := (U \otimes W)_{a_{\tau(1)} a_{\tau(2)}; a_{\tau(3)} a_{\tau(4)}}$$
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- We consider $\mathcal{A}_n = (\mathbb{C}_{\langle n \rangle} \otimes \mathbb{C}_{\langle n \rangle}) \oplus (\mathbb{C}_{\langle n \rangle} \otimes_\tau \mathbb{C}_{\langle n \rangle})$ with product

$$[(U \otimes_\vartheta W) \star (P \otimes_\alpha Q)]_{ab;cd} := (U \otimes_\vartheta W)_{ab;xy} (P \otimes_\alpha Q)_{yx;cd} ,$$

where $\alpha, \vartheta$ stand for either $\tau$ or an empty label.

# Algebraic structure (dictated by the proof of the FRGE)

- $\mathcal{A}_n = \mathbb{C}_{\langle n \rangle}^{\otimes 2} \oplus \mathbb{C}_{\langle n \rangle}^{\otimes_\tau 2}$ is an associative algebra satisfying

$$
\begin{aligned}
(U \otimes_\tau W) \star (P \otimes_\tau Q) &= PU \otimes_\tau WQ\,, \\
(U \otimes W) \star (P \otimes_\tau Q) &= U \otimes PWQ\,, \\
(U \otimes_\tau W) \star (P \otimes Q) &= WPU \otimes Q\,, \\
(U \otimes W) \star (P \otimes Q) &= \underbrace{\mathrm{Tr}(WP)}_{\varphi:\mathcal{A}\longrightarrow\mathbb{C}\text{ (a state)}}\, U \otimes Q
\end{aligned}
$$

for $U, W, P, Q \in \mathbb{C}_{\langle n \rangle}$

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for $U, W, P, Q \in \mathbb{C}_{\langle n \rangle}$

- $\mathcal{A}_n$ is also unital, $\mathbf{1} = 1 \otimes_\tau 1$

- One is particularly interested in $M_n(\mathcal{A}_n) \supset$ NC Hessians.

$$\mathcal{Q} = (\mathcal{Q}_{ij|ab;cd})_{\substack{i,j=1,\dots,n \\ a,b,c,d=1,\dots,N}} \in M_n(\mathbb{C}) \otimes \left[\mathbb{C}_{\langle n\rangle}^{\otimes 2} \oplus \mathbb{C}_{\langle n\rangle}^{\otimes_\tau 2}\right] = M_n(\mathcal{A}_n)$$

with "supertrace" (no relation to SUSY)

$$\mathrm{STr} = \mathrm{Tr}_n \otimes \mathrm{Tr}_{\mathcal{A}_n} : M_n(\mathcal{A}_n) \to \mathbb{C}$$

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- Twisted products are thus merged. Example:

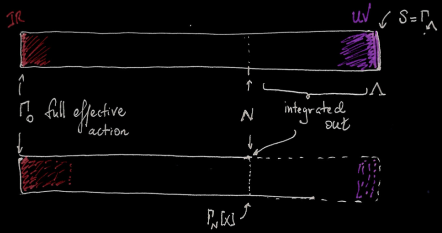
$$\mathrm{STr}\begin{pmatrix} 1 \otimes A^4 & * \\ * & B^2 \otimes_\tau B^2 \end{pmatrix} = \mathrm{Tr}_{\mathcal{A}_2}(1 \otimes A^4 + B^2 \otimes_\tau B^2)$$

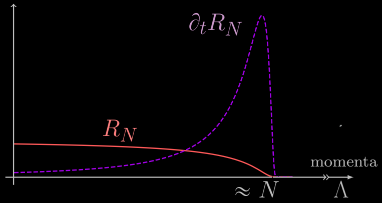
$$= N\,\mathrm{Tr}(A^4) + \mathrm{Tr}(B^4)$$

# Sketching the Functional Renormalisation Group (scalar $\varphi$)

- The splitting $\mathcal{Z} = \int \mathcal{D}[\varphi_L] \underbrace{\mathcal{D}[\varphi_H] \mathrm{e}^{-S[\varphi_H+\varphi_L]}}_{\exp(-S_{\text{eff}}[\varphi_L])}$ in high/low dof's

  implemented smoothly by an IR-regulator $\Delta S_N[\varphi] = \frac{1}{2} R_N \varphi^2$



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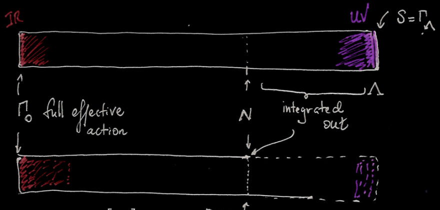


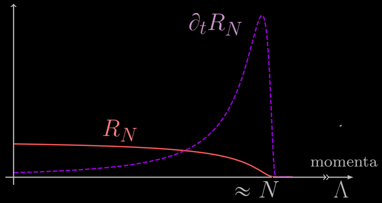


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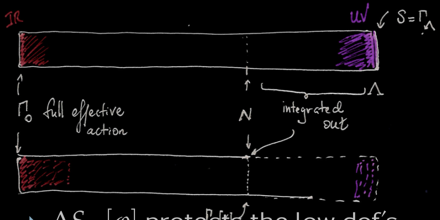


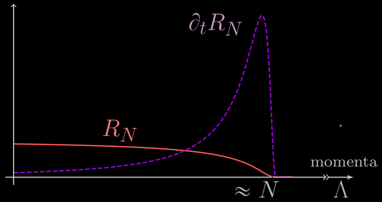


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$$\mathrm{e}^{\mathcal{W}_N[J]} = \int \mathrm{e}^{-S[\varphi] - \Delta S_N[\varphi] + (J \cdot \varphi)} \mathcal{D}[\varphi]$$

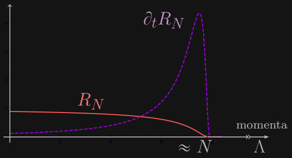
- For the 'classical' field $\langle \varphi \rangle_J = X$, the interpolating eff. action

$$\Gamma_N[X] := \sup_J \{ J \cdot X - \mathcal{W}_N[J] \} - (\Delta S_N)[X] \, .$$

[C.P. '20]

Influenced by [Brézin–Zinn-Justin, *Phys.Lett. B* '92] [Eichhorn-Koslowski, PRD, '13]

- Bare action $S$ for $n$ matrices $\varphi^i$ of size $\Lambda \times \Lambda$ is IR-regulated by a mass term $R_N = r_N \cdot 1 \otimes_\tau 1$:




$$\Delta S_N[\varphi] = \frac{1}{2}\sum_{i=1}^{n} e_i \,\mathrm{Tr}_\Lambda^{\otimes 2}((\varphi^i \otimes_\tau \varphi^i) \star R_N) \sim \frac{r_N}{2}\varphi^2$$

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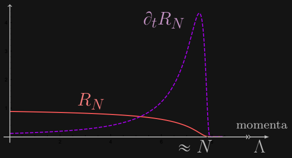
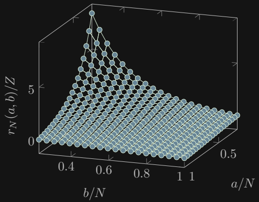



- IR-regulated partition function

$$\mathcal{Z}_N[J] = \mathrm{e}^{\mathcal{W}_N[J]} = \int_{\mathcal{M}_N^{p,q}} \mathrm{e}^{-S[\varphi]-\Delta S_N[\varphi]+\mathrm{Tr}(J\cdot\varphi)} \mathrm{d}\mu_\Lambda(\varphi)$$




$$r_N(a,b) = Z\cdot\left[\frac{N^2}{a^2+b^2} - 1\right] \cdot \Theta_{\mathbb{D}_N}(a,b)$$

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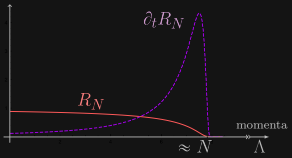


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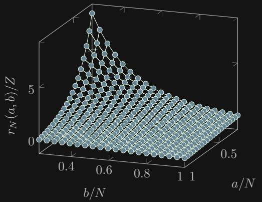


- Interpolating effective action

$$\Gamma_N[X] := \sup_J \Big( \operatorname{Tr}(J \cdot X) - \mathcal{W}_N[J] \Big) - (\Delta S_N)[X]$$

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## Functional Renormalisation Group Equation [Wetterich, Morris]

$$\partial_t \Gamma_N[X] = \frac{1}{2}\mathrm{STr}\left(\frac{\partial_t R_N}{\mathrm{Hess}_\sigma \Gamma_N[X] + R_N}\right) \quad t = \log N$$

where

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$$\partial_t \Gamma_N[X] = \frac{1}{2}(\mathrm{Tr}_n \otimes \mathrm{Tr}_N^{\otimes 2})\{-\tilde{h}_1(N)F[X] + \tilde{h}_2(N)\big(F[X]\big)^{\star 2} + \ldots\}$$

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- dependence on $R_N$ through $h_k = \lim_{N\to\infty} \sum_{a,b,c,d=1}^{N} \frac{(\partial_t R_N)_{ab;cd}}{N^2 P_{ab;cd}^{(k+1)}}$ with

$$h_1 = \frac{\pi}{24}(6 - 5\eta),\ h_2 = \frac{\pi}{48}(8 - 7\eta),\ h_3 = \frac{\pi}{80}(10 - 9\eta),\ \eta = -\partial_t \log Z$$

# Optional: The one-matrix model

Choosing a truncation for the effective action

$$\Gamma_N[X] = \mathrm{Tr} \otimes \mathrm{Tr} \Big\{ \frac{Z}{2N} 1_N \otimes X^2 + \frac{\bar{g}_4}{4N} 1_N \otimes X^4 + \frac{\bar{g}_6}{6N} 1_N \otimes X^6 + \frac{\bar{g}_{2|2}}{8} X^2 \otimes X^2 + \frac{\bar{g}_{2|4}}{8} X^2 \otimes X^4 \Big\}$$

one needs

$$\frac{1}{2N} \partial^2 \mathrm{Tr}_{\mathcal{A}_{1,N}} \left(1_N \otimes X^2\right) = 1_N \otimes 1_N$$

$$\frac{1}{4N} \partial^2 \mathrm{Tr}_{\mathcal{A}_{1,N}} \left(1_N \otimes X^4\right) = X \otimes X + 1_N \otimes X^2 + X^2 \otimes 1_N$$

$$\frac{1}{8} \partial^2 \mathrm{Tr}_{\mathcal{A}_{1,N}} \left(X^2 \otimes X^2\right) = X \otimes_\tau X + 1_N \otimes 1_N \mathrm{Tr}_N \left(\frac{X^2}{2}\right)$$

$$\frac{1}{6N} \partial^2 \mathrm{Tr}_{\mathcal{A}_{1,N}} \left(1_N \otimes X^6\right) = X \otimes X^3 + 1_N \otimes X^4 + X^2 \otimes X^2 + X^3 \otimes X + X^4 \otimes 1_N$$

# The quantum fluctuations & $\beta_I = \partial_t g_I$-functions

$$
\begin{aligned}
\partial_t \Gamma_N[X] = &-\frac{1}{2}\frac{\tilde{h}_1}{N^2}\Big\{(N^2+2)\bar{g}_{2|2}+4N\bar{g}_4\Big\}\,\mathrm{Tr}_N\left(\frac{X^2}{2}\right)\\
&+\Big\{-\frac{\tilde{h}_1}{N^2}\Big((4+\frac{N^2}{2})\bar{g}_{2|4}+4N\bar{g}_6\Big)\\
&\quad+\frac{\tilde{h}_2}{N^2}(12\bar{g}_{2|2}\bar{g}_4+4N\bar{g}_4^2)\Big\}\,\mathrm{Tr}_N\left(\frac{X^4}{4}\right)\\
&+\Big\{\frac{\tilde{h}_2}{N^2}((8+N^2)\bar{g}_{2|2}^2+8N\bar{g}_{2|2}\bar{g}_4+12\bar{g}_4^2)\\
&\quad-\frac{\tilde{h}_1}{N^2}(4N\bar{g}_{2|4}+4\bar{g}_6)\Big\}\,\frac{1}{8}\,\mathrm{Tr}_N^2(X^2)\\
&+\Big\{\frac{\tilde{h}_2}{N^2}(36\bar{g}_{2|4}\bar{g}_4+30\bar{g}_{2|2}\bar{g}_6+12N\bar{g}_4\bar{g}_6)\\
&\quad-\frac{\tilde{h}_3}{N^2}(81\bar{g}_{2|2}\bar{g}_4^2+6N\bar{g}_4^3)\Big\}\,\mathrm{Tr}_N\left(\frac{X^6}{6}\right)\\
&+\Big\{\frac{\tilde{h}_2}{N^2}(\bar{g}_{2|4}((38+N^2)\bar{g}_{2|2}+12N\bar{g}_4)+8N\bar{g}_{2|2}\bar{g}_6+48\bar{g}_4\bar{g}_6)\\
&\quad-\frac{\tilde{h}_3}{N^2}(72\bar{g}_{2|2}^2\bar{g}_4+12N\bar{g}_{2|2}\bar{g}_4^2+48\bar{g}_4^3)\Big\}\,\mathrm{Tr}_N\left(\frac{X^2}{2}\right)\mathrm{Tr}_N\left(\frac{X^4}{4}\right),
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& \quad - \frac{\tilde{h}_1}{N^2}(4N\bar{g}_{2|4} + 4\bar{g}_6)\Big\}\,\frac{1}{8}\,\mathrm{Tr}_N^2(X^2) \\
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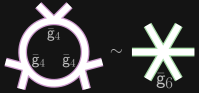
\end{aligned}
$$

$\bar{g}_4^3 \sim$ 

Extracting the coeff's in the large $N$,

$$\eta = h_1\left(\frac{1}{2}g_{2|2} + 2g_4\right),$$

$$\beta_4 = (1+2\eta)g_4 + 4h_2 g_4^2 - h_1\left(4g_6 + \frac{g_{2|4}}{2}\right),$$

$$\beta_{2|2} = (2+2\eta)g_{2|2} - 4h_1(g_{2|4} + g_6) + h_2(g_{2|2}^2 + 8g_{2|2}g_4 + 12g_4^2),$$

$$\beta_6 = (2+3\eta)g_6 + 12g_4 g_6 h_2 - 6g_4^3 h_3,$$

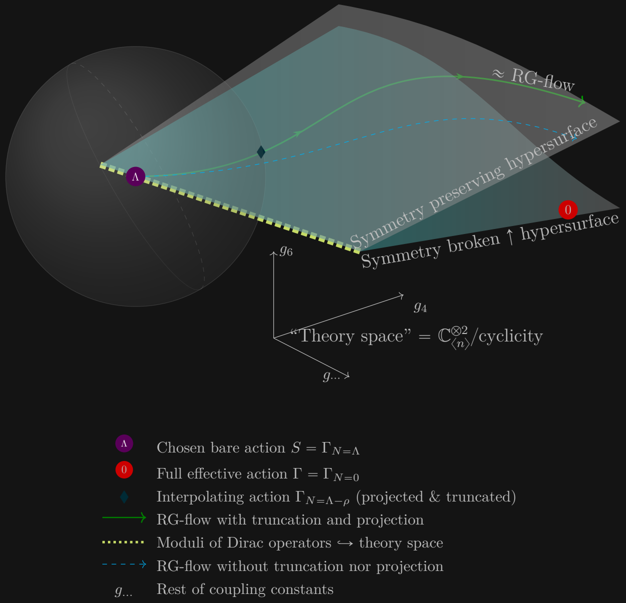
$$\beta_{2|4} = (3+3\eta)g_{2|4} + h_2(g_{2|2}g_{2|4} + 8g_{2|2}g_6 + 12g_{2|4}g_4 + 48g_4g_6) - h_3\left(12g_{2|2}g_4^2 + 48g_4^3\right).$$

with solution

$$\eta^\diamond = -0.2494, \qquad g_4^\diamond = -0.08791, \qquad \left(g_4^{\text{exact}} = -\frac{1}{12} = -0.083\bar{3}\right)$$

$$g_6^\diamond = -0.003386, \qquad g_{2|4}^\diamond = -0.02423, \qquad g_{2|2}^\diamond = -0.17415.$$

# The FRGE for multi-matrix models motivated by random NCG



# The two-matrix models from random NCG

## Flowing operators for $\operatorname{Tr} f(D)$ with $f(x) = \frac{1}{4}\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6}\right)$

| Degree | Operators | Coupling constant | Scalings |
|---|---|---|---|
| Quadratic | $1_N \otimes (AA)$ | $\frac{1}{2} Z_a e_a$ | $*$ |
| | $1_N \otimes (BB)$ | $\frac{1}{2} Z_b e_b$ | $*$ |
| | $A \otimes A$ | $\frac{1}{2} \bar{d}_{1\|1}$ | $1/N$ |
| | $B \otimes B$ | $\frac{1}{2} \bar{d}_{01\|01}$ | $1/N$ |

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| | $A \otimes A$ | $\frac{1}{2} \bar{d}_{1\|1}$ | $1/N$ |
| | $B \otimes B$ | $\frac{1}{2} \bar{d}_{01\|01}$ | $1/N$ |
| Quartic | $1_N \otimes (AAAA)$ | $\frac{1}{4} \bar{a}_4$ | $1/N$ |
| | $1_N \otimes (BBBB)$ | $\frac{1}{4} \bar{b}_4$ | $1/N$ |
| | $1_N \otimes (AABB)$ | $\bar{c}_{22} e_a e_b$ | $1/N$ |
| | $1_N \otimes (ABAB)$ | $-\frac{1}{2} \bar{c}_{1111} e_a e_b$ | $1/N$ |
| | $(AB) \otimes (AB)$ | $\bar{d}_{11\|11}$ | $1/N^2$ |
| | $(AA) \otimes (BB)$ | $2\bar{d}_{2\|02} e_a e_b$ | $1/N^2$ |
| | $A \otimes (AAA)$ | $\bar{d}_{1\|3} e_a$ | $1/N^2$ |
| | $A \otimes (ABB)$ | $\bar{d}_{1\|12} e_b$ | $1/N^2$ |
| | $B \otimes (AAB)$ | $\bar{d}_{01\|21} e_a$ | $1/N^2$ |
| | $B \otimes (BBB)$ | $\bar{d}_{01\|03} e_b$ | $1/N^2$ |
| | $(AA) \otimes (AA)$ | $3\bar{d}_{2\|2}$ | $1/N^2$ |
| | $(BB) \otimes (BB)$ | $3\bar{d}_{02\|02}$ | $1/N^2$ |

| Sextic Operators | NCG coefficient value | Coupling constant | Scalings |
|---|---|---|---|
| $1_N \otimes (AAAAAA)$ | $e_\mathsf{a}$ | $\bar{\mathsf{a}}_6$ | $1/N^2$ |
| $1_N \otimes (AAAABB)$ | $6e_\mathsf{b}$ | $\bar{\mathsf{c}}_{42}$ | $1/N^2$ |
| $1_N \otimes (AAABAB)$ | $-6e_\mathsf{b}$ | $\bar{\mathsf{c}}_{3111}$ | $1/N^2$ |
| $1_N \otimes (AABAAB)$ | $3e_\mathsf{b}$ | $\bar{\mathsf{c}}_{2121}$ | $1/N^2$ |
| $1_N \otimes (BBBBBB)$ | $e_\mathsf{b}$ | $\bar{\mathsf{b}}_6$ | $1/N^2$ |
| $1_N \otimes (AABBBB)$ | $6e_\mathsf{a}$ | $\bar{\mathsf{c}}_{24}$ | $1/N^2$ |
| $1_N \otimes (ABBBAB)$ | $-6e_\mathsf{a}$ | $\bar{\mathsf{c}}_{1311}$ | $1/N^2$ |
| $1_N \otimes (ABBABB)$ | $3e_\mathsf{a}$ | $\bar{\mathsf{c}}_{1212}$ | $1/N^2$ |
| $A \otimes (AAAAA)$ | $2$ | $\bar{\mathsf{d}}_{1\vert 5}$ | $1/N^3$ |
| $A \otimes (ABBBB)$ | $2$ | $\bar{\mathsf{d}}_{1\vert 14}$ | $1/N^3$ |
| $A \otimes (AAABB)$ | $6e_\mathsf{a}e_\mathsf{b}$ | $\bar{\mathsf{d}}_{1\vert 32}$ | $1/N^3$ |
| $A \otimes (AABAB)$ | $-2e_\mathsf{a}e_\mathsf{b}$ | $\bar{\mathsf{d}}_{1\vert 2111}$ | $1/N^3$ |
| $B \otimes (AAAAB)$ | $2$ | $\bar{\mathsf{d}}_{01\vert 41}$ | $1/N^3$ |
| $B \otimes (AABBB)$ | $6e_\mathsf{a}e_\mathsf{b}$ | $\bar{\mathsf{d}}_{1\vert 23}$ | $1/N^3$ |
| $B \otimes (ABBAB)$ | $-2e_\mathsf{a}e_\mathsf{b}$ | $\bar{\mathsf{d}}_{01\vert 1211}$ | $1/N^3$ |
| $B \otimes (BBBBB)$ | $2$ | $\bar{\mathsf{d}}_{01\vert 05}$ | $1/N^3$ |
| $(AB) \otimes (AAAB)$ | $8e_\mathsf{a}$ | $\bar{\mathsf{d}}_{11\vert 31}$ | $1/N^3$ |
| $(AB) \otimes (ABBB)$ | $8e_\mathsf{b}$ | $\bar{\mathsf{d}}_{11\vert 13}$ | $1/N^3$ |
| $(AA) \otimes (AABB)$ | $8e_\mathsf{b}$ | $\bar{\mathsf{d}}_{2\vert 22}$ | $1/N^3$ |
| $(AA) \otimes (ABAB)$ | $-2e_\mathsf{b}$ | $\bar{\mathsf{d}}_{2\vert 1111}$ | $1/N^3$ |
| $(AA) \otimes (AAAA)$ | $5e_\mathsf{a}$ | $\bar{\mathsf{d}}_{2\vert 4}$ | $1/N^3$ |
| $(AA) \otimes (BBBB)$ | $e_\mathsf{a}$ | $\bar{\mathsf{d}}_{2\vert 04}$ | $1/N^3$ |
| $(BB) \otimes (AABB)$ | $8e_\mathsf{a}$ | $\bar{\mathsf{d}}_{02\vert 22}$ | $1/N^3$ |
| $(BB) \otimes (ABAB)$ | $-2e_\mathsf{a}$ | $\bar{\mathsf{d}}_{02\vert 1111}$ | $1/N^3$ |
| $(BB) \otimes (BBBB)$ | $5e_\mathsf{b}$ | $\bar{\mathsf{d}}_{02\vert 04}$ | $1/N^3$ |
| $(BB) \otimes (AAAA)$ | $e_\mathsf{b}$ | $\bar{\mathsf{d}}_{02\vert 4}$ | $1/N^3$ |
| $(AAA) \otimes (AAA)$ | $\frac{10}{3}$ | $\bar{\mathsf{d}}_{3\vert 3}$ | $1/N^3$ |
| $(ABB) \otimes (AAA)$ | $4e_\mathsf{a}e_\mathsf{b}$ | $\bar{\mathsf{d}}_{12\vert 3}$ | $1/N^3$ |
| $(AAB) \otimes (AAB)$ | $6$ | $\bar{\mathsf{d}}_{21\vert 21}$ | $1/N^3$ |
| $(BBB) \otimes (BBB)$ | $\frac{10}{3}$ | $\bar{\mathsf{d}}_{03\vert 03}$ | $1/N^3$ |
| $(AAB) \otimes (BBB)$ | $4e_\mathsf{a}e_\mathsf{b}$ | $\bar{\mathsf{d}}_{21\vert 03}$ | $1/N^3$ |
| $(ABB) \otimes (ABB)$ | $6$ | $\bar{\mathsf{d}}_{12\vert 12}$ | $1/N^3$ |

| Geometry | Signature | KO-dim. | # Operators in the RG-flow | # Operators with duality |
|---|---|---|---|---|
| 'Double time' | $(+,+)$ | 6 | 48 | 26 |
| 2D Lorentzian | $(+,-)$ | 0 | 41 | — |
| Riemannian | $(-,-)$ | 2 | 34 | 19 |

Forbidden: $A \cdot B$ (Ising 2-matrix model), $A \cdot A \cdot A \cdot B, \ldots,$

$$A \cdot A \cdot A \cdot A \cdot A \cdot B, \quad A \cdot A \cdot A \cdot B \cdot B \cdot B, \quad A \cdot A \cdot B \cdot A \cdot B \cdot B,$$
$$A \cdot A \cdot B \cdot B \cdot A \cdot B, \quad A \cdot B \cdot A \cdot B \cdot A \cdot B, \quad A \cdot B \cdot B \cdot B \cdot B \cdot B.$$

or inserting $\otimes$ anywhere inside.

# The $\beta$-functions for the 2-geometries

For the 2-dimensional fuzzy geometry with signature $\mathrm{diag}(e_a, e_b)$,

$$2h_1(a_4 + c_{22} + 2d_{2|02} + 6d_{2|2}) = \eta_a$$
$$2h_1(b_4 + c_{22} + 6d_{02|02} + 2d_{2|02}) = \eta_b$$
$$-h_1[e_a(a_4 - c_{1111}) + 2d_{1|12} + 6d_{1|3}] + d_{1|1}(\eta + 1) = \beta(d_{1|1})$$
$$-h_1[e_b(b_4 - c_{1111}) + 6d_{01|03} + 2d_{01|21}] + d_{01|01}(\eta + 1) = \beta(d_{01|01})$$

# The $\beta$-functions for the 2-geometries

For the 2-dimensional fuzzy geometry with signature $\mathrm{diag}(e_a, e_b)$,

$$2h_1(a_4 + c_{22} + 2d_{2|02} + 6d_{2|2}) = \eta_a$$
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$$-h_1[e_b(b_4 - c_{1111}) + 6d_{01|03} + 2d_{01|21}] + d_{01|01}(\eta + 1) = \beta(d_{01|01})$$

The next block encompasses the connected quartic couplings:

$$\begin{aligned} &h_2(4a_4^2 + 4c_{22}^2) + a_4(2\eta + 1) \\ &-h_1(24a_6e_a + 4c_{42}e_b + 4d_{02|4}e_b + 4d_{2|4}e_a) = \beta(a_4) \end{aligned}$$

$$\begin{aligned} &h_2(4b_4^2 + 4c_{22}^2) + b_4(2\eta + 1) \\ &-h_1(24b_6e_b + 4c_{24}e_a + 4d_{02|04}e_b + 4d_{2|04}e_a) = \beta(b_4) \end{aligned}$$

$$\begin{aligned} &-h_1(2e_ac_{1212} + e_b2c_{2121} + 3e_ac_{24} + 3e_bc_{42} + e_ad_{02|22} + e_bd_{2|22}) \\ &+h_2(2a_4c_{22} + 2b_4c_{22} + 2e_ae_bc_{1111}^2 + 2e_ae_bc_{22}^2) + c_{22}(2\eta + 1) = \beta(c_{22}) \end{aligned}$$

$$\begin{aligned} &8e_ae_bc_{1111}c_{22}h_2 + c_{1111}(2\eta + 1) \\ &+h_1(4e_ac_{1311} + 4e_bc_{3111} + 2e_ad_{02|1111} + 2e_bd_{2|1111}) = \beta(c_{1111}) \end{aligned}$$

(+ others fitting in 5 pages)

$$\partial_t \Gamma_N[X] = \frac{1}{2}(\mathrm{Tr}_n \otimes \mathrm{Tr}_N^{\otimes 2})\{-h_1(N)F[X] + h_2(N)\big(F[X]\big)^{\star 2} + \dots\}$$

| Operator | Its $\mathrm{Hess}_\sigma$ |
|---|---|
| $\mathrm{Tr}(A^4)$ | $\begin{pmatrix} 4e_a(1\otimes A^2 + A^2\otimes 1 + A\otimes A) & 0 \\ 0 & 0 \end{pmatrix}$ |
| $\mathrm{Tr}^2 B$ | $\begin{pmatrix} 0 & 0 \\ 0 & 2e_b 1\otimes_\tau 1 \end{pmatrix}$ |
| $\mathrm{Tr}(ABAB)$ | $\begin{pmatrix} 2e_a B\otimes B & 2(1\otimes BA + AB\otimes 1) \\ 2(1\otimes AB + BA\otimes 1) & 2e_b A\otimes A \end{pmatrix}$ |
| $\mathrm{Tr}(A)\,\mathrm{Tr}(A^3)$ | $\begin{pmatrix} 3e_a[\mathrm{Tr}(A)(A\otimes 1 + 1\otimes A) \\ +1\otimes_\tau A^2 + A^2\otimes_\tau 1] & 0 \\ 0 & 0 \end{pmatrix}$ |

Some Hessians of second and fourth order operators

# Quantum fluctuations

- from $h_1 \, \mathrm{Tr}_{M_2(\mathcal{A}_2)}(F)$ ...

$$\begin{aligned}
\ldots + \mathrm{Tr}_N(A\cdot A) \times \big(&2e_a e_b N\bar{d}_{01|21} + 4N^2 e_a \bar{d}_{2|02} + 12N^2 e_a \bar{d}_{2|2} \\
&+2e_a N\bar{a}_4 + 2e_a N\bar{c}_{22} + 6N\bar{d}_{1|3}\big) \\
+\mathrm{Tr}_N(B\cdot B) \times \big(&2e_a e_b N\bar{d}_{1|12} + 12N^2 e_b \bar{d}_{02|02} + 4N^2 e_b \bar{d}_{2|02} \\
&+2e_b N\bar{b}_4 + 2e_b N\bar{c}_{22} + 6N\bar{d}_{01|03}\big) \\
+\mathrm{Tr}_N(A\cdot A\cdot A\cdot A) \times \big(&2N^2 e_a \bar{d}_{2|4} + 12e_a N\bar{a}_6 + 10e_a N\bar{d}_{1|5} \\
&+2N^2 e_b \bar{d}_{02|4} + 2e_b N\bar{c}_{42} + 2e_b N\bar{d}_{01|41}\big) \\
+\mathrm{Tr}_N(B\cdot B\cdot B\cdot B) \times \big(&2N^2 e_a \bar{d}_{2|04} + 2e_a N\bar{c}_{24} + 2e_a N\bar{d}_{1|14} \\
&+2N^2 e_b \bar{d}_{02|04} + 12e_b N\bar{b}_6 + 10e_b N\bar{d}_{01|05}\big) \\
+\mathrm{Tr}_N(A\cdot A\cdot B\cdot B) \times \big(&2N^2 e_a \bar{d}_{2|22} + 2e_a N\bar{c}_{42} + 2e_a N\bar{d}_{1|32} \\
&+2N^2 e_b \bar{d}_{02|22} + 2e_b N\bar{c}_{24} + 2e_b N\bar{d}_{01|23}\big) \\
+\mathrm{Tr}_N(A\cdot B\cdot A\cdot B) \times \big(&2N^2 e_a \bar{d}_{2|1111} + 2e_a N\bar{c}_{3111} + 2e_a N\bar{d}_{1|2111} \\
&+2e_b N^2 \bar{d}_{02|1111} + 2e_b N\bar{c}_{1311} + 2e_b N\bar{d}_{01|1211}\big) + \ldots
\end{aligned}$$

# Quantum fluctuations

- from $h_1 \operatorname{Tr}_{M_2(\mathcal{A}_2)}(F)$ ...

$$\ldots + \operatorname{Tr}_N(A\cdot A) \times \big(2e_a e_b N\bar{d}_{01|21} + 4N^2 e_a \bar{d}_{2|02} + 12N^2 e_a \bar{d}_{2|2}$$
$$+2e_a N\bar{a}_4 + 2e_a N\bar{c}_{22} + 6N\bar{d}_{1|3}\big)$$
$$+\operatorname{Tr}_N(B\cdot B) \times \big(2e_a e_b N\bar{d}_{1|12} + 12N^2 e_b \bar{d}_{02|02} + 4N^2 e_b \bar{d}_{2|02}$$
$$+2e_b N\bar{b}_4 + 2e_b N\bar{c}_{22} + 6N\bar{d}_{01|03}\big)$$
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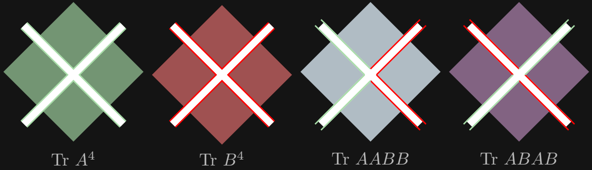
$$+2N^2 e_b \bar{d}_{02|4} + 2e_b N\bar{c}_{42} + 2e_b N\bar{d}_{01|41}\big)$$
$$+\operatorname{Tr}_N(B\cdot B\cdot B\cdot B) \times \big(2N^2 e_a \bar{d}_{2|04} + 2e_a N\bar{c}_{24} + 2e_a N\bar{d}_{1|14}$$
$$+2N^2 e_b \bar{d}_{02|04} + 12e_b N\bar{b}_6 + 10e_b N\bar{d}_{01|05}\big)$$
$$+\operatorname{Tr}_N(A\cdot A\cdot B\cdot B) \times \big(2N^2 e_a \bar{d}_{2|22} + 2e_a N\bar{c}_{42} + 2e_a N\bar{d}_{1|32}$$
$$+2N^2 e_b \bar{d}_{02|22} + 2e_b N\bar{c}_{24} + 2e_b N\bar{d}_{01|23}\big)$$
$$+\operatorname{Tr}_N(A\cdot B\cdot A\cdot B) \times \big(2N^2 e_a \bar{d}_{2|1111} + 2e_a N\bar{c}_{3111} + 2e_a N\bar{d}_{1|2111}$$
$$+2e_b N^2 \bar{d}_{02|1111} + 2e_b N\bar{c}_{1311} + 2e_b N\bar{d}_{01|1211}\big) + \ldots$$

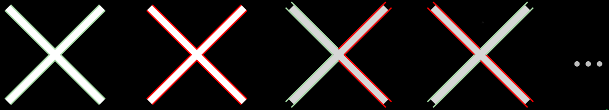
- for $h_2 \operatorname{Tr}_{M_2(\mathcal{A}_2)}[F^{\star 2}]$, multiply the $(48-2)\times(48-2)$ Hessians

# Ribbon graphs interpretation



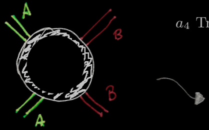

$$\beta(c_{22}) = -h_1\big(2e_a c_{1212} + e_b 2c_{2121} + 3e_a c_{24} + 3e_b c_{42} + e_a d_{02|22} + e_b d_{2|22}\big) + h_2\big(2a_4 c_{22} + 2b_4 c_{22} + 2e_a e_b c_{1111}^2 + 2e_a e_b c_{22}^2\big) + c_{22}(2\eta + 1)$$

# Beta functions of 2-matrix models: ribbon graph interpreted

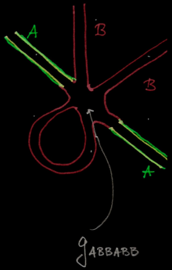


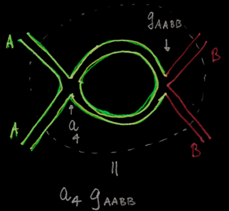
$a_4\ \mathrm{Tr}\,(A^4)$ $b_4\ \mathrm{Tr}\,(B^4)$ $g_{AABB}\ \mathrm{Tr}\,(AABB)$ $g_{ABAB}\ \mathrm{Tr}\,(ABAB)$ ...

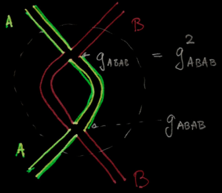
(filling of the ribbons has no meaning)

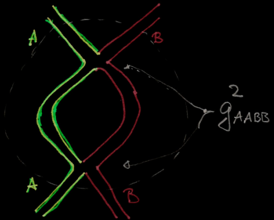



$$\beta(g_{AABB}) = -h_1\big(2g_{ABBABB} + 2g_{BAABAA} + 3g_{AABBBB} + 3g_{BBAAAA} + \text{disconn.}\big) + h_2\big(2a_4 \cdot g_{AABB} + 2b_4 g_{AABB} + 2g^2_{ABAB} + 2g^2_{AABB}\big) + g_{AABB}(2\eta + 1)$$



# Results for the (2,0)-geometry

- We obtain a unique solution leading to a single positive eigenvalue of the stability matrix $(-\partial\beta_I/\partial g_J)_{IJ}$,

$$\theta = +0.2749$$

and the corresponding fixed point has the coupling constants:

$$\begin{aligned}
\eta^\lozenge &= -0.3625 & a_4^\lozenge &= -0.07972 & a_6^\lozenge &= 0 & c_{1111}^\lozenge &= 0\\
c_{22}^\lozenge &= -0.03986 & c_{2121}^\lozenge &= 0 & c_{3111}^\lozenge &= 0 & c_{42}^\lozenge &= 0\\
d_{2|02}^\lozenge &= -0.01337 & d_{2|04}^\lozenge &= 0 & d_{2|1111}^\lozenge &= 0 & d_{1|5}^\lozenge &= 0\\
d_{2|2}^\lozenge &= -0.005156 & d_{2|22}^\lozenge &= 0 & d_{2|4}^\lozenge &= 0 & d_{12|3}^\lozenge &= 0\\
d_{1|12}^\lozenge &= -0.00985 & d_{3|3}^\lozenge &= 0 & d_{21|21}^\lozenge &= 0 & d_{1|14}^\lozenge &= 0\\
d_{1|3}^\lozenge &= -0.00985 & d_{1|2111}^\lozenge &= 0 & d_{1|32}^\lozenge &= 0 & &\\
d_{01|01}^\lozenge &= -0.2543 & d_{11|11}^\lozenge &= -0.004201 & d_{11|31}^\lozenge &= 0\,. & &
\end{aligned}$$

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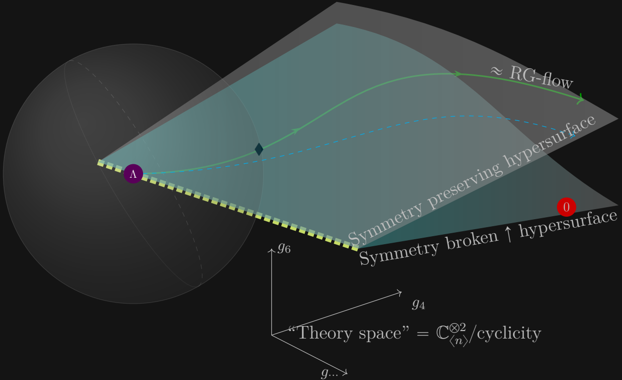
\end{aligned}$$

- Agreement: $-0.07972 \approx -\frac{1}{4\pi}$, so (after flipped sign convention)

$$a_4^\lozenge = 1.0018 \times (a_4^\lozenge)_{\text{Kazakov–Zinn-Justin}}$$

and $2c_{22}^\lozenge = -\frac{1}{4\pi}$ (after normalization convention)

# The RG-flow and the space of Dirac operators




- Λ — Chosen bare action $S = \Gamma_{N=\Lambda}$
- 0 — Full effective action $\Gamma = \Gamma_{N=0}$
- ♦ — Interpolating action $\Gamma_{N=\Lambda-\rho}$ (projected & truncated)
- ⟶ — RG-flow with truncation and projection
- ······ — Moduli of Dirac operators ↪ theory space
- - - → — RG-flow without truncation nor projection
- $g_{\dots}$ — Rest of coupling constants

- (0,2)-geometry We obtain a unique solution leading to a single positive eigenvalue of the stability matrix $(-\partial\beta_I/\partial g_J)_{IJ}$, $\theta = +0.2749$ fixed point:

$$\eta^{\Diamond} = -0.3625 \qquad a_4^{\Diamond} \approx -\frac{1}{4\pi} \qquad a_6^{\Diamond} = 0 \qquad c_{1111}^{\Diamond} = 0$$

$$c_{2121}^{\Diamond} = 0 \qquad c_{22}^{\Diamond} \approx -\frac{1}{8\pi} \qquad c_{3111}^{\Diamond} = 0 \qquad c_{42}^{\Diamond} = 0$$

$$d_{2|02}^{\Diamond} \approx -\frac{1}{24\pi} \qquad d_{2|04}^{\Diamond} = 0 \qquad d_{2|1111}^{\Diamond} = 0 \qquad d_{12|3}^{\Diamond} = 0$$

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- Rescalable Dirac with $g_4^{1/4}$ in $g_2D^2 + g_4D^4$, then $g_2 \to g_2/\sqrt{g_4}$ [Glaser, *J.Phys. A* 2017].
**Speculating** (projection of the fixed point to the moduli of $D$'s)

$$g^\lozenge = -\frac{|g_2^{\mathrm{BG}}|}{\sqrt{g_4^{\mathrm{BG}}}} = -\underbrace{\frac{g_2/8}{\sqrt{g_4/16}}}_{\text{my convention}} \overset{\lozenge}{\approx} -\frac{1}{2}(\langle g_4\rangle)^{-1/2} \approx -2.992 \qquad \text{(rough estimate)}$$

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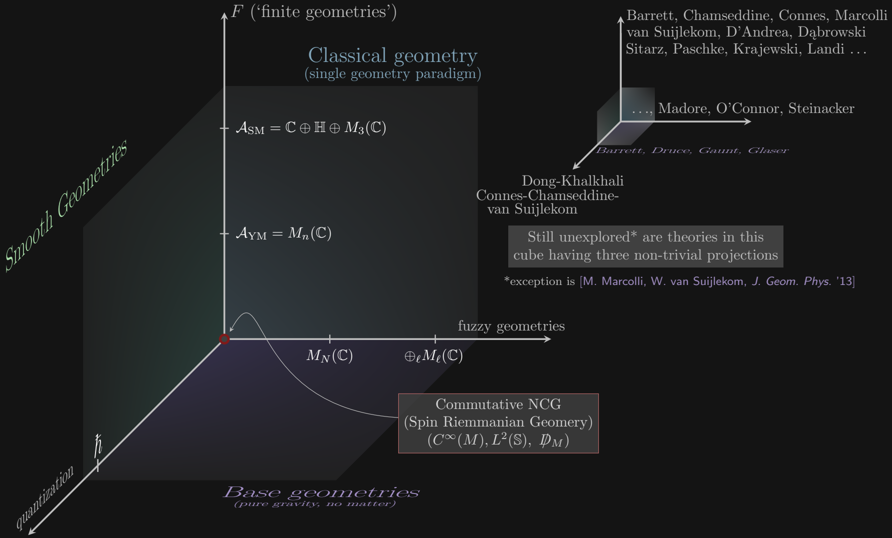
($g^\lozenge \approx -2.238$ if sum is wheighted)

- Parenthetically, since $\sqrt{\pi}$ appears, the correct units *might* be

$$\sqrt{4\pi} \approx \underbrace{\left(\frac{5\sqrt{2}}{2}\right)}_{\text{[Khalkhali-Pagliaroli, '20, 1-dim]}}$$

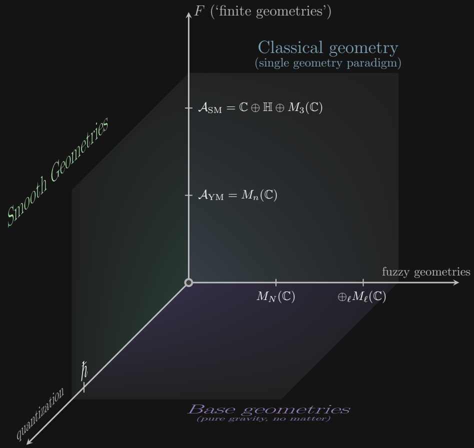
# A landscape




Still unexplored* are theories in this cube having three non-trivial projections

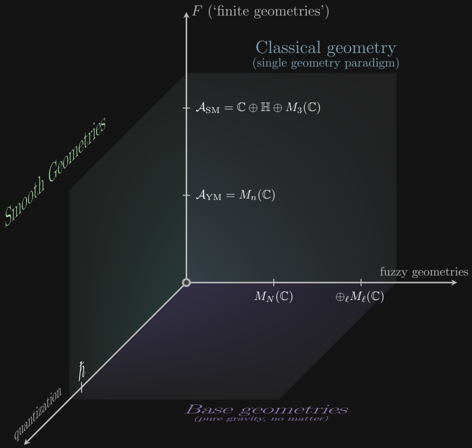
*exception is [M. Marcolli, W. van Suijlekom, *J. Geom. Phys.* '13]

# Further directions on Random NCG




- precision results for critical exponents
- solvability via Eynard-Orantin Topological Recursion
- extend upwards to add matter fields
- use the FRGE to compute critical exponents with matter

# Further directions on Random NCG

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Thank you