Combinatorial aspects of tensor models

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in collaboration with:

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arXiv:1408.5725, *Electronic J. Comb.* (2015)

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arXiv:2109.07238, *J. Phys.* **A** (2022)

Plan

1. Introduction and motivation
   - Combinatorial Physics
   - A standard combinatorial tool - generating functions
   - Random matrices and combinatorics
   - Motivation for sums over random surfaces
2. The multi-orientable (MO) model
   - Ribbon jackets and the asymptotic expansion
   - The dominant order
   - The sub-dominant order
   - The general order of the expansion
     - Dipoles, chains and (reduced) schemes
     - Finiteness of the set of reduced schemes
     - Generating functions
     - Dominant schemes
   - The double scaling limit
3. The $O(N)^3-$invariant model
4. Conclusions and perspectives

Combinatorial Physics

- problems in Theoretical Physics successfully tackled using Combinatorics methods
- problems in Combinatorics successfully tackled using Theoretical Physics methods

this talk: example of the first case

Combinatorics - what is a generating function?

In combinatorics, a generating function is a way of encoding an infinite sequence of numbers (a_n) by treating them as the coefficients of a formal power series.

This series is called the generating function of the sequence.

What is a generating function?

"A generating function is a device somewhat similar to a bag. Instead of carrying many little objects detachedly, which could be embarrassing, we put them all in a bag, and then we have only one object to carry, the bag."

George Pólya, *"Mathematics and plausible reasoning"* (1954)

"A generating function is a clothesline on which we hang up a sequence of numbers for display."

Herbert Wilf, *Generatingfunctionology* (1994)

Generating functions; definition

The (ordinary) generating function of a sequence a_n:

$$G(a_n; u) = \sum_{n=0}^{\infty} a_n u^n.$$

Example of generating functions

1. the generating function of the sequence $(1, 1, \ldots)$ is $\frac{1}{1-u}$

$$\frac{1}{1-u} = 1 + u + u^2 + \ldots$$

$u^2 \frac{1}{1-u} = u^2 + u^3 + \ldots$

2. the generating function of the sequence $(1, 0, 1, 0, \ldots)$ is $\frac{1}{1-u^2}$

$$\frac{1}{1-u^2} = 1 + u^2 + u^4 + \ldots$$

$u^2 \frac{1}{1-u^2} = \frac{u^2}{1-u^2} = u^2 + u^4 + u^6 + \ldots$
$u^3 \frac{1}{1-u^2} = \frac{u^3}{1-u^2} = u^3 + u^5 + u^6 + \ldots$

generating functions explictly used to study the double scaling limit of tensor models

Random matrices & combinatorics

A random matrix is a matrix of given type and size whose entries consist of random numbers from some specified distribution.

counting maps theorems (*via* matrix integral techniques)

$$\int f(\text{matrix of dim } N) \quad = \quad \sum_g N^{2-2g} A_g$$

A_g - some weighted sum encoding maps of genus g
(this depends on the choice of f - the physical model)

A. Zvonkine, in "*Computers & Math. with Applications: Math. & Computer Modelling*", 1997

J. Bouttier, in "*The Oxford Handbook of Random Matrix Theory", 2011, arXiv:1104.3003*

Ph. Di Francesco *et. al.*, *Phys. Rept.* (1995), arXiv:hep-th/9306153

Sums over random surfaces

Matrix models - sums over surfaces

"There are methods and formulae in science, which serve as master-keys to many apparently different problems. The resources of such things have to be refilled from time to time. In my opinion at the present time we have to develop an art of handling sums over random surfaces. These sums replace the old-fashioned (and extremely useful) sums over random paths."

A. M. Polyakov, *Quantum geometry of bosonic strings, Phys. Lett.* (1981)

(a physics sociology bracket: 4100+ citations ...)

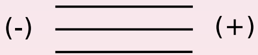
A new (QFT-inspired) simplification of tensor models

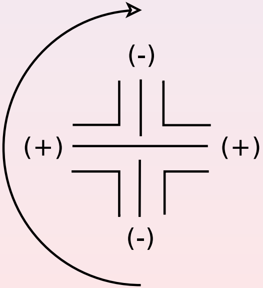
Multi-orientable (MO) models

A. Tanasă, J. Phys. **A** (2012) arXiv:1109.0694[math.CO]

non-commutative QFT inspired idea

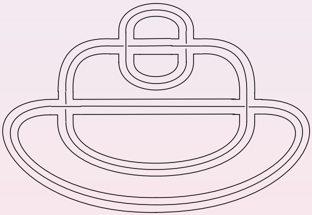
edge and (valence 4) vertex of the model:





(Feynman) MO tensor graphs

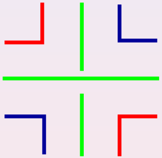
Example of a MO tensor graph:



Combinatorial and topological tools - jacket ribbon subgraphs

S. Dartois *et. al.*, *Annales Henri Poincaré* (2014)

three pairs of opposite corner strands



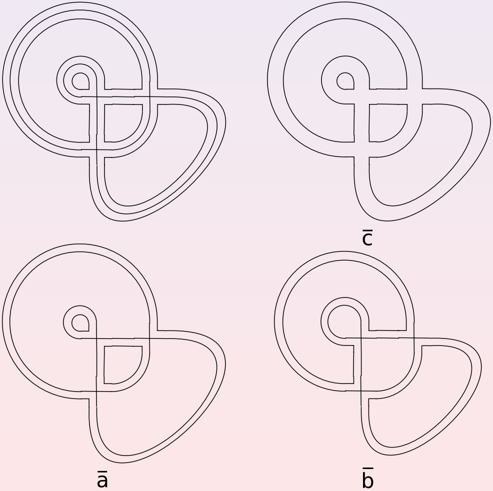
Definition

A **jacket of an MO graph** is the graph made by excluding one type of strands throughout the graph. The *outer* jacket $\bar{c}$ is made of all outer strands, or equivalently excludes the inner strands (the green ones); jacket $\bar{a}$ excludes all strands of type a (the red ones) and jacket $\bar{b}$ excludes all strands of type b (the blue ones).

$\hookrightarrow$ such a splitting is always possible

Example of jacket subgraphs

A MO graph with its three jackets $\bar{a}$, $\bar{b}$, $\bar{c}$



Euler characteristic & degree of MO tensor graphs

ribbon graphs can represent orientable or non-orientable surfaces.

Euler characteristic formula:

$$\chi(\mathcal{J}) = V_{\mathcal{J}} - E_{\mathcal{J}} + F_{\mathcal{J}} = 2 - k_{\mathcal{J}},$$

$k_{\mathcal{J}}$ is the non-orientable genus,
$V_{\mathcal{J}}$ is the number of vertices,
$E_{\mathcal{J}}$ the number of edges and
$F_{\mathcal{J}}$ the number of faces.

If the surface is orientable, k is even and equal to twice the orientable genus g

Given an MO graph $\mathcal{G}$, its degree $\delta(\mathcal{G})$ is defined by

$$\delta(\mathcal{G}) := \sum_{\mathcal{J}} \frac{k_{\mathcal{J}}}{2} = 3 + \frac{3}{2} V_{\mathcal{G}} - F_{\mathcal{G}},$$

the sum over $\mathcal{J}$ running over the three jackets of $\mathcal{G}$.

Asymptotic expansion of the MO tensor model

generalization of the random matrix asymptotic expansion in N

One needs to count the number of faces of the tensor graph

This can be achieved using the graph's jackets (ribbon subgraphs)

The tensor partition function writes as a formal series in $1/N$:

$$\sum_{\delta \in \mathbb{N}/2} C^{[\delta]}(\lambda) N^{3-\delta},$$

$$C^{[\delta]}(\lambda) = \sum_{\mathcal{G}, \delta(\mathcal{G})=\delta} \frac{1}{s(\mathcal{G})} \lambda^{v_{\mathcal{G}}}.$$

the role of the genus is played by the degree

Dominant graphs of the large N expansion

dominant graphs:

$$\delta = 0.$$

Theorem

The MO model admits a $1/N$ expansion whose dominant graphs are the "melonic" ones.

The general term of the expansion

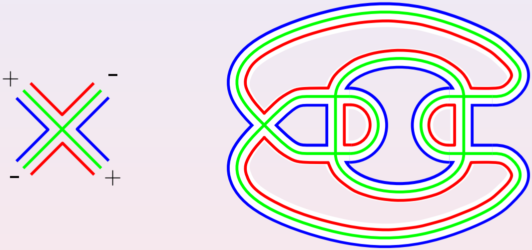
E. Fusy and A. Tanasă, arXiv:1408.5725[math.CO], *Elec. J. Comb.* (2015)

adaptation of the Gurău-Schaeffer combinatorial approach for the MO case

R. Gurău and G. Schaeffer, arXiv:1307.5279[math.CO],

Annales IHP **D** *Comb., Phys. & their Interactions* (2016)

(Types of) strands



An external strand is called **left (L)** if it is on the left side of a positive half-edge or on the right side of a negative half-edge.
An external strand is called **right (R)** if it is on the right side of a positive half-edge or on the left side of a negative half-edge.

(L - blue, straight (S) - green, R - red)

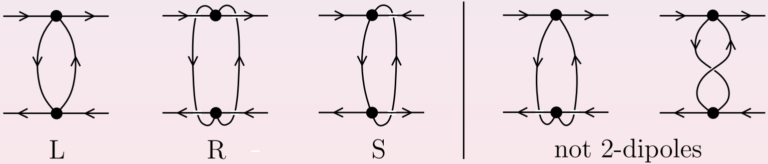
Main issue of a combinatorial analysis

Problem: There exists an infinite number of melon-free graphs of a given degree.

Nevertheless, some configurations can be repeated without increasing the degree.

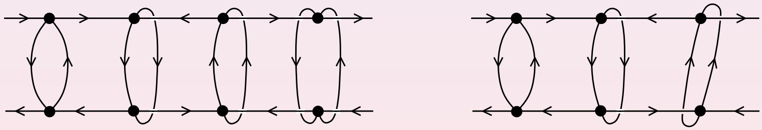
Dipoles

A **(two-)dipole** is a subgraph formed by a couple of vertices connected by two parallel edges which has a face of length two, which, if the graph is rooted, does not pass through the root.



Chains

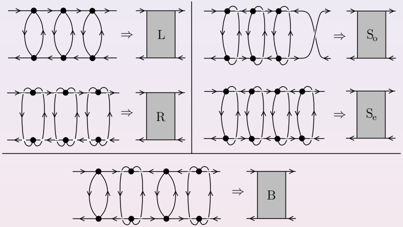
In a (possibly rooted) graph, define a **chain** as a sequence of dipoles $d_1 \ldots, d_p$ such that for each $1 \leq i < p$, d_i and d_{i+1} are connected by two edges involving two half-edges on the same side of d_i and two half-edges on the same side of d_{i+1}.



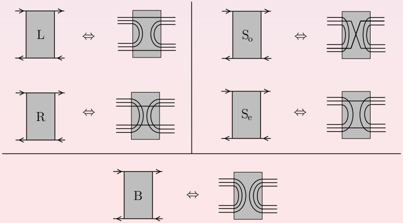
Some more definitions - (un)broken chains

- A chain is called **unbroken** if all the p dipoles are of the same type.
- A **proper chain** is a chain of at least two dipoles.
- A proper chain is called **maximal** if it cannot be extended into a larger proper chain.

Chains, chain-vertices and their strand configurations

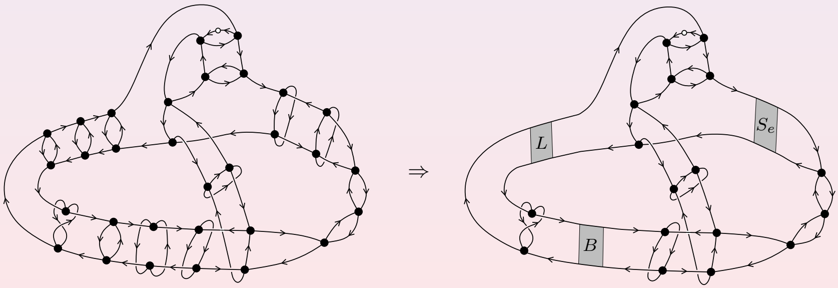


strand configurations:



Schemes

Let G be a rooted melon-free MO-graph. The **scheme** of G is the graph obtained by simultaneously replacing any maximal proper chain of G by a *chain-vertex*.



A **reduced scheme** is a rooted melon-free MO-graph with chain-vertices and with no proper chain.

By construction, the scheme of a rooted melon-free MO-graph (with no chain-vertices) is a reduced scheme.

Proposition

Every rooted melon-free MO-graph is uniquely obtained as a reduced scheme where each chain-vertex is consistently substituted by a chain of at least two dipoles (consistent means that if the chain-vertex is of type L, then the substituted chain is an unbroken chain of L-dipoles, etc).

Degree conservation

Proposition

Let G be an MO-graph with chain-vertices and let G' be an MO-graph with chain-vertices obtained from G by consistently substituting a chain-vertex by a chain of dipoles. Then the degrees of G and G' are the same.

Proof. Carefully counting the number of faces, vertices and connected components and using the formula:

$$2\delta = 6c + 3V - 2F.$$

Finiteness of the set of reduced schemes of a given degree

Theorem

For each $\delta \in \frac{1}{2}\mathbb{Z}_+$, the set of reduced schemes of degree δ is finite.

Proof.

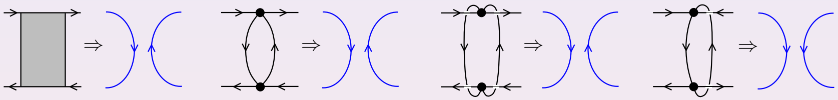
Lemma

For each reduced scheme of degree δ, the sum $N(G)$ of the numbers of dipoles and chain-vertices satisfies $N(G) \leq 7\delta - 1$.

Lemma

For $k \geq 1$ and $\delta \in \frac{1}{2}\mathbb{Z}_+$, there is a constant $n_{k,\delta}$ s. t. any connected unrooted MO-graph of degree δ with at most k dipoles has at most $n_{k,\delta}$ vertices.

Proof - dipole and chain-vertex reductions



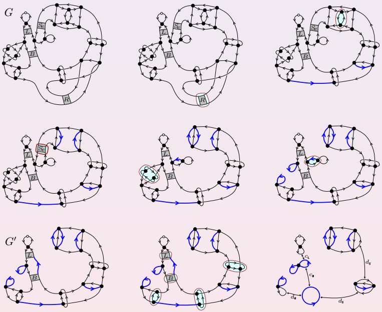
- removal of a chain-vertex (of any type)
- removal of a dipole of type L, R and S.

2 types of chain-vertices (and dipoles):

1. separating
2. non-separating

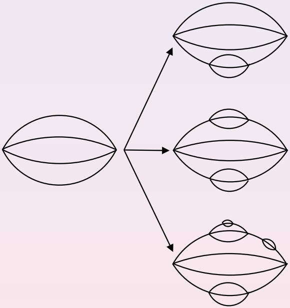
(if the number of connected components is conserved or not after removal)

Iterative removal of dipoles and chain-vertices



The removal chain-vertices and dipoles (first the non-separating ones and then the separating ones) leads to a tree of components.

Some analytic combinatorics - melonic generating function



the *generating function of melonic graphs*:

$$T(z) = 1 + z\,(T(z))^4\,.$$

Generating functions of our objects

u marks half the number of vertices
(i.e., for $p \in \frac{1}{2}\mathbb{Z}_+$, u^p weight given to a MO Feynman graph with $2p$ vertices)

generating function for:

- unbroken chains of type L (or R)

$$u^2 \frac{1}{1-u} = u^2 + u^3 + \dots$$

- even straight chains

$$u^2 \frac{1}{1-u^2} = \frac{u^2}{1-u^2} = u^2 + u^4 + u^6 + \dots$$

- odd straight chains

$$u^3 \frac{1}{1-u^2} = \frac{u^3}{1-u^2} = u^3 + u^5 + u^6 + \dots$$

etc.

More generating functions

putting together the generating functions of all contributions
$\Longrightarrow G_S^{(\delta)}(u)$ - the generating function of rooted melon-free MO-graphs of reduced scheme S of degree δ,

$$G_S^{(\delta)}(u) = u^p \frac{u^{2a}}{(1-u)^a} \frac{u^{2s_e}}{(1-u^2)^{s_e}} \frac{u^{3s_o}}{(1-u^2)^{s_o}} \frac{6^b u^{2b}}{(1-3u)^b (1-u)^b}.$$

b - the number of broken chain-vertices
a - the number of unbroken chain-vertices of type L or R
s_e - the number of even straight chain-vertices,
s_o - the number of odd straight chain-vertices.

Generating functions

This simplifies to

$$G_S^{(\delta)}(u) = \frac{6^b u^{p+2c+s_o}}{(1-u)^{c-s}(1-u^2)^s(1-3u)^b}.$$

c - the total number of chain-vertices

$s = s_e + s_o$ - the total number of straight chain-vertices

MO generating functions

$F_S^{(\delta)}(z)$ - the generating function of graphs of reduced scheme S

$$F_S^{(\delta)}(z) = T(z)\frac{6^b U(z)^{p+2c+s_o}}{(1-U(z))^{c-s}(1-U(z)^2)^s(1-3U(z))^b},$$

$U(z) := zT(z)^4 = T(z) - 1$

$F^{(\delta)}(z)$ - the generating function of rooted MO-graphs of degree δ

$$F^{(\delta)}(z) = \sum_{S\in\mathcal{S}_\delta} F_S^{(\delta)}(z).$$

$\mathcal{S}_\delta$ - the (finite) set of reduced schemes of degree δ.

Singularity order - dominant schemes

$T(z)$ has its main singularity at

$$z_0 := 3^3/2^8,$$

$T(z_0) = 4/3$, and $1 - 3U(z) \sim_{z\to z_0} 2^{3/2}3^{-1/2}(1 - z/z_0)^{1/2}$.

R. Gurău and G. Schaeffer, arXiv:1307.5279[math.CO]

$$\Longrightarrow (1 - 3U(z))^{-b} \sim_{z\to z_0} (1 - z/z_0)^{-b/2}$$

$\Longrightarrow$ the dominant terms are those for which b is maximized.

the larger b, the larger the singularity order

A reduced scheme S of degree $\delta \in \frac{1}{2}\mathbb{Z}_+$ is called **dominant** if it maximizes (over reduced schemes of degree δ) the number b of broken chain-vertices.

Bound on the number of broken vertices

$$b \leq 4\delta - 1.$$

Proof. Iterative removal of broken chains, leading (again) to some tree T.

If $b = 4\delta - 1$, then:

- all broken chain-vertices are separating
- the component containing the root has degree 0
- all the components of positive degree and the component containing the root are leaves of T, and the other components of degree 0 have 3 neighbors in T
- all positive degree components have degree $1/2$

MO dominant schemes and rooted binary trees

Theorem

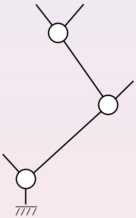
For $\delta \in \frac{1}{2}\mathbb{Z}_+^*$, the dominant schemes arise from rooted binary trees with

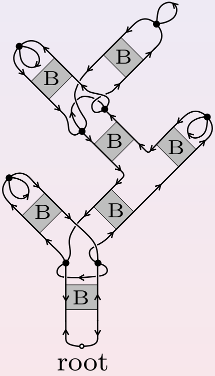
- $2\delta + 1$ *leaves,*
- $2\delta - 1$ *inner nodes, and*
- $4\delta - 1$ *edges,*

where

- *the root-leaf is occupied by the rooted cycle-graph,*
- *the 2δ other leaves are occupied by (cw or ccw) infinity graphs,*
- *the $4\delta - 1$ edges are occupied by separating broken chain-vertices.*

$\delta = 2$ example





- (a) A rooted binary tree (5 leaves, 3 internal nodes, 7 edges)
- (b) A dominant scheme associated to the tree (a)

The double scaling limit of the MO tensor model

R. Gurău, A. Tanasă, D. Youmans, Europhys. Lett. (2015)

The dominant configurations in the double scaling limit are the dominant schemes

The successive coefficient functions $Z_g(\lambda)$ as well diverge at the same critical value of the coupling $\lambda = \lambda_c$
contributions from higher degree are enhanced as $\lambda \to \lambda_c$

$\kappa^{-1} := N^{\frac{1}{2}}(1 - \lambda/\lambda_c)$
the partition function expansion:

$$Z = \sum_{\bar{\omega}} N^{3-\bar{\omega}} f_{\bar{\omega}}$$

double scaling limit: $N \to \infty$, $\lambda \to \lambda_c$ while holding fixed κ

contribution from all degree tensor graphs

similar behaviour to the matrix model double scaling limit

The $O(N)^3$-invariant tensor model

Main complication:
two types of quartic invariant interactions (and hence two coupling constants) are considered

The $O(N)^3$-tensor model

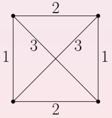
S. Carrozza, A. T., 2015 (arXiv:1512.06718) Lett. Math. Phys. (2016)

- The tensor ϕ_{abc} is invariant under the action of $O(N)^3$:

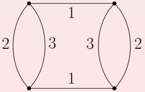
$$\phi_{abc} \to \phi'_{a'b'c'} = \sum_{a,b,c=1}^{N} O^1_{a'a} O^2_{b'b} O^3_{c'c} \phi_{abc} \qquad O^i \in O(N)$$

- Two different quartic invariants:

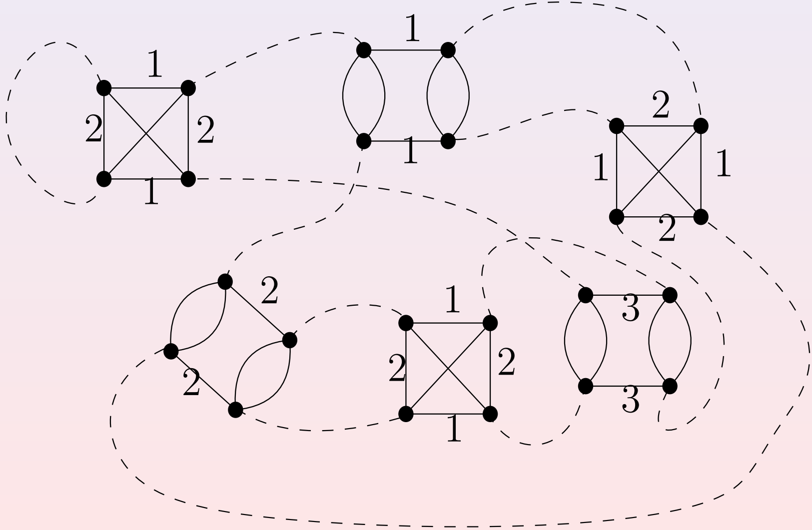
$$I_t(\phi) = \sum_{a,a',b,b',c,c'} \phi_{abc}\phi_{ab'c'}\phi_{a'bc'}\phi_{a'b'c} =$$



$$I_{p,1}(\phi) = \sum_{a,a',b,b',c,c'} \phi_{abc}\phi_{a'bc}\ \phi_{ab'c'}\phi_{a'b'c'} =$$



An example of Feynman graph of the model



The large N limit expansion

The free energy admits a large N expansion

$$F_N(\lambda_1, \lambda_2) = \ln Z_N(\lambda_1, \lambda_2) = \sum_{\bar{\mathcal{G}} \in \bar{\mathbb{G}}} N^{3-\omega(\bar{\mathcal{G}})} \mathcal{A}(\bar{\mathcal{G}}). \qquad (1)$$

where the degree is:

$$\omega(\bar{\mathcal{G}}) = 3 + \frac{3}{2} n_t(\bar{\mathcal{G}}) + 2 n_p(\bar{\mathcal{G}}) - F(\bar{\mathcal{G}}) \qquad (2)$$

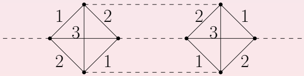
Two types of melonic graphs

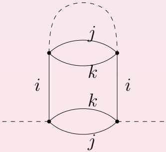
$$\omega(\bar{\mathcal{G}}) = 3 + \frac{3}{2} n_t(\bar{\mathcal{G}}) + 2 n_p(\bar{\mathcal{G}}) - F(\bar{\mathcal{G}})$$

Definition

Melons are the graphs of vanishing degree $\omega(\bar{\mathcal{G}}) = 0$

two types of interaction $\rightarrow$ two types of melonic graphs:

Type I: 

Type II: 

Again on schemes

- Recall that a scheme (of degree ω) is a "blueprint" that tells us how to obtain graphs of the same degree ω.

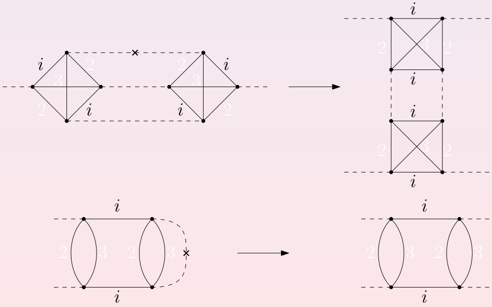
Recall the general idea: Identify operations that leave the degree invariant and use them to repackage all the graphs that differ only by the applications of these operations

Melonic moves are such graphic operations.

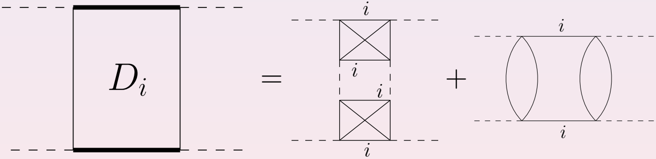
Dipoles

Definition

A dipole is a 4-point graph obtained by cutting an edge in an elementary melon.



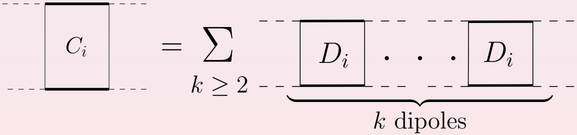
Dipoles



(3)

Definition

Chains are the 4-point functions obtained by connecting an arbitrary number of dipoles.



(4)

Schemes

Definition

The scheme $\mathcal{S}$ of a 2-point graph $\mathcal{G}$ is obtained by

1. Removing all melonic 2-point subgraphs in $\mathcal{G}$
2. Replacing all maximal chains with chain-vertices and all dipoles with dipole-vertex of the same color.

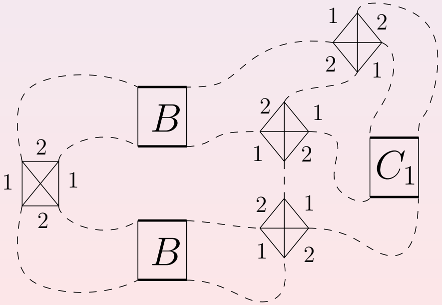


Figure: An example of scheme

Finiteness of the number of schemes

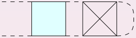
Theorem

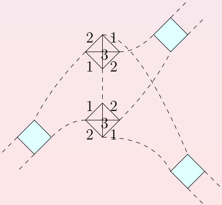
The set of schemes of a given degree is finite in the quartic $O(N)^3$-invariant tensor model.

Structure of the dominant schemes

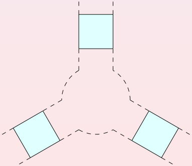
Theorem

The dominant schemes of degree ω are given bijectively by rooted plane binary trees with $4\omega - 1$ edges, s. t.

- *The root of the tree corresponds to the two external legs of the 2-point function.*
- *Edges of the tree correspond to broken chains.*
- *The leaves are tadpoles:* 
- *There are two types of internal nodes,*



,



Generating function of dominant scheme

The generating function associated to a dominant schemes is

$$\begin{aligned} G^{\omega}_{\mathcal{T}}(t,\mu) &= (3t^{\frac{1}{2}})^{2\omega}(1+6t)^{2\omega-1}B(t,\mu)^{4\omega-1} \\ &= (3t^{\frac{1}{2}})^{2\omega}(1+6t)^{2\omega-1}\frac{6^{4\omega-1}U^{8\omega-2}}{((1-U)(1-3U))^{4\omega-1}} \end{aligned} \quad (5)$$

where B is the generation functions of broken chains and U is th generation function of dipoles.

Summing over the different trees and taking into account melonic insertions at the root gives

$$\begin{aligned} G^{\omega}_{\text{dom}}(t,\mu) &= M(t,\mu) \sum_{\substack{\mathcal{T} \\ 2\omega \text{ leaves}}} G^{\omega}_{\mathcal{T}}(t,\mu) \\ &= \text{Cat}_{2\omega-1}\, M(t,\mu) G^{\omega}_{\mathcal{T}}(t,\mu) \end{aligned} \quad (6)$$

where M is the generation functions of melons.

Double scaling parameter

Near critical point

$$G^{\omega}_{dom}(t,\mu) \underset{t\to t_c(\mu)}{\sim} N^{3-\omega} M_c(\mu)\, \mathrm{Cat}_{2\omega-1}\, 9^{\omega}\, t_c^{\omega}\, (1+6t_c)^{2\omega-1}$$
$$\times \left(\frac{1}{\left(1-\frac{4}{3}t_c(\mu)\mu M_c(\mu)\right) K(\mu)\sqrt{1-\frac{t}{t_c(\mu)}}} \right)^{4\omega-1} \quad (7)$$

- The double scaling parameter $\kappa(\mu)$ is the quantity to hold fixed when sending $N \to +\infty, t \to t_c(\mu)$.
- dominant schemes of all degree ω contribute in the double scaling limit

One has

$$\kappa(\mu)^{-1} = \frac{1}{3}\frac{1}{t_c(\mu)^{\frac{1}{2}}\,(1+6t_c(\mu))}\left(\left(1-\frac{4}{3}t_c(\mu)\mu M_c(\mu)\right)K(\mu)\right)^2\left(1-\frac{t}{t_c(\mu)}\right)N^{\frac{1}{2}} \quad (8)$$

2−point function in the double scaling limit

$$G_2^{DS}(\mu) = N^{-3} \sum_{\omega \in \mathbb{N}/2} G_{dom}^{\omega}(\mu)$$
$$= M_c(\mu) \left(1 + N^{-\frac{1}{4}} \sqrt{3} \frac{t_c(\mu)^{\frac{1}{4}}}{(1 + 6t_c(\mu))^{\frac{1}{2}}} \frac{1 - \sqrt{1 - 4\kappa(\mu)}}{2\kappa(\mu)^{\frac{1}{2}}} \right) \quad (9)$$

convergent for $\kappa(\mu) \leq \frac{1}{4}$.

tensor double scailing limit is summable
(different behaviour with respect to the celebrated matrix models case)

Implementation of this approach for other models

D. Benedetti *et. al.*, *Annales IHP* **D** (in press)

double-scaling limit mechanism of $U(N)^2 \times O(D)$ multi-matrix models

Conclusion and perspectives

- Bottom line:
  purely combinatorial techniques can be used to study physical mechanisms, such as the double scaling limit for various tensor models

- Perspectives - Implementation of this combinatorial approach for other models:
  - the multi-matrix models with several types of interactions (see Victor Nador's talk on Friday!)
  - the prismatic tensor model
    (work in progress with Thomas Krajewski and Thomas Muller (Thomas Muller Masters internship))

    $O(N)^3$ tensor model with prismatic interaction

    S. Giombi *et. al.*, arXiv:1808.04344, *Phys.Rev.* **D** (2018)

Danke für Ihre Aufmerksamkeit!

Comparison with the colored case

The dominant schemes differ:

for the colored model, for degree $\delta \in \mathbb{Z}_+$, the dominant schemes are associated to rooted binary trees with $\delta + 1$ leaves (and $\delta - 1$ inner nodes), where the root-leaf is occupied by a root-melon, while the δ non-root leaves are occupied by the unique scheme of degree 1.