FROM VECTOR AND MATRICES TO TENSORS: COMMENTS & QUESTIONS

Carlos I. Pérez Sánchez, Univ. Heidelberg & ESI, Vienna
"Invitation to Recursion, Resurgence and Combinatorics"
OIST, Okinawa, Japan. April 2023

To rethink some of the geometric tools of 'tensor models' invites us:

> "The construction of tensor models was motivated by the idea of generalizing to higher dimensions the familiar relation of matrix models to random two-dimensional geometries (...) The status of this program is unclear, since it is not clear that the rather special Feynman diagrams that are generated by (...)
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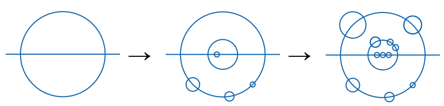
> 
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> describe a useful class of random (...) geometries" E. Witten *J. Phys. A* '16

Can we import (further) tools from matrix/vector models?

1. Discrete surfaces

- To address the enumeration of surfaces constructed by 'gluings of polygons', we first address an simpler problem: count gluings of a rooted polygon of $2p$ sides. By a gluing, we mean pairings $\pi \in \mathcal{P}_2(2p)$ of its sides. We think of π as chords inside the polygon; 'rooted' means that the polygon is fixed while $\mathbb{Z}_{2p}$ rotates the chord diagram
- from the $(2p-1)!! = (2p)!/2^p p! = \#\mathcal{P}_2(2p)$ gluings, let $c_g(p)$ be the number of those having genus g. Call $Q_p(N)$ the generating series (a polynomial in this case) in the sense

$$Q_p(N) = \frac{1}{N^2}\sum_{g\geqslant 0} c_g(p) N^{2-2g} \,,$$

where the scalings in N (still just a formal variable to be clarified) are by convenience. Notice that for $g>0$, $c_g(p)$ are higher genus generalizations of the Catalan number $c_0(p) = \frac{1}{p+1}\binom{2p}{p}$. For instance, $N^2 Q_3(N) = 5N^2 + 10N^0$

- a further step is dropping the restriction of the polygons having $2p$-sides and summing over the number of sides

$$1 + 2zN + 2z\sum_{p\geqslant 1}\frac{Q_p(N)}{(2p-1)!!}(Nz)^p = \left[\frac{1+z}{1-z}\right]^N \qquad (1)$$

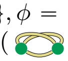
J. Harer & D. Zagier *Invent. Math.* '86. This generating function contains all the information, since the coefficient $[z^{p+1}, N^{p+1-2g}]\frac{1}{2}$RHS gives the genus-$g$ fraction of gluings of $2p$-agons for arbitrary p

- a matrix integral representation was relevant in one of the many proofs of Eq. 1. With the trace $\operatorname{Tr}(H) = \sum_{a=1}^N H_{a,a}$,

$$Q_p(N) = \textstyle\int_{M_N(\mathbb{C})_{\text{s.a.}}} \frac{1}{N}\operatorname{Tr}(H^{2p})\,\mathrm{d}\mu(H) =: \langle \frac{1}{N}\operatorname{Tr} H^{2p}\rangle_{\mathrm{G}},$$

where $\mathrm{d}\mu(H)$ is the normalized Gaußian measure $\mathrm{d}\mu(H) = K_N \exp\left[-(N/2)\operatorname{Tr} H^2\right]\mathrm{d}H$. While in order to get Formula 1 one has to work more, the matrix integral representation is readily obtained via $\langle H_{a,b}H_{c,d}\rangle_{\mathrm{G}} = \frac{1}{N}\delta_{a,d}\delta_{b,c}$ and

$$\langle H_{a_1,b_1}\cdots H_{a_{2p},b_{2p}}\rangle_{\mathrm{G}} = \sum_{\pi\in\mathcal{P}_2(2p)}\prod_{(i,j)\in\pi}\langle H_{a_i,b_i}H_{a_j,b_j}\rangle_{\mathrm{G}}$$

- if one allows connected 'gluings' of several polygons, the natural concept is *combinatorial map* $G = (J,\phi,\tau)$, where $J = \{1,\dots,h\}$ is the set of $h \in 2\mathbb{N}$ half-edges and $\phi,\tau \in \mathfrak{S}_h = \mathrm{S}_h$, being τ free from fixed points and $\tau^2 = 1$. The faces, edges and vertices of the map are the cycles (denoted $\mathcal{C}$) of ϕ,τ and $\upsilon = \phi\circ\tau$, respectively. Thus $\#\mathcal{C}(\upsilon) - \#\mathcal{C}(\tau) - \#\mathcal{C}(\phi) = \chi(G) = 2-2g$. For instance, $J = \{1,\dots,6\}$, $\phi = (162435)$, $\tau = (14)(25)(36)$ describe a map with $\chi($  $) = 0$, since $\upsilon = (132)(465)$
- to generate maps, one introduces a potential $V(x) = \sum_{0<k\leqslant d} t_k x^k/k$ which yields a new partition function $\mathcal{Z} = C_N\int_{M_N(\mathbb{C})_{\text{s.a.}}} \mathrm{e}^{-NV(H)}\mathrm{d}H$. Maps (with ∂) are counted by

$$\langle \operatorname{Tr} H^{\ell_1}\cdots\operatorname{Tr} H^{\ell_n}\rangle =: \textstyle\sum_{g\geqslant 0} N^{2-2g-n}\mathcal{T}^{(g)}_{\ell_1,\ell_2,\dots,\ell_n}$$

where the LHS is computed with $\langle P(H)\rangle := \mathcal{Z}^{-1}\int P(H)\mathrm{e}^{NV(H)}\mathrm{d}H$. These can be obtained when $\partial_\ell := \partial/\partial t_\ell$ hits the partition function $\mathcal{Z}$

- with $\ell = (\ell_2,\dots,\ell_n)$, *Tutte Equations*$\big|_{t_1=0,t_2=-1}$ read

$$\mathcal{T}^{(g)}_{\ell_1+1,\ell} = \sum_{j=3}^{d} t_j\cdot\mathcal{T}^{(g)}_{\ell_1+j-1,\ell} + \sum_{c=2}^{k}\ell_c\cdot\mathcal{T}^{(g)}_{\ell_1+\ell_c-1,\ell_2,\dots,\widehat{\ell_c},\dots,\ell_n}$$
$$+ \sum_{\substack{p,q\text{ with}\\ p+q=\ell_1-1}}\Big\{\sum_{\substack{h_1+h_2=g\\ I\cup J=\ell}}\mathcal{T}^{(h_1)}_{p,I}\times\mathcal{T}^{(h_2)}_{q,J} + \mathcal{T}^{(g-1)}_{p,q,\ell}\Big\}$$

- to obtain these (with $t_1, t_2+1 \neq 0$) one can use *Schwinger-Dyson equations* (SDE), sketched next: from $\int \mathrm{d}(X\mathrm{e}^{-S/\hbar}) = 0$, it holds $\int\left[\operatorname{div} X - \frac{1}{\hbar}\mathrm{d}S(X)\right]\mathrm{e}^{-S/\hbar} = 0$. So $\langle \operatorname{grad} S(X)\rangle = \hbar\langle \operatorname{div} X\rangle$. For $\hbar\to 0$, the SDE yield $\langle$classical EOM$\rangle$
- Tutte Equations can be restated as differential operators, $\mathcal{L}_k$, $k = -1,0,1,2\dots$ that annihilate the partition function $\mathcal{Z}' = \exp(N^2 t_0)\mathcal{Z}$, $\mathcal{L}_k\mathcal{Z}' = 0$. Omitting the cases $k = 0,\pm 1$, $\mathcal{L}_k$ for $k>1$ is given by

$$\sum_{j=1}^{k-1}\frac{j(k-j)}{N^2}\partial_j\partial_{k-j} + \frac{2k}{N^2}\partial_k\partial_0 + \sum_{j\in\mathbb{N}}(j+k)t_j\partial_{j+k},$$

which satisfies the Witt algebra $\mathfrak{w}$, $[\mathcal{L}_p,\mathcal{L}_q] = (p-q)\mathcal{L}_{p+q}$ (if $\mathcal{L}_{-1,0,1}$ are added) for $p,q\in\mathbb{Z}_{\geqslant -1}$ i.e. non-central $\mathfrak{vir}$

2. 'Geometry' and tensor models

- Let's begin by geometry: having gravitation as purpose, we are interested in PL-manifolds, as there the path integral seems better controlled. In fact, Alexander theorem—and refined (independent) versions by Ramirez, Montesinos and Hilden—states that connected, closed orientable 3-manifolds are *covers* over S^3 *branched* along a link/knot (which is far from unique). This holds in general dimension D, the branching over S^D taking place always at codimension-2 subset of that sphere

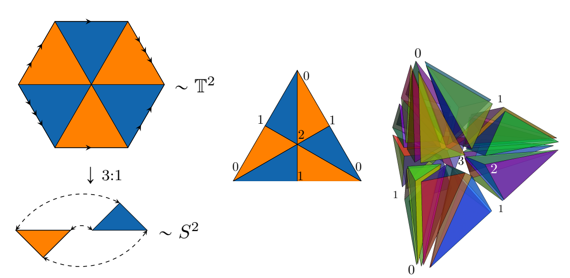
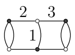


Fig. 1 (L) $\mathbb{T}^2$ as branched over S^2 (C&R) Barycentric subdivision $D = 2,3$

- one can always 'color' those triangulations by barycentric subdivision. Tensor Models computes integrals of the form

$$\int B(\phi,\bar\phi)\mathrm{e}^{-N^2\bar\phi_{pqr}\phi_{pqr}}\mathrm{d}\phi\,\mathrm{d}\bar\phi$$

which are performed over functions $\phi,\bar\phi:\{1,\dots,N\}^3\to\mathbb{C}$ satisfying that for each argument $1\leqslant p,q,r\leqslant N$ the evaluation $\phi_{pqr}=\phi(p,q,r)$ transforms independently under $\mathrm{U}(N)$. Interesting for $B(\phi,\bar\phi)$ are $\mathrm{U}(N)^3$-invariants, aka 'bubbles'



$= \phi_{a_1a_2a_3}\bar\phi_{a_1b_2c_3}\phi_{b_1b_2b_3}\bar\phi_{b_1a_2b_3}\phi_{c_1c_2c_3}\bar\phi_{c_1c_2a_3}$

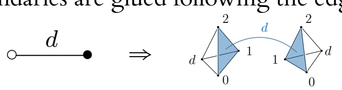
which we rather specify via (regularly) 3-colored (vertex-bipartite) graphs

- those invariants form a basis for the interactions, $S(\phi,\bar\phi)=\sum_{\mathcal{B},\mathrm{U}(N)^3\text{-inv.}}t_{\mathcal{B}}\,\mathcal{B}(\phi,\bar\phi)$ being $t_{\mathcal{B}}$ formal variables.

$$\mathcal{Z}_N=\int\exp[-N^2S(\phi,\bar\phi)]\mathrm{d}\phi\,\mathrm{d}\bar\phi\in\mathbb{C}[[\{t_{\mathcal{B}}\}_{\mathcal{B},\text{tricolored}}]]$$

⋆ Q0: *Is there a Harer-Zagier function for tensor integrals analogous to $(1+z)^N/(1-z)^N$ for matrix integrals?*

- amplitude of a Feynman graph $\mathcal{G}$, now having $D+1$ colors, scales $\sim N^{\#\text{faces}(\mathcal{G})-D(D+1)\#\text{vertices}(\mathcal{G})/4}$ and *interpreting* $\mathcal{G}$ as the gluing $\Delta(\mathcal{G})$ of $\#V(\mathcal{G})$ of equilateral D-simplices whose boundaries are glued following the edges of $\mathcal{G}$



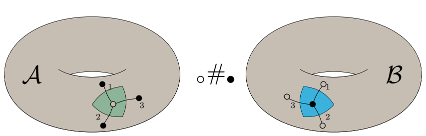
one can relate the amplitudes to Regge-Einstein-Hilbert action (which in some units reads) $S_{\mathrm{REH}}(\Delta(\mathcal{G}))=\sum_{D\text{ simplices }\sigma}\mathrm{vol}(\sigma)-\sum_{\text{codim 2 simpl. }\tau}\delta(\tau)\mathrm{vol}(\tau)$, being $\delta(\tau)$ the deficiency angle at τ

⋆ Q1: While in $D=2$ the equilateral condition is irrelevant, in $D\geqslant3$ it seems only extremely convenient. *Can one for $D\geqslant3$ find an association of geometric parameters $\boldsymbol{P}_D(a_1,\dots,a_D)\in\{$angles, areas, lengths,....$\}$ that respects the $1/N$ expansion?*

- For invariants (bubbles) $\mathcal{A},\mathcal{B}$ and $v\in V_\bullet(\mathcal{A})$, $w\in V_\bullet(\mathcal{B})$,

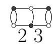
$$\mathcal{A}*\mathcal{B}=\sum_{u\in V_\circ(\mathcal{A})}(\mathcal{A}\sqcup\mathcal{B})\backslash\{u,w\}\big|_{\text{glue colorwise}}\qquad\text{(rooted at } v)$$

which in topologically boils down to connected sums:

$\mathcal{A}*\mathcal{B}$ '=' $\sum_{\circ\in\mathcal{A}}$


- if $\mathcal{A}_{v,w}=\mathcal{A}\backslash\{v,w\}\big|^{\text{broken edges}}_{\text{glued by color}}$ for $w\in V_\circ(\mathcal{A})$, the loop or Schwinger-Dyson Equations can be expressed as $\mathcal{L}_{A,v}\mathcal{Z}_N=0$, where

$$\mathcal{L}_{\mathcal{A},v}=\sum_{w\in V_\circ(A)}N^{\#\text{Edges}(w\leftrightarrow v)}\prod_{\rho\in\pi_0(\mathcal{A}_{v,w})}\Big[-\frac{1}{N^2}\frac{\partial}{\partial t_\rho}\Big]+\sum_{\mathcal{B}\in S}t_{\mathcal{B}}\frac{\partial}{t_{\mathcal{B}*\mathcal{A}}}$$

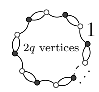
so for connected $\mathcal{A}$, $\mathcal{L}_{\mathcal{A},v}$ can be of second degree, e.g. for $\mathcal{A}=$ 

, if v,w are the middle vertices

- Similar graph operations describe the Polchinski equation T. Krajewski & R. Toriumi *J. Phys. A* '16 (Wetterich's is w.i.p.)
- by R. Gurău *Nucl. Phys. B* '12 these operators satisfy

$$[\mathcal{L}_{A,v},\mathcal{L}_{B,w}]:=\mathcal{L}_{A*B,v}-\mathcal{L}_{B*A,w}$$

- this Gurău algebra restricts to $\mathfrak{w}_{\geqslant1}$ algebra $[\mathcal{L}_{C_p},\mathcal{L}_{C_q}]=(p-q)\mathcal{L}_{C_{p+q}}$ for $p,q\in\mathbb{Z}_{\geqslant1}$ if we restrict to cycles,

$C_q :=$ 

⋆ Q2: *Can one find a recursion (satisfied by the correlators) in tensor models, even though it is not topological?*

3. Airy structures and Topological Recursion (TR)

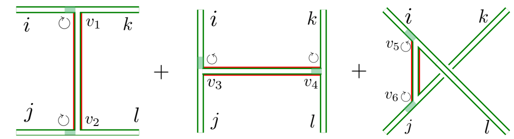
Airy structures M. Kontsevich & Y. Soibelman '17 capture the essence of TR

- Let W^* be the vector space with basis $\{t_j\}_{j=1}^d$. If $\hbar$ is a formal variable, a (quantum) *Airy structure* on W is a family of operators $\{L_k\}_k$ on $\mathrm{Sym}(W^*)[[\hbar,\hbar^{-1}]]$ where L_k reads

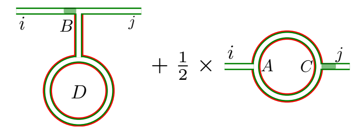
$$-\frac{1}{2}(\mathbf{t},A^k\mathbf{t})-\hbar(\mathbf{t},B^k\partial)-\frac{\hbar^2}{2}(\partial,C^k\partial)+\hbar(\partial_k-D_k)$$

and such that $[L_i,L_j]=\hbar\sum_k f^k_{i,j}L_k$, being f^k,A^k, $B^k,C^k\in M_d(\mathbb{C})$, where A^k and C^k are symmetric, while f^k is skew-symmetric for each k (not a matrix index nor exponent, abusing on notation)

- The Lie algebra condition implies that A, seen as a tensor, is fully symmetric; that $f^k_{i,j}=B^i_{j,k}-B^j_{i,k}$; and three IHX-relations described next. To the six vertices one associates letters. Red edges have indices that run. Further, the indices of each letter O at the vertices $O_{\boxed{2}\boxed{3}}^{\boxed{1}}$ is determined in the sense of the arrow, starting at the shaded edge. The IHX-relation for $(v_1,v_2,\dots,v_6)=(B,B,B,B,C,A)$ is that $\sum_{a=1}^d B^i_{j,a}A^a_{k,l}+B^i_{k,a}A^j_{a,l}+B^i_{l,a}A^j_{a,k}$ is $(i\leftrightarrow j)$-symmetric. Similar relations hold for $(v_1,v_2,\dots,v_6)=(B,A,B,B,B,A)$ and (C,B,B,C,C,B).



(2a)

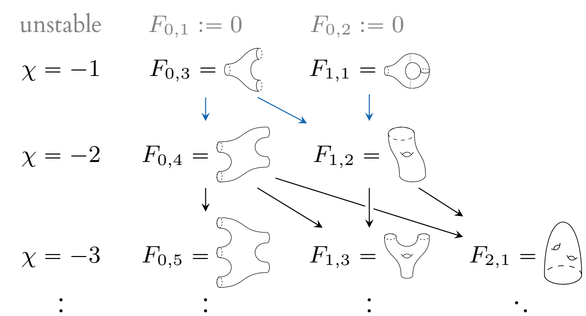


(2b)

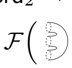
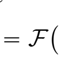
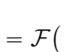
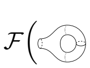
- Thm (Kontsevich-Soibelman) *There exists a unique $\hbar^{-1}F\in\mathrm{Sym}(W^*)[[\hbar]]$ such that $\{L_j\mathrm{e}^F=0\}_{j=1,\dots,d}$*
Proof sketch of uniqueness (cf. G. Borot *Rev. Math. Phys.* '20). Expand $F=\sum_{g\geqslant0}\hbar^{g-1}\sum_{n\geqslant1}\sum_{I,\#I=n}F_{g,n}[I]t_I/n!$, with $I=(i_1,\dots,i_n)\in\{1,\dots,d\}^n$ and $t_I=t_{i_1}\cdots t_{i_n}$ in multi-index notation, and read off the coefficient of $\hbar^g\times t_{i_2}\cdots t_{i_n}/(n-1)!$ in $\exp(-F)L_{i_1}\exp(F)=0$. This yields $F_{0,3}[i_1,i_2,i_3]=A^{i_1}_{i_2,i_3}$ and $F_{1,1}[i_1]=D_{i_1}$ for $\chi=-1$, while for higher $-\chi$, $F_{g,n}[i_1,i_2,\dots,i_n]$ is determined by recursion and equals (cf. last page)

$$\sum_{m=2}^{n}B^{i_1}_{i_m,a}F_{g,n-1}[a,i_2,\dots,\widehat{i_m},\dots,i_n]\quad(j_q:=\#J_q,\,q=1,2)$$
$$+\frac{1}{2}C^{i_1}_{a,b}\Big\{F_{g-1,n+1}[a,b,i_2,\dots,i_n]$$
$$+\sum_{\substack{h_1+h_2=g\\J_1\sqcup J_2=\{i_2,\dots,i_n\}}}F_{h_1,1+j_1}[a,J_1]\times F_{h_2,1+j_2}[b,J_2]\Big\}\qquad(3)$$

- adj. 'topological' explained by excisions of 'pair of pants'



- the boundaries above are not oriented; but, parenthetically, the ABCD-terms could stem from a TQFT $\mathcal{F}:\mathsf{Bord}_2\to\mathsf{Vect}_{\mathbb{C}}$

$A=\mathcal{F}($  $)$, $B=\mathcal{F}($  $)$, $C=\mathcal{F}($  $)$, $D=\mathcal{F}($  $)$

4. The volume of the moduli space $\mathcal{M}_{g,n}(L)$

- $\mathscr{T}_{g,n}(L) = \{\text{metrics on } \Sigma_{g,n} : \text{length of boundary } b_j = L_j\}/\{\text{conformal maps}\}$, with $L = (L_1, \ldots, L_n)$
- $\Gamma_{g,n} = \{\mathrm{Diff}(\Sigma_{g,n}) \text{ that keep labels}\}/\{\text{isotopies to } \mathrm{id}_{\Sigma_{g,n}}\}$
- $\mathcal{M}_{g,n}(L) = \mathscr{T}_{g,n}(L)/\Gamma_{g,n}$ = Teichmüller/mapping class
- decomposition of a stable surface $\Sigma_{g,n}$ in simple closed curves yields p Y-pieces, each having Euler number -1, so $p = -\chi(\Sigma_{g,n})$. From the $3p$ geodesic boundaries, n are *not* glued, so there are $\frac{1}{2}(p-n) = 3g+n-3 := d_{g,n}$ inner pairings of cycles, whose lengths ℓ_j can coincide. The twisting angle θ_j of one cycle with respect to the other is another parameter
- $\{\ell_j, \theta_j\}_{j=1,\ldots,3g+n-3}$ are in fact the *Fenchel-Nielsen* coordinates of $\mathscr{T}_{g,n}$. The form $\omega_{\text{WP}} = \sum_j \mathrm{d}\ell_j \wedge \mathrm{d}\theta_j$ is $\Gamma_{g,n}$-invariant, as shown by Wolpert '85, and $\omega_{\text{WP}}^{\wedge d_{g,n}}/d_{g,n}!$ defines the volume form of $\mathcal{M}_{g,n}(L)$ and $V_{g,n}(L) = \mathrm{vol}[\mathcal{M}_{g,n}(L)]$
- Mirzakhani *JAMS* '07 TR states that $V_{g,n+1}(L_0, L)$ equals

$$\sum_{m>0} \int_{\mathbb{R}_+} B_{\text{Mirz}}(L_0, L_m, \ell) V_{g,n}(\ell, L_1, \ldots, \widehat{L_m}, \ldots, L_n)\,\mathrm{d}\ell$$
$$+ \frac{1}{2} \int_{\mathbb{R}_+^2} C_{\text{Mirz}}(L_0, \ell, \ell') \Big[V_{g-1,n+2}(\ell, \ell', L_1, \ldots, L_n)$$
$$+ \sum_{\substack{h_1+h_2=g \\ J_1 \sqcup J_2 = \{L_j\}_{j=0}^n}} V_{h_1,1+j_1}(\ell, J_1) V_{h_2,1+j_2}(\ell', J_2) \Big]\, \mathrm{d}\ell \mathrm{d}\ell',$$

where $B_{\text{Mirz}}(L_1, L_2, L_3)$ and $C_{\text{Mirz}}(L_1, L_2, L_3)$ are given by

$$\frac{L_3}{L_1} \log \frac{[1+\mathrm{e}^{(L_3+L_2-L_1)/2}][1+\mathrm{e}^{(L_3-L_2-L_1)/2}]}{[1+\mathrm{e}^{(L_3+L_2+L_1)/2}][1+\mathrm{e}^{(L_3-L_2+L_1)/2}]}$$

and $\quad 2\frac{L_2 L_3}{L_1} \log \frac{1+\mathrm{e}^{(L_3+L_2-L_1)/2}}{1+\mathrm{e}^{(L_3+L_2+L_1)/2}}$, respectively

- keeping the coefficients of the volumes as amplitudes,

$$V_{g,n}(L) = \sum_{a_1,\ldots,a_n \geqslant 0} F_{g,n}[a_1, \ldots, a_n] \prod_{j=1}^n L_j^{2a_j},$$

Mirzakhani's TR and (3) lead to the Airy structure:

$$B^i_{j,k} = \frac{(2k+1)!}{(2i+1)!(2j+1)!} (2j+1)\,\theta_{k-j-i} \qquad \text{(4a)}$$
$$C^i_{j,k} = \frac{(2j+1)!(2k+1)!}{(2i+1)!} \theta_{k+j+1-i}, \qquad \text{(4b)}$$

where $\sum_{k+1\geqslant 0} z^{2k}\theta_k/\mathrm{d}z = 4\pi/\sin(2\pi z)\mathrm{d}z^2 =: 1/y(z)\mathrm{d}z^2$. But we need the initial A and D terms

- the remarkable formula in M. Kontsevich *Anal. and Appl,* '91 uses maps, or ribbon graphs, to compute also intersection numbers

$$\sum_{\substack{a_j \in \mathbb{Z}_{\geqslant 0}, \text{for all } j \\ a_1+\ldots+a_n = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n}}} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdot \psi_2^{a_2} \cdots \psi_n^{a_n} \prod_{j=1}^n \frac{(2a_j-1)!!}{\lambda_j^{2a_j+1}}$$
$$= \sum_{G \text{ trivalent, of topology } (g,n)} \frac{2^{2g-2+n}}{\#\mathrm{Aut}(G)} \prod_{e \in E(G)} \frac{1}{\lambda_{L(e)} + \lambda_{R(e)}}$$

- e.g. there is a unique $(1,1)$-graph:

$G =$  $\Rightarrow \frac{1}{\lambda^3} \int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{2^1}{\#\mathrm{Aut}(G)} \frac{1}{(2\lambda)^3}$

$$\mathrm{Aut}(G) = \{\psi \in \mathfrak{S}_6 : \text{ commuting with } \phi \text{ and } \tau\}$$
$$= \{\mathrm{id}, (123)(456), (132)(465), \tau, \phi, \phi^{-1}\}$$

then $\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = 1/24$, so $D_1 = 1/24$. Penner '85 computed $V_{1,1}(0) = \zeta(2)$ implying $D_0 = \pi^2/6$. Else $D_k = 0$ for $k > 1$. Four graphs of $(0,3)$-type $\Rightarrow A^{i_1}_{i_2,i_3} \neq 0$ iff $i_* = 0$ ($A^0_{0,0} = 1$)

★ Q3: *Is there a geometric object enumerated by tensor models?*

5. Towards Borel summability in $1/N$

As a perspective for Borel summability in tensor models it is convenient to mention two essential techniques which together are the essence of the Loop Vertex Expansion V. Rivasseau *JHEP* '07. In L. Ferdinand, R. Gurău, C.P & F. Vignes-Tourneret 2209.09045 BS in $1/N$ is addressed for the cumulants of the vector model (defined below).

- The Hubbard-Stratonovich transformation

$$e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}y\, e^{-\frac{y^2}{2} + \mathrm{i}xy}$$

allows to transform the N-dimensional integral in the partition function of the $\mathrm{O}(N)$-vector model

$$Z(g, 1/N, J) = \int_{\mathbb{R}^N} \mathrm{e}^{-\frac{1}{2}\phi\cdot\phi - \frac{g}{8N}(\phi\cdot\phi)^2 - \sqrt{N} J\cdot\phi} \frac{\mathrm{d}^N\phi}{(2\pi)^{N/2}}$$

into an integral where N appears only as a parameter, and not as the dimension of the dimension of the integration domain, thus allowing for analytic continuation. With $R(\sigma, z) = (1 - \sqrt{z}\sigma)^{-1}$

$$Z(g, 1/N; J) = \int \mathrm{e}^{\frac{N}{2} \log R(\sigma, g/N) + \frac{N}{2} R(\sigma, g/N)\, J\cdot J} \frac{e^{-\frac{\sigma^2}{2}} \mathrm{d}\sigma}{\sqrt{2\pi}}$$

- Brydges-Kennedy–Abdesselam-Rivasseau proved

$$\Phi(\mathbf{1}) - \Phi(\mathbf{0}) = \sum_{\substack{\text{non-empty forests} \\ F \text{ with } n \text{ vertices}}} \int \frac{\partial \Phi}{\partial F} \mathrm{d}F \qquad \text{(BKAR)}$$

$\mathbf{1}, \mathbf{0} \in \mathbb{R}^{\binom{n}{2}}$ the vectors with 1's and 0's as constant entries, respectively. The BKAR formula holds for any smooth function $\Phi : \mathbb{R}^{\binom{n}{2}} \to \mathbb{C}$, where the 'measure' over a forest and a 'forest' derivative are given by

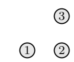
$$\mathrm{d}F := \prod_{e \in E(F)} \mathrm{d}u_e \qquad \text{(5a)}$$
$$\frac{\partial \Phi}{\partial F} := \Big\{ \Big[\prod_{(l,m) \in E(F)} \frac{\partial}{\partial x_{lm}} \Big] \Phi(x) \Big\} \Big|_{x = w(F)} \qquad \text{(5b)}$$

with parameters given by

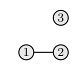
$$w(F)_{ij} := \min\{u_e\}_{e \in E(F),\ e \text{ along the path } i \to j} \qquad \text{(5c)}$$

and $w(F)_{ij} = 0$ if such path does not exist. For instance, for $n = 3$, the forest expansion of $\Phi(1,1,1)$ reads

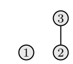
$$\Phi(0,0,0) \qquad \text{(6)}$$



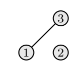
$$+ \int_0^1 \mathrm{d}u_{12} \frac{\partial \Phi(x)}{\partial x_{12}} \Big|_{x=(u_{12},0,0)}$$



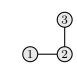
$$+ \int_0^1 \mathrm{d}u_{23} \frac{\partial \Phi(x)}{\partial x_{23}} \Big|_{x=(0,u_{23},0)}$$



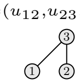
$$+ \int_0^1 \mathrm{d}u_{23} \frac{\partial \Phi(x)}{\partial x_{23}} \Big|_{x=(0,0,u_{13})}$$



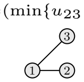
$$+ \int_0^1 \int_0^1 \mathrm{d}u_{23}\mathrm{d}u_{12} \frac{\partial^2 \Phi(x)}{\partial x_{12} \partial x_{23}} \Big|_{x=(u_{12},u_{23},\min\{u_{12},u_{23}\})}$$



$$+ \int_0^1 \int_0^1 \mathrm{d}u_{23}\mathrm{d}u_{13} \frac{\partial^2 \Phi(x)}{\partial x_{23} \partial x_{13}} \Big|_{x=(\min\{u_{23},u_{13}\},u_{23},u_{13})}$$



$$+ \int_0^1 \int_0^1 \mathrm{d}u_{13}\mathrm{d}u_{12} \frac{\partial^2 \Phi(x)}{\partial x_{12} \partial x_{13}} \Big|_{x=(u_{12},\min\{u_{12},u_{13}\},u_{13})}$$

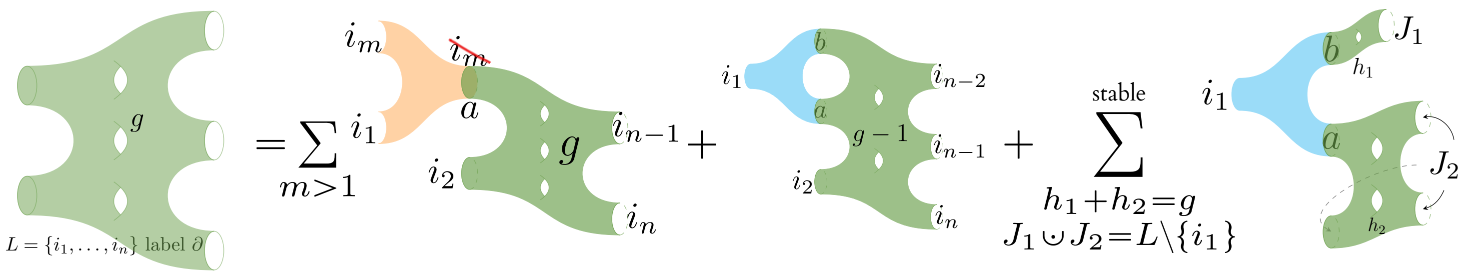


- to have the cumulants, one needs the logarithm of this partition function. Performing Gaußian integration

$$Z(g, 1/N; J) = \sum_{n \geqslant 0} \frac{N^n}{(2)^n n!} \Big[\exp\Big(\frac{\langle \partial, \partial \rangle_{X(x)}}{2N} \Big)$$
$$\prod_{i=1}^n \Big\{ \log R(\sigma^{(i)}, g) + R(\sigma^{(i)}, g) J\cdot J \Big\} \Big]_{\substack{\sigma^{(i)}=0 \\ x_{ij}=1}},$$

where $[X(x)]_{ii} = 1$ and $[X(x)]_{ij} = x_{ij}$ for $i \neq j$, one can use for the function in square-brackets BKAR-formula. Then $\log Z$ will be the restriction from forests to trees.

★ Q4: *Are tensor models $1/N$ Borel summable?*



g
$L = \{i_1, \ldots, i_n\}$ label ∂
$= \sum_{m>1}$
i_m
i_1
i_m
a
i_2
g
i_{n-1}
i_n
$+$
i_1
b
a
$g-1$
i_{n-2}
i_{n-1}
i_2
i_n
$+$
$\sum^{\text{stable}}_{h_1+h_2=g,\ J_1 \cup J_2 = L \setminus \{i_1\}}$
i_1
b
h_1
J_1
a
J_2
h_2