STRUCTURES
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# Yang-Mills on a certain 'quantum space'

*Random Geometry in Heidelberg*, May 2022

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Based on 1912.13288*; 2007.10914*; 2102.06999*; 2105.01025*,†; 2111.02858†,*

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# Introduction

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  This can be governed by a classical action.

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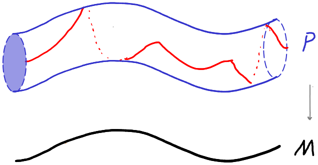
  A model of space~~time~~ not based on a smooth manifold.
  This can be governed by a classical action.

- Classical SU($n$)-Yang-Mills theory is geometrically modelled on connections on a SU($n$)-principal bundle on a smooth space $M$.




We want to replace $M$ by a ‘quantum space’ based on *noncommutative geometry* (NCG).

- NCG and physics: The Standard Model from the Spectral Action

$$
\begin{gathered}
-\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4} g_s^2 f^{abc} f^{ade} g^b_\mu g^c_\nu g^d_\mu g^e_\nu + \\
\tfrac{1}{2} i g_s^2 (\bar{q}^\sigma_i \gamma^\mu q^\sigma_j) g^a_\mu + \bar{G}^a \partial^2 G^a + g_s f^{abc} \partial_\mu \bar{G}^a G^b g^c_\mu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - \\
M^2 W^+_\mu W^-_\mu - \tfrac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c_w^2} M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \\
\tfrac{1}{2}\partial_\mu H \partial_\mu H - \tfrac{1}{2} m_h^2 H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+ \phi^- - \tfrac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \\
\tfrac{1}{2c_w^2} M \phi^0 \phi^0 - \beta_h [\tfrac{2M^2}{g^2} + \tfrac{2M}{g} H + \tfrac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)] + \tfrac{2M^4}{g^2}\alpha_h - \\
i g c_w [\partial_\nu Z^0_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + \\
Z^0_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)] - i g s_w [\partial_\nu A_\mu (W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - A_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu (W^+_\nu \partial_\nu W^-_\mu - \\
W^-_\nu \partial_\nu W^+_\mu)] - \tfrac{1}{2} g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \tfrac{1}{2} g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + \\
g^2 c_w^2 (Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2 (A_\mu W^+_\mu A_\nu W^-_\nu - \\
A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w [A_\mu Z^0_\nu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - \\
2 A_\mu Z^0_\mu W^+_\nu W^-_\nu] - g\alpha [H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-] - \tfrac{1}{8} g^2 \alpha_h [H^4 + \\
(\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2 H^2] - \\
g M W^+_\mu W^-_\mu H - \tfrac{1}{2} g \tfrac{M}{c_w^2} Z^0_\mu Z^0_\mu H - \tfrac{1}{2} i g [W^+_\mu (\phi^0 \partial_\mu \phi^- - \phi^- \partial_\mu \phi^0) - \\
W^-_\mu (\phi^0 \partial_\mu \phi^+ - \phi^+ \partial_\mu \phi^0)] + \tfrac{1}{2} g [W^+_\mu (H \partial_\mu \phi^- - \phi^- \partial_\mu H) - \\
W^-_\mu (H \partial_\mu \phi^+ - \phi^+ \partial_\mu H)] + \tfrac{1}{2} g \tfrac{1}{c_w} (Z^0_\mu (H \partial_\mu \phi^0 - \phi^0 \partial_\mu H) - \\
i g \tfrac{s_w^2}{c_w} M Z^0_\mu (W^+_\mu \phi^- - W^-_\mu \phi^+) + i g s_w M A_\mu (W^+_\mu \phi^- - W^-_\mu \phi^+) - \\
i g \tfrac{1 - 2c_w^2}{2c_w} Z^0_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) + i g s_w A_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) - \\
\tfrac{1}{4} g^2 W^+_\mu W^-_\mu [H^2 + (\phi^0)^2 + 2\phi^+\phi^-] - \tfrac{1}{4} g^2 \tfrac{1}{c_w^2} Z^0_\mu Z^0_\mu [H^2 + (\phi^0)^2 + \\
2(2s_w^2 - 1)^2 \phi^+\phi^-] - \tfrac{1}{2} g^2 \tfrac{s_w^2}{c_w} Z^0_\mu \phi^0 (W^+_\mu \phi^- + W^-_\mu \phi^+) -
\end{gathered}
$$

$$
\begin{gathered}
\tfrac{1}{2} i g^2 \tfrac{s_w^2}{c_w} Z^0_\mu H (W^+_\mu \phi^- - W^-_\mu \phi^+) + \tfrac{1}{2} g^2 s_w A_\mu \phi^0 (W^+_\mu \phi^- + \\
W^-_\mu \phi^+) + \tfrac{1}{2} i g^2 s_w A_\mu H (W^+_\mu \phi^- - W^-_\mu \phi^+) - g^2 \tfrac{s_w}{c_w} (2c_w^2 - \\
1) Z^0_\mu A_\mu \phi^+ \phi^- - g^1 s_w^2 A_\mu A_\mu \phi^+ \phi^- - \bar{e}^\lambda (\gamma\partial + m_e^\lambda) e^\lambda - \bar{\nu}^\lambda \gamma\partial \nu^\lambda - \\
\bar{u}^\lambda_j (\gamma\partial + m_u^\lambda) u^\lambda_j - \bar{d}^\lambda_j (\gamma\partial + m_d^\lambda) d^\lambda_j + i g s_w A_\mu [-(\bar{e}^\lambda \gamma^\mu e^\lambda) + \\
\tfrac{2}{3} (\bar{u}^\lambda_j \gamma^\mu u^\lambda_j) - \tfrac{1}{3} (\bar{d}^\lambda_j \gamma^\mu d^\lambda_j)] + \tfrac{ig}{4c_w} Z^0_\mu [(\bar{\nu}^\lambda \gamma^\mu (1 + \gamma^5) \nu^\lambda) + \\
(\bar{e}^\lambda \gamma^\mu (4s_w^2 - 1 - \gamma^5) e^\lambda) + (\bar{u}^\lambda_j \gamma^\mu (\tfrac{4}{3} s_w^2 - 1 - \gamma^5) u^\lambda_j) + (\bar{d}^\lambda_j \gamma^\mu (1 - \\
\tfrac{8}{3} s_w^2 - \gamma^5) d^\lambda_j)] + \tfrac{ig}{2\sqrt{2}} W^+_\mu [(\bar{\nu}^\lambda \gamma^\mu (1 + \gamma^5) e^\lambda) + (\bar{u}^\lambda_j \gamma^\mu (1 + \\
\gamma^5) C_{\lambda\kappa} d^\kappa_j)] + \tfrac{ig}{2\sqrt{2}} W^-_\mu [(\bar{e}^\lambda \gamma^\mu (1 + \gamma^5) \nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\lambda\kappa} \gamma^\mu (1 + \\
\gamma^5) u^\lambda_j)] + \tfrac{ig}{2\sqrt{2}} \tfrac{m_e^\lambda}{M} [-\phi^+ (\bar{\nu}^\lambda (1 - \gamma^5) e^\lambda) + \phi^- (\bar{e}^\lambda (1 + \gamma^5) \nu^\lambda)] - \\
\tfrac{g}{2} \tfrac{m_e^\lambda}{M} [H (\bar{e}^\lambda e^\lambda) + i\phi^0 (\bar{e}^\lambda \gamma^5 e^\lambda)] + \tfrac{ig}{2M\sqrt{2}} \phi^+ [-m_d^\kappa (\bar{u}^\lambda_j C_{\lambda\kappa} (1 - \\
\gamma^5) d^\kappa_j) + m_u^\lambda (\bar{u}^\lambda_j C_{\lambda\kappa} (1 + \gamma^5) d^\kappa_j] + \tfrac{ig}{2M\sqrt{2}} \phi^- [m_d^\lambda (\bar{d}^\lambda_j C^\dagger_{\lambda\kappa} (1 + \\
\gamma^5) u^\kappa_j) - m_u^\kappa (\bar{d}^\lambda_j C^\dagger_{\lambda\kappa} (1 - \gamma^5) u^\kappa_j] - \tfrac{g}{2} \tfrac{m_u^\lambda}{M} H (\bar{u}^\lambda_j u^\lambda_j) - \\
\tfrac{g}{2} \tfrac{m_d^\lambda}{M} H (\bar{d}^\lambda_j d^\lambda_j) + \tfrac{ig}{2} \tfrac{m_u^\lambda}{M} \phi^0 (\bar{u}^\lambda_j \gamma^5 u^\lambda_j) - \tfrac{ig}{2} \tfrac{m_d^\lambda}{M} \phi^0 (\bar{d}^\lambda_j \gamma^5 d^\lambda_j)
\end{gathered}
$$

...this 'fits' in

$$\mathrm{Tr}(f(D/\Lambda)) + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle$$

Num. of generations and $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \longrightarrow$ NCG $\longrightarrow$ *Classical* Lagrangian of the Standard Model

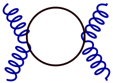
[Chamseddine-Connes-Marcolli *ATMP* '07 (Euclidean); J. Barrett *J. Math. Phys.* '07 (Lorenzian)]

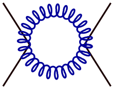
go to sketch of proof ▻

- Why do we want SU($n$)-Yang-Mills theory on that space?

  The natural quantum theory to consider could be 'gravity + matter'

  [Donà, Eichhorn, Percacci, *PRD* 2014]:





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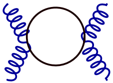

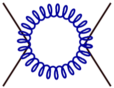

- **What is the present 'quantum space' about?**

  This talk's model is based on Connes' *noncommutative (nc) geometry* = nc topology [Gelfand, Najmark *Mat. Sbornik* '43] + metric [A. Connes, *NCG* '94]

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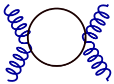

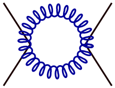

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$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

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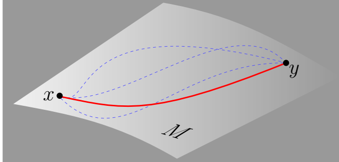
Replace spin manifold $(M, g)$ by $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

## Connes' geodesic distance




$\gamma : \mathbb{R} \to M$

$$\inf_{\gamma \text{ as above}} \{ \int_\gamma \mathrm{d}s \} = d(x, y)$$

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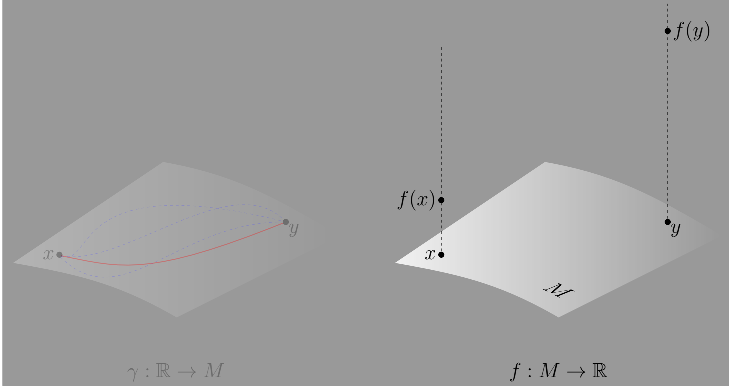

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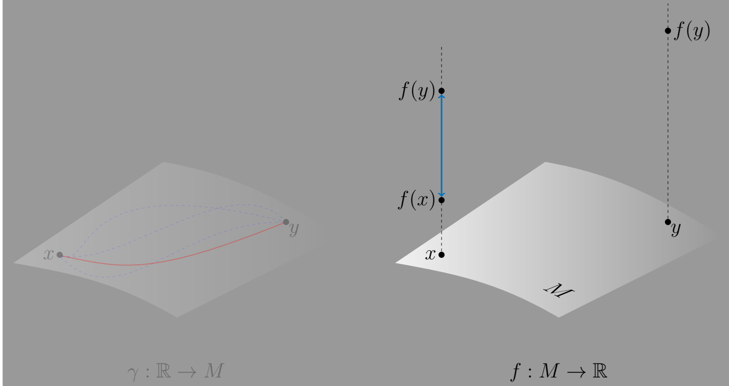


$$|f(x) - f(y)|$$

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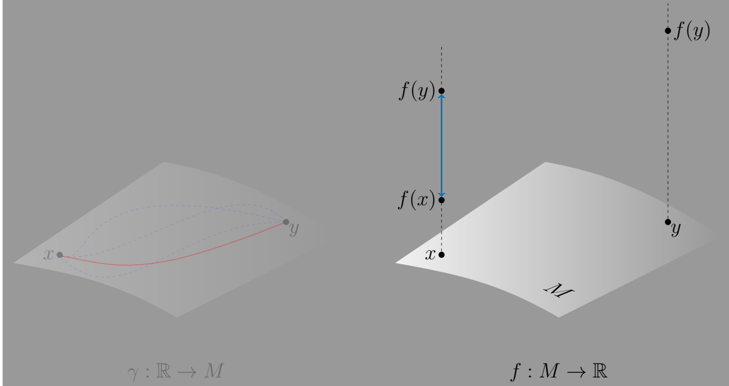


$$\sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| \ : \ ||D_M f - f D_M|| \leqslant 1 \}$$

Replace spin manifold $(M, g)$ by $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

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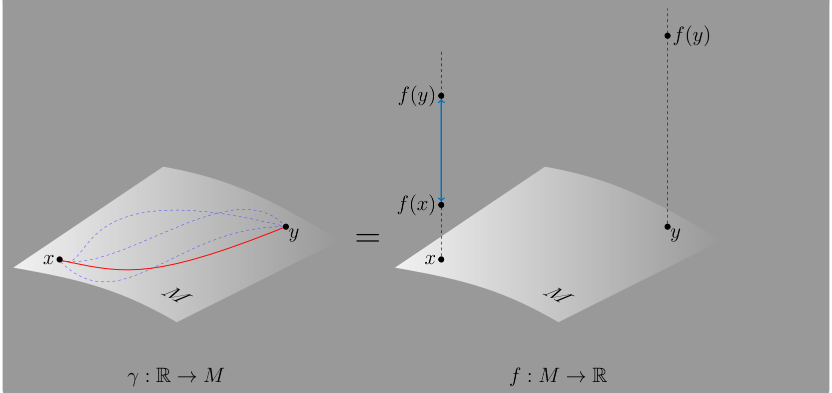
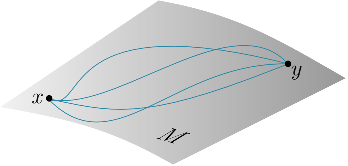



$$\inf_{\gamma \text{ as above}} \{\textstyle\int_\gamma \mathrm{d}s\} = d(x, y) = \sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| : ||D_M f - f D_M|| \leqslant 1\}$$

go to examples ∇

# Possible application to (Euclidean) quantum gravity



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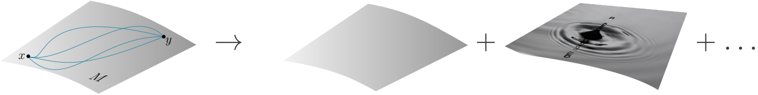
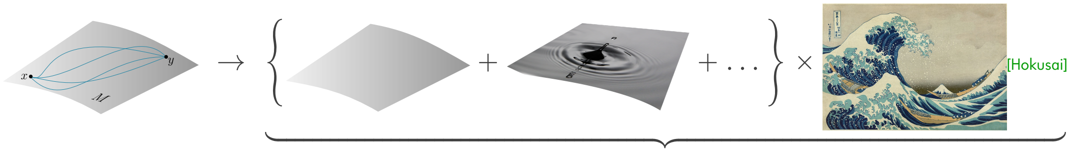


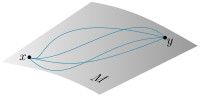

Quantum superposition of geometries

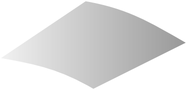
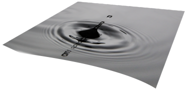
# Possible application to (Euclidean) quantum gravity




Quantum superposition of geometries (small perturbations + instantons)

# Possible application to (Euclidean) quantum gravity




$\to \Big\{$  $+$  $+ \dots \Big\} \times$  [Hokusai]

Quantum superposition of geometries

$$\text{functional integral} \xrightarrow{\text{paradigm shift}} \text{operator integral}$$

$$\int_{\text{metric}} e^{-\frac{1}{\hbar} S_{\text{EH}}[g]} \, \mathrm{d}g \xrightarrow{\text{Einstein-Hilbert} \,\longrightarrow\, \text{spectral}} \int_{\text{Dirac}} e^{-\frac{1}{\hbar} \mathrm{Tr}\, f(D)} \, \mathrm{d}D$$

(hard to define for manifolds)

$f : \mathbb{R} \longrightarrow \mathbb{R}$ makes $f(D)$ traceclass

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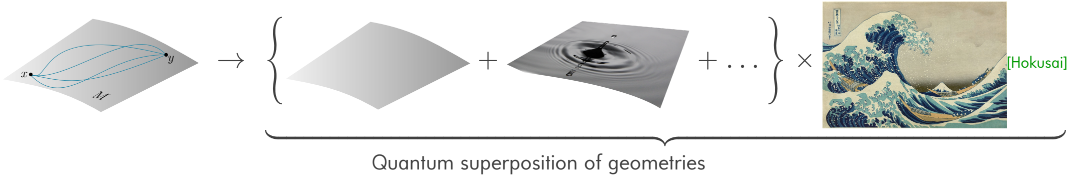


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« The far distant goal is to set up a functional integral evaluating spectral observables $\mathcal{S}$ as

(1.892) $$\langle \mathcal{S} \rangle = \mathcal{N} \int \mathcal{S} \, e^{-\text{Tr}(f(D/\Lambda)) - \frac{1}{2}\langle J\psi, D\,\psi\rangle - \rho(e,D)} \, \mathcal{D}[\psi]\mathcal{D}[D]\mathcal{D}[e],$$ »

[Connes Marcolli, *NCG, QFT and motives*, 2007]

# Commutative spectral triples

A spin manifold $M$ yields $(A_M, H_M, D_M)$

- $A_M = C^\infty(M)$ is a comm. $*$-algebra
- $H_M := L^2(M, \mathbb{S})$ a repr. of $A_M$
- $D_M = -\mathrm{i}\gamma^\mu(\partial_\mu + \omega_\mu)$ is self-adjoint
- for each $a \in A_M$, $[D_M, a]$ is bounded, and in fact $[D_M, x^\mu] = -\mathrm{i}\gamma^\mu$

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...

go to more on spin geometry ▷

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$\cdots$

go to more on spin geometry ▻

A *spectral triple* $(A, H, D)$ consists of

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The 'commutative case' motivates

$$\Omega^1_D(A) := \Big\{ \sum_{\text{(finite)}} b[D, a] \mid a, b \in A \Big\}$$

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go to more on spin geometry ⊳

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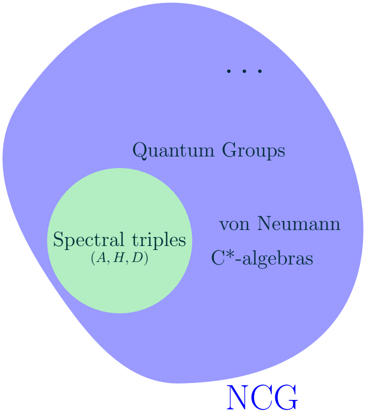
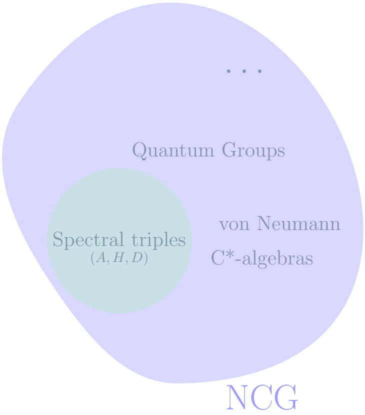
**Reconstruction Theorem:** [A, Connes, *JNCG* '13] *(quite roughly formulated)*
Commutative spectral triples$^{+\textit{some more axioms}}$ are Riemannian manifolds.


. . .
Quantum Groups
von Neumann
Spectral triples
$(A, H, D)$
C*-algebras
NCG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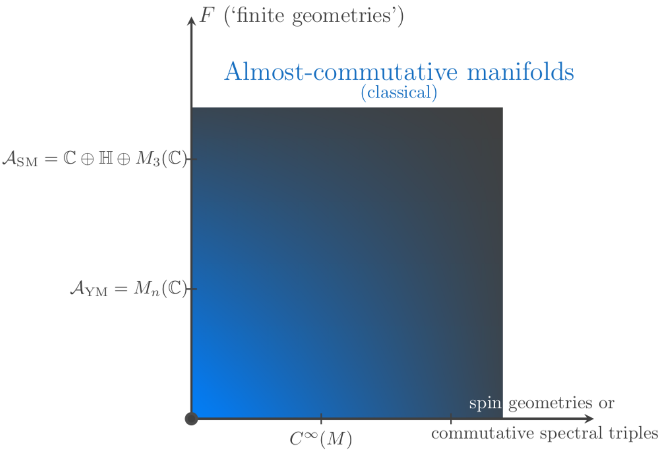
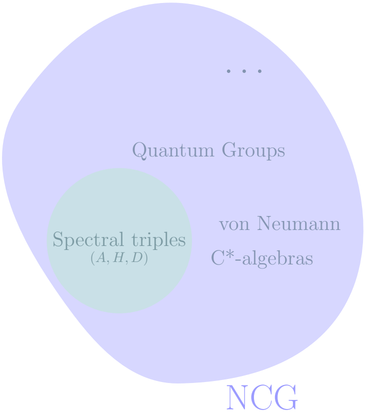

$F$ ('finite geometries')
Almost-commutative manifolds
(classical)
$\mathcal{A}_{\mathrm{SM}} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$
$\mathcal{A}_{\mathrm{YM}} = M_n(\mathbb{C})$
$C^\infty(M)$
spin geometries or
commutative spectral triples



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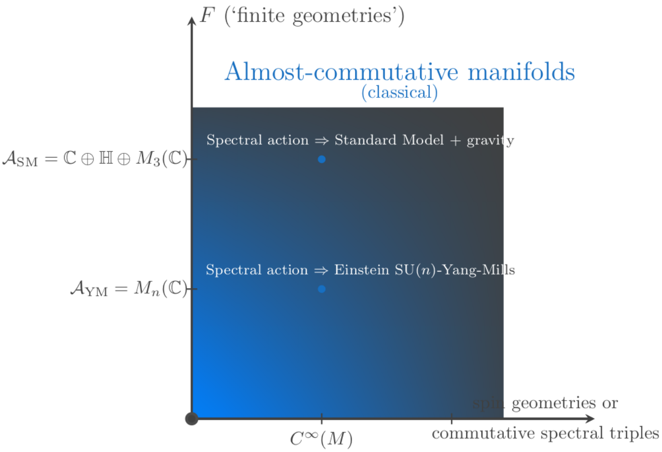

$F$ ('finite geometries')
Almost-commutative manifolds
(classical)
Spectral action ⇒ Standard Model + gravity
$\mathcal{A}_{\mathrm{SM}} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$
Spectral action ⇒ Einstein SU($n$)-Yang-Mills
$\mathcal{A}_{\mathrm{YM}} = M_n(\mathbb{C})$
geometries or
commutative spectral triples
$C^\infty(M)$

# NCG toolkit in high energy physics

- On a spectral triple $(A, H, D)$ the (bosonic) classical action is given by

  $$S(D) = \mathrm{Tr}_H\, f(D/\Lambda)$$ [Chamseddine-Connes *CMP* '97]

  for a bump function $f$, $\Lambda$ a scale. It's computed with heat kernel expansion

  [P. Gilkey, *J. Diff. Geom.* '75]

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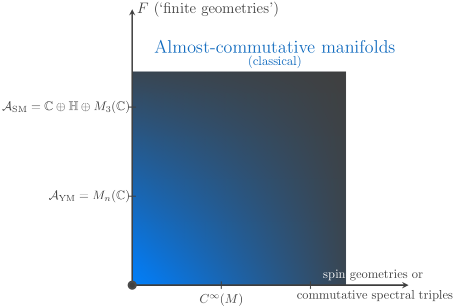

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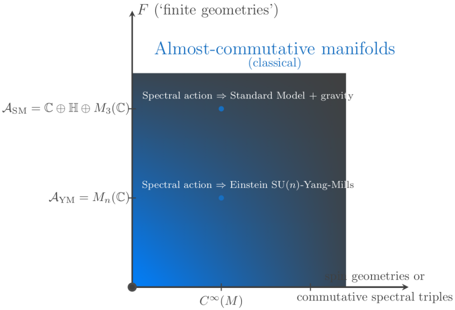

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  $$\mathbb{A}' = u\mathbb{A}u^{-1} + u\mathrm{d}u^{-1} \qquad u \in \mathrm{Maps}(M, G)$$

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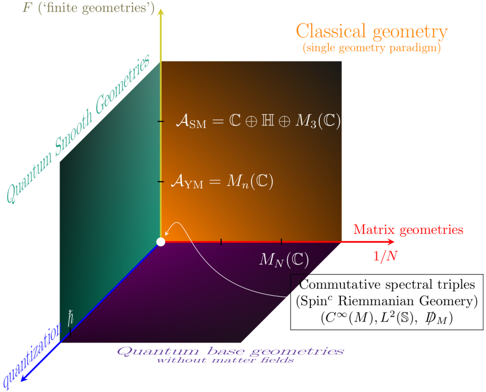
- given $(A, H, D)$ and a Morita equivent algebra $B$ (i.e. $\mathrm{End}_A(E) \cong B$) yields new $(B, E \otimes_A H, D')$. For $A = B$, in fact a tower

$$\left\{(A, H, D + \omega \pm J\omega J^{-1})\right\}_{\omega \in \Omega^1_D(A)}$$

$$D_\omega \mapsto \mathrm{Ad}(u) D_\omega \mathrm{Ad}(u)^* = D_{\omega_u}$$

$$\omega \mapsto \omega_u = u\omega u^* + u[D, u^*] \qquad u \in \mathcal{U}(A)$$ skip cube

# Organization



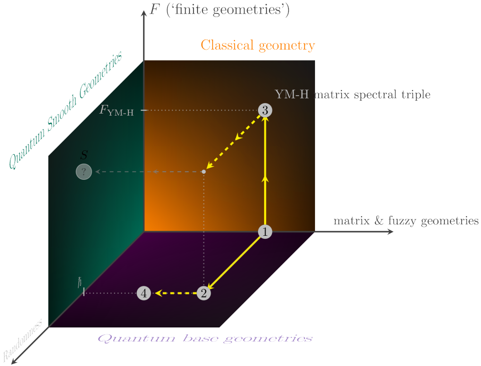

**Aim:** Make sense of

$$\mathcal{Z} = \int_{\text{Dirac}} e^{-\operatorname{Tr}_H f(D)} \mathrm{d}D$$

- *Plane* $(\hbar, 1/N, 0)$ *of 'base geometries'*
- *Plane* $(\hbar, 0, F) = \lim_{N\to\infty}(\hbar, 1/N, F)$
- *Plane* $(0, 1/N, F) = \lim_{\hbar\to 0}(\hbar, 1/N, F)$ *of classical geometries*

[CP 2105.01025]

# Organization




1. Matrix Geometries
   [J. Barrett, *J. Math. Phys.* 2015]
2. Dirac ensembles [J. Barrett, L. Glaser, *J. Phys. A* 2016] and how to compute the spectral action [CP 1912.13288]
3. Gauge matrix spectral triples *(this talk)*
   [CP 2105.01025]
4. Functional Renormalization [CP 2007.10914] and [CP 2111.02858]

# II. Fuzzy Geometries and Multimatrix Models

A *fuzzy geometry* of signature $(p, q)$, so $\eta = \mathrm{diag}(+_p, -_q)$, consists of

- $A = M_N(\mathbb{C})$
- $H = \mathbb{S} \otimes M_N(\mathbb{C})$, with $\mathbb{S}$ a $\mathbb{C}\ell(p, q)$-module

. . . (axioms for $D$ omitted, `go to axioms` $\triangledown$) . . .

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- Fixing conventions for $\gamma$'s, characterization of $D$ in even dimensions:

$$D = \sum_J \Gamma^J_{\text{s.a.}} \otimes \{H_J, \ \cdot\ \} + \sum_J \Gamma^J_{\text{anti.}} \otimes [L_J, \ \cdot\ ]$$

  multi-index $J$ monot. increasing, $|J|$ odd [J. Barrett, *J. Math. Phys.* '15], $H_J^* = H_J$, $L_J^* = -L_J$

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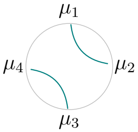
- Examples: [J. Barrett, L. Glaser, *J. Phys. A* 2016]
  - $D_{(1,1)} = \gamma^1 \otimes [L, \,\cdot\,] + \gamma^2 \otimes \{H, \,\cdot\,\}$
  - $D_{(0,4)} = \sum_\mu \gamma^\mu \otimes [L_\mu, \,\cdot\,] + \gamma^{\hat\mu} \otimes \{H_{\hat\mu}, \,\cdot\,\}$

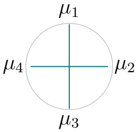
  so we will get double traces from $\mathrm{Tr}_H = \mathrm{Tr}_{\mathbb{S}} \otimes \mathrm{Tr}_{M_N(\mathbb{C})} = \mathrm{Tr}_{\mathbb{S}} \otimes \mathrm{Tr}_N^{\otimes 2}$

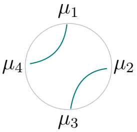
Notation: $\mathrm{Tr}_V\, X$ is the trace on operators $X : V \to V$, $\mathrm{Tr}_V\, 1 = \dim V$. So $\mathrm{Tr}_N\, 1 = N$ but $\mathrm{Tr}_{M_N(\mathbb{C})}(1) = N^2$.

- A tool to organize the fuzzy spectral action is chord diagrams:

$$\mathrm{Tr}_{\mathbb{S}}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}) = \dim \mathbb{S}\big(\overbrace{\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}}^{\theta^{\mu_1\mu_2\mu_3\mu_4}} + \overbrace{(-)\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}}^{\xi^{\mu_1\mu_2\mu_3\mu_4}} + \overbrace{\eta^{\mu_2\mu_3}\eta^{\mu_1\mu_4}}^{\zeta^{\mu_1\mu_2\mu_3\mu_4}}\big)$$

$\theta^{\mu_1\mu_2\mu_3\mu_4} =$ 


$\xi^{\mu_1\mu_2\mu_3\mu_4} =$ 


$\zeta^{\mu_1\mu_2\mu_3\mu_4} =$ 

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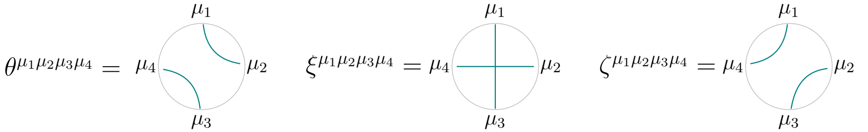
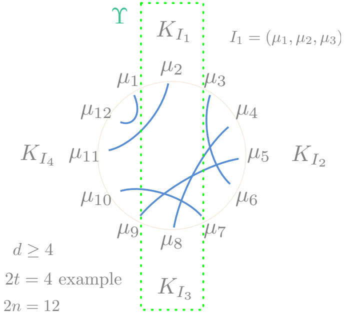



- for dimension-$d$ geometries, the combinatorial formula [CP '19] reads

$$\frac{1}{\dim \mathbb{S}}\mathrm{Tr}(D^{2t}) = \overbrace{\sum_{I_1,\ldots,I_{2t}\in\Lambda_d^-}}^{\text{if } J\in\Lambda_d,\, \mathrm{d}x^J \neq 0 \text{ on } \mathbb{R}^d} \Bigg\{ \overbrace{\sum_{\substack{\chi\in\mathrm{CD}_{2n}\\ 2n=\sum_i |I_i|}} \chi^{I_1\ldots I_{2t}}}^{\text{decorated chord diags}}$$

$$\times \Big( \sum_{\Upsilon\in\mathscr{P}_{2t}} \mathrm{sgn}(I_\Upsilon) \times \mathrm{Tr}_N(K_{I_{\Upsilon^c}}) \times \mathrm{Tr}_N\big[(K^T)_{I_\Upsilon}\big]\Big)\Bigg\}$$

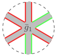


# Multimatrix models with multitraces & ribbon graphs

- The chord-diagram description holds in general dimension and signature [CP '19]

$$\mathcal{Z} = \int_{\text{Dirac}} e^{-\operatorname{Tr}_H f(D)} \mathrm{d}D \quad (\hbar = 1)$$
$$= \int_{M_{p,q}} e^{-N \operatorname{Tr}_N P - \operatorname{Tr}_N^{\otimes 2}(Q_{(1)} \otimes Q_{(2)})} \mathrm{d}\mathbb{X}_{\text{Leb}}$$

- $\mathbb{X} \in M_{p,q}$ = products of $\mathfrak{su}(N)$ and $\mathcal{H}_N$
- $\mathrm{d}\mathbb{X}_{\text{Leb}}$ is the Lebesgue measure on $M_{p,q}$
- $P$, $Q_{(i)}$ in $\mathbb{C}_{\langle k \rangle} = \mathbb{C}\langle X_1, \ldots, X_k \rangle = \mathbb{C}\langle \mathbb{X} \rangle$ are certain noncommutative polynomials
- $\mathcal{Z}_{\text{formal}}$ leads to colored ribbon graphs

$\bar{g}_1 \operatorname{Tr}_N(ABBBAB) \quad \leftrightarrow$ 

$\bar{g}_2 \operatorname{Tr}_N^{\otimes 2}(AABABA \otimes AA) \quad \leftrightarrow$ 

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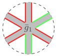
$\bar{g}_1 \operatorname{Tr}_N(ABBBAB) \quad \leftrightarrow$ 

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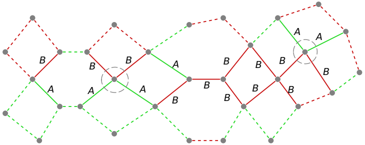
- Multitrace: 'touching interactions' [Klebanov, *PRD* '95], wormholes [Ambjørn-Jurkiewicz-Loll-Vernizzi, *JHEP* '01], 'stuffed maps' [G. Borot *Ann. Inst. Henri Poincaré Comb. Phys. Interact.* '14], AdS/CFT [Witten, `hep-th/0112258`]

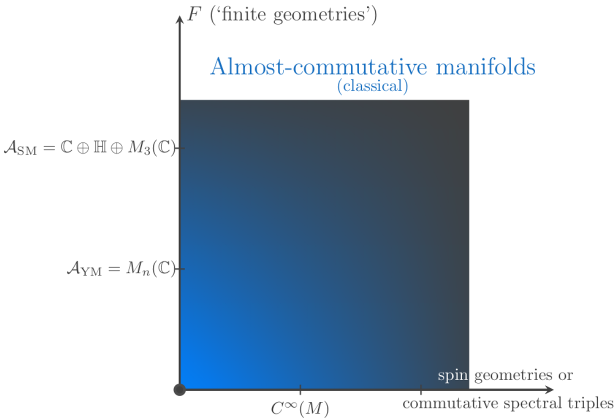
- Ribbon graphs: Enumeration of maps [Brezin, Itzykson, Parisi, Zuber, *CMP* '78], here 'face-worded'



& intersection num. of $\psi$-classes [Kontsevich, *CMP*, '92]

$$\sum_{a_1+\ldots+a_n=\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdot \psi_2^{a_2} \cdots \psi_n^{a_n} \prod_{j=1}^{n} \frac{(2a_j - 1)!!}{s_j^{2a_j+1}}$$
$$= \sum_{G \text{ trivalent of type } (g,n)} \frac{2^{2g-2+n}}{\#\mathrm{Aut}(G)} \prod_{e \in G} \frac{1}{s_{L(e)} + s_{R(e)}}$$

# III. Yang-Mills-Higgs matrix theory



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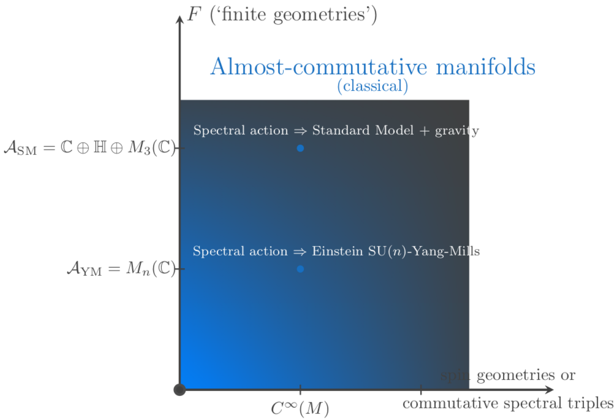


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(hard for almost-commutative manifolds)

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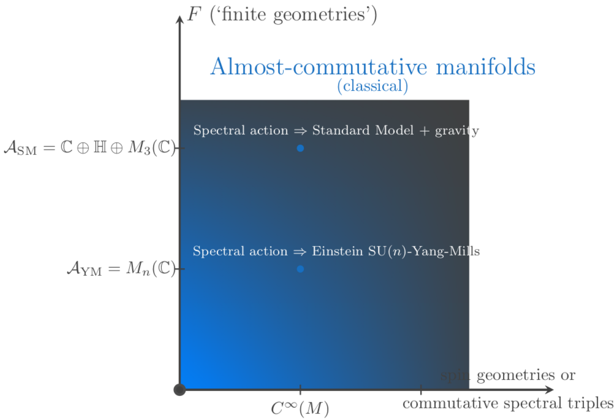


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**Definition** [CP 2105.01025] We define a *gauge matrix spectral triple* $G_\ell \times F$ as the spectral triple product of a fuzzy geometry $G_\ell$ with a finite geometry $F = (A_F, H_F, D_F)$, $\dim A_F < \infty$.

**Lemma-Definition** [CP 2105.01025] Consider a gauge matrix spectral triple $G_\ell \times F$ with

$$F = (M_n(\mathbb{C}), M_n(\mathbb{C}), D_F)$$

and $G_\ell$ Riemannian ($d = 4$) fuzzy geometry on $M_N(\mathbb{C})$, whose fluctuated Dirac op. is

$$D_\omega = \sum_{\mu=0}^{3} \overbrace{\gamma^\mu \otimes (\mathscr{\ell}_\mu + \mathscr{a}_\mu) + \gamma^{\hat\mu} \otimes (\mathscr{x}_\mu + \mathscr{s}_\mu)}^{D_{\text{gauge}}} + \overbrace{\gamma \otimes \Phi}^{D_{\text{Higgs}}}, \qquad \mathscr{a}_\mu = \text{'gauge potential'}, \mathscr{x}_\mu = \text{spin connection?}$$

The *field strength* is given by

$$\mathscr{F}_{\mu\nu} := [\overbrace{\mathscr{\ell}_\mu + \mathscr{a}_\mu}^{d_\mu}, \mathscr{\ell}_\nu + \mathscr{a}_\nu] =: [\mathsf{F}_{\mu\nu}, \ \cdot\ ]$$

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The content of the Spectral Action ...

| Meaning | Random matrix case, flat $d = 4$ Riem. | Smooth operator |
| --- | --- | --- |
| | Tr = trace of ops. $M_N \otimes M_n \to M_N \otimes M_n$ | |
| Derivation | $\mathscr{\ell}_\mu = [L_\mu \otimes 1_n, \ \cdot\ ]$ | $\partial_i$ |
| Gauge potential | $\mathscr{a}_\mu = [A_\mu, \ \cdot\ ]$ | $\mathbb{A}_i$ |
| Higgs field | $\Phi$ | $h$ |
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| Field strength | $[d_\mu, d_\nu] = \overbrace{[\ell_\mu, \ell_\nu]}^{\not\equiv 0} + [\ell_\mu, a_\nu] - [\ell_\nu, a_\mu] + [a_\mu, a_\nu]$ | $[\mathbb{D}_i, \mathbb{D}_j] = \overbrace{[\partial_i, \partial_j]}^{\equiv 0} + \partial_i \mathbb{A}_j - \partial_j \mathbb{A}_i + [\mathbb{A}_i, \mathbb{A}_j]$ |
| Higgs potential | $\mathrm{Tr}(f_2 \Phi^2 + f_4 \Phi^4)$ | $\int_M \left(f_2 \lvert h\rvert^2 + f_4 \lvert h\rvert^4\right) \mathrm{vol}$ |
| Gauge-Higgs coupling | $-\mathrm{Tr}(d_\mu \Phi d^\mu \Phi)$ | $-\int_M \lvert\mathbb{D}_i h\rvert^2 \mathrm{vol}$ |
| Yang-Mills action | $-\frac{1}{4}\mathrm{Tr}(\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu})$ | $-\frac{1}{4}\int_M \mathrm{Tr}_{\mathfrak{su}(n)}(\mathbb{F}_{ij}\mathbb{F}^{ij})\mathrm{vol}$ |

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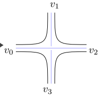
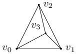
| Higgs field | $\Phi$ | $h$ |
| Covariant derivative | $d_\mu=\ell_\mu+a_\mu$ | $\mathbb{D}_i=\partial_i+\mathbb{A}_i$ |
| Field strength | $[d_\mu,d_\nu]=\overbrace{[\ell_\mu,\ell_\nu]}^{\not\equiv 0}+[\ell_\mu,a_\nu]-[\ell_\nu,a_\mu]+[a_\mu,a_\nu]$ | $[\mathbb{D}_i,\mathbb{D}_j]=\overbrace{[\partial_i,\partial_j]}^{\equiv 0}+\partial_i\mathbb{A}_j-\partial_j\mathbb{A}_i+[\mathbb{A}_i,\mathbb{A}_j]$ |
| Higgs potential | $\mathrm{Tr}(f_2\Phi^2+f_4\Phi^4)$ | $\int_M\left(f_2\lvert h\rvert^2+f_4\lvert h\rvert^4\right)\mathrm{vol}$ |
| Gauge-Higgs coupling | $-\mathrm{Tr}(d_\mu\Phi d^\mu\Phi)$ | $-\int_M\lvert\mathbb{D}_i h\rvert^2\mathrm{vol}$ |
| Yang-Mills action | $-\frac{1}{4}\mathrm{Tr}(\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu})$ | $-\frac{1}{4}\int_M\mathrm{Tr}_{\mathfrak{su}(n)}(\mathbb{F}_{ij}\mathbb{F}^{ij})\mathrm{vol}$ |

+ Propagators and $\sim(\ell_\mu)_{ij}(\ell_\nu)_{jm}(\ell^\mu)_{ml}(\ell^\nu)_{li}\leftrightarrow$ 

$\sim$ 

skip FRG ▷

# IV. FRG for multimatrix models with multitraces

## Motivation from '2D-Quantum Gravity'

| | | |
|---|---|---|
| discrete surfaces | $\leftrightarrow$ | matrix integrals $\mathcal{Z}(\lambda)$ <br> [B. Eynard, *Counting Surfaces* '16] |
| smooth surface | $\leftrightarrow$ | $\langle \text{area} \rangle$ finite <br> & infinitesimal mesh $\mathfrak{a}$ <br> $\langle \text{area} \rangle_g \sim \frac{\mathfrak{a}^2 (2-2g)}{\lambda/\lambda_c - 1}$ |
| all topologies | $\leftrightarrow$ | $\mathcal{Z}(\lambda) = \sum_g N^{2-2g} \mathcal{Z}_g(\lambda)$, with $\mathcal{Z}_g(\lambda) \wr (\lambda_c - \lambda)^{(2-2g)/\theta}$ |
| $\Uparrow$ | | |
| double-scaling limit | | $N(\lambda_c - \lambda)^{1/\theta} = C$ |
| lin. RG-flow near a fixed point | $\leftrightarrow$ | $\lambda(N) = \lambda_c + (N/C)^{-\theta}$ <br> $\theta = -(\partial \beta / \partial \lambda)\vert_{\lambda_c}$ <br> [Eichhorn-Koslowski, PRD, '13] |

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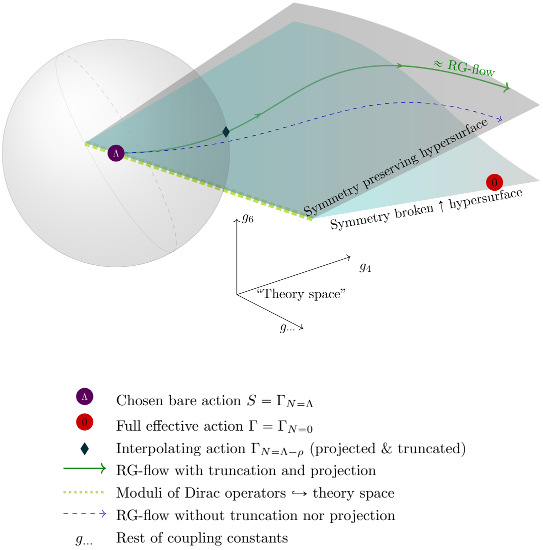

# Two approaches

## 1. Mathematical construction:

[CP 2007.10914, *Ann. Henri Poincaré* 2021]

- prove Wetterich Equation, or FRGE; its proof determines the algebra that governs geometric series in Hess $\Gamma$ [D. Benedetti, K. Groh, P. F. Machado and F. Saueressig, *JHEP* 2011]
- scalings of the couplings with $Z$ and $N$ based on [A. Eichhorn, T. Koslowski, *Phys. Rev. D* '13], (but our proof of the FRGE dictates an algebra not reported there)
- fixed-point solution to $\beta$-equations for a sextic truncation (48 running operators)
- the unique real solution $g^*$ leading to a single relevant direction (positive e.v. of $-(\partial\beta_i/\partial g_j)_{i,j}|_{g^*}$) yields an $R_N$-dependent

$$g^*_{A^4} = 1.002 \times \left( g^*_{A^4}|_{\text{[Kazakov-Zinn-Justin, Nucl. Phys. B '99]}} \right)$$

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## 2. Algebraic-graphic approach:

[CP 2111.02858, *Lett. Math. Phys.*, to appear]

- write down Wetterich Equation
- assume an expansion of its rhs in unitary-invariant operators ($\neq$ exact RG, but it's «what people do»)
- impose the one-loop structure
- determine from it the 'algebra of functional renormalization'; it is unique and the one reported in [CP 2007.10914]

# Some Feynman graphs of multimatrix $\phi^4$-theory...

Several-loop graph

[pic by 'Princi19skydiver', Wikipedia]

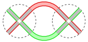
One-loop graph

[pic by 'Wojciech Kielar', Wikipedia]

# Functional Renormalization for $k$-matrix models (w/multitrace-measures)

Quantum theories 'flow' with energy, here in RG-time $t = \log N$, $1 \ll N < \mathcal{N}$. E.g. for $k = 2$ and with bare action

$$S[A, B] = \mathcal{N}\,\mathrm{Tr}_{\mathcal{N}} \left\{ \frac{1}{2}A^2 + \frac{1}{2}B^2 + g_{A^4}\frac{1}{4}A^4 + g_{B^4}\frac{1}{4}B^4 + \frac{1}{2}g_{ABAB}ABAB \right\}$$

radiative corrections 'generate' *effective vertices*. For instance  generates $\mathcal{N}\,\mathrm{Tr}_{\mathcal{N}}(ABBA)$.

$$\Gamma_N[A, B] = \mathrm{Tr}_N \Bigg\{ \overbrace{\frac{Z_A}{2}A^2 + \frac{Z_B}{2}B^2 + \bar{g}_{A^4}\frac{1}{4}A^4 + \bar{g}_{B^4}\frac{1}{4}B^4 + \frac{1}{2}\bar{g}_{ABAB}ABAB}^{\text{operators from the bare action (but with 'running couplings')}} + \overbrace{\frac{1}{2}\bar{g}_{ABBA}ABBA + \frac{1}{2}\bar{g}_{A|A}\,\mathrm{Tr}_N(A) \times A + \ldots}^{\text{radiative corrections}} \Bigg\}$$

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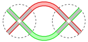
radiative corrections 'generate' *effective vertices*. For instance  generates $\mathcal{N}\, \mathrm{Tr}_{\mathcal{N}}(ABBA)$.

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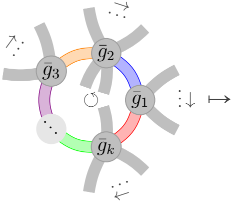
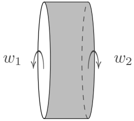
We are interested in *one-loop graphs*. The *effective vertex* $O_G^{\mathrm{eff}}$ of such a graph is formed by reading off each word $w_i$ traveling around all ribbon edges (propagators) by both sides:

$$O_G^{\mathrm{eff}} = \overbrace{\mathrm{Tr}_N(w_1) \times \mathrm{Tr}_N(w_2) \times \cdots \times \mathrm{Tr}_N(w_s)}^{\text{from vertices contracted with propagators}}$$
$$\underbrace{\times\, \mathrm{Tr}_N(U_1) \times \mathrm{Tr}_N(U_2) \dots \times \mathrm{Tr}_N(U_r)}_{\text{from vertices uncontracted with propagators}}$$




gral. situation (yet, without multitraces)




inner word $w_1 \otimes$ outer word $w_2$

- *nc-derivative* $\partial_A : \mathbb{C}_{\langle k\rangle} \to \mathbb{C}_{\langle k\rangle}^{\otimes 2}$ sums over 'replacements of $A$ by $\otimes$' [Rota-Sagan-Stein+Voiculescu]:

$$\partial_A(PAAR) = P \otimes AR + PA \otimes R\,, \text{ but}$$
$$\partial_A(ALGEBRA) = 1 \otimes LGEBRA + ALGEBR \otimes 1$$

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$$\partial_A(PAAR) = P \otimes AR + PA \otimes R\,, \text{ but}$$
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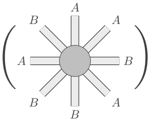
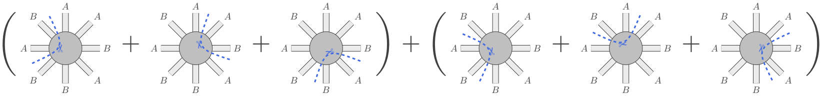
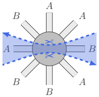
- $W \in \mathbb{C}_{\langle k \rangle}$, the *nc-Hessian* [CP 2007.10914] $\operatorname{Hess}\operatorname{Tr}_N W \in M_k(\mathbb{C}_{\langle k \rangle} \otimes \mathbb{C}_{\langle k \rangle})$ has entries are $\operatorname{Hess}_{b,a} \operatorname{Tr} W = (\partial_{X_b} \circ \partial_{X_a}) \operatorname{Tr}_N W$. Are computed by 'cuts': e.g. $W = ABAABABB$

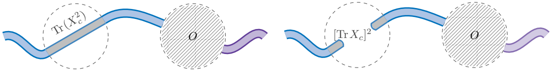
$\partial_B \partial_A \Big($  $\Big)$

go to examples of nc-Hessians ∇

$= 1_N \otimes \Big($  $\Big) \otimes 1_N + \ldots$

in ellipsis $\sum_{\text{cuts}}$  $\to BAA \otimes ABB$

- products of traces $\Rightarrow$ extend by $\boxtimes$, $\mathcal{A}_k = (\mathbb{C}_{\langle k \rangle} \otimes \mathbb{C}_{\langle k \rangle}) \oplus (\mathbb{C}_{\langle k \rangle} \boxtimes \mathbb{C}_{\langle k \rangle})$

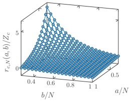



$$\mathrm{Hess}_{a,b}(\mathrm{Tr}\, P \cdot \mathrm{Tr}\, Q) = \mathrm{Tr}\, P \cdot \mathrm{Hess}_{a,b}[\mathrm{Tr}\, Q] + (\partial_{X_a} \mathrm{Tr}\, P) \boxtimes (\partial_{X_b} \mathrm{Tr}\, Q) + (P \leftrightarrow Q)$$

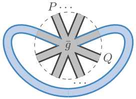
- Wetterich Eq. governs the functional RG $t = \log N$

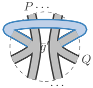
$$\partial_t \Gamma_N[\mathbb{X}] = \frac{1}{2} \mathrm{STr} \left\{ \frac{\partial_t R_N}{\mathrm{Hess}\, \Gamma_N[\mathbb{X}] + R_N} \right\}$$

$$\overset{assume}{=} \sum_{k=0}^{\infty} \bar{h}_k(N, \eta_1, \ldots, \eta_n) \times \underbrace{\frac{1}{2}(-1)^k \mathrm{STr} \left\{ (\mathrm{Hess}\, \Gamma_N^{\mathrm{Int}}[\mathbb{X}])^{\star k} \right\}}_{\text{regulator-independent part}}$$




- $\mathrm{STr} = \mathrm{Tr}_k \otimes \mathrm{Tr}_{\mathcal{A}_n}$. Tadpoles

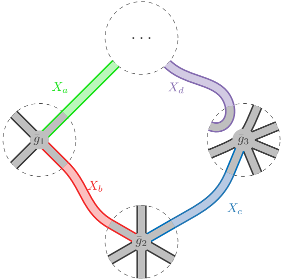



imply

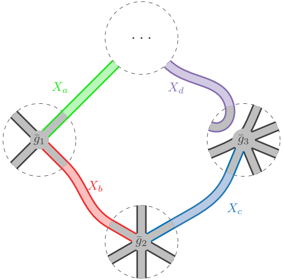
$$\mathrm{Tr}_{\mathcal{A}_n}(P \otimes Q) = \mathrm{Tr}_N\, P \cdot \mathrm{Tr}_N\, Q, \qquad \mathrm{Tr}_{\mathcal{A}_n}(P \boxtimes Q) = \mathrm{Tr}_N(PQ)$$

# Finding $\star$

Want: 

$\subset \mathrm{Hess}_{a,b}\, O_1 \star \mathrm{Hess}_{b,c}\, O_2 \star \mathrm{Hess}_{c,d}\, O_3 \star \ldots \star \mathrm{Hess}_{*,a}\, O_\ell$

# Finding ⋆

Want: 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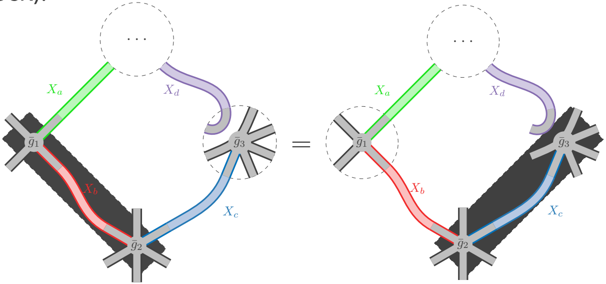
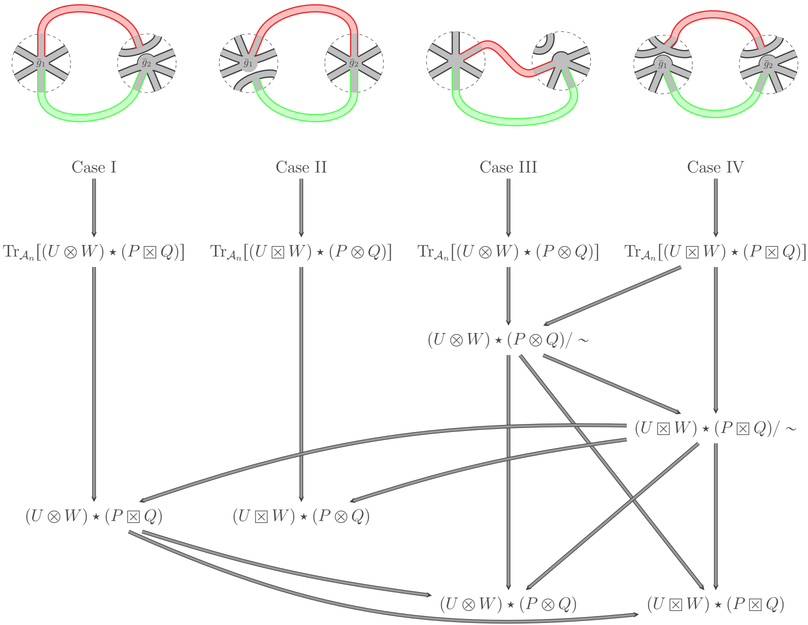
Associativity (trivial check):




$\bar{g}_1$
$\bar{g}_2$
Case I
Case II
Case III
Case IV
$\mathrm{Tr}_{\mathcal{A}_n}[(U \otimes W) \star (P \boxtimes Q)]$
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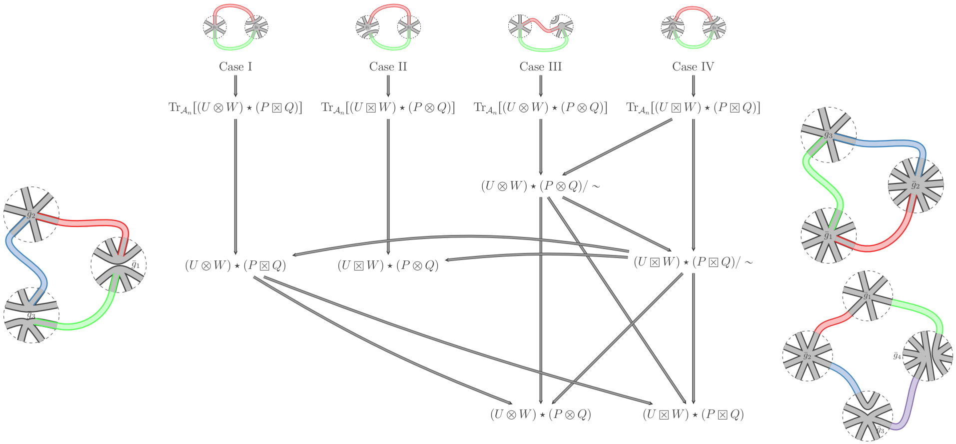
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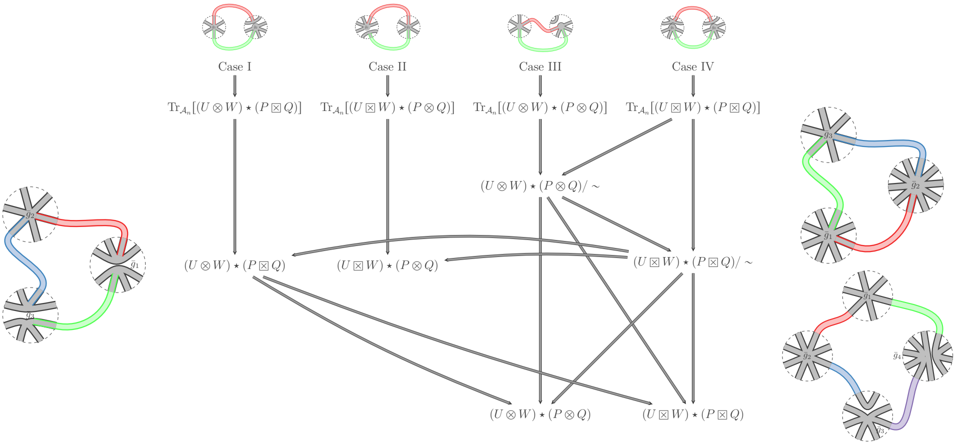


**Thm.** [CP 2111.02858] If the RG-flow is computable in terms of $\mathrm{U}(N)$-invariants, the algebra of Functional Renormalization is $M_k(\mathcal{A}_{N,k},\star)$ where $\mathcal{A}_{N,k} = (\mathbb{C}^{(N)}_{\langle k\rangle}\otimes\mathbb{C}^{(N)}_{\langle k\rangle})\oplus(\mathbb{C}^{(N)}_{\langle k\rangle}\boxtimes\mathbb{C}^{(N)}_{\langle k\rangle})$ whose product in homogeneous elements reads:

$$(U\otimes W)\star(P\otimes Q) = PU\otimes WQ\,,$$
$$(U\boxtimes W)\star(P\otimes Q) = U\boxtimes PWQ\,,$$
$$(U\otimes W)\star(P\boxtimes Q) = WPU\boxtimes Q\,,$$
$$(U\boxtimes W)\star(P\boxtimes Q) = \mathrm{Tr}(WP)U\boxtimes Q\,.$$

# Example: a Hermitian 3-matrix model

Consider two operators $O_1 = \frac{\bar{g}_1}{2}[\mathrm{Tr}_N(\frac{A^2}{2})]^2$ and $O_2 = \bar{g}_2 \,\mathrm{Tr}_N(ABC)$. We compute $g_1 g_2^2$-coefficients:

$$\mathrm{Hess}_{I,J}\, O_1 = \delta_I^J \delta_I^A \bar{g}_1 \{ \underbrace{\mathrm{Tr}_N(A^2/2)\cdot[1_N \otimes 1_N]} + \underbrace{A \boxtimes A} \},$$

where a 'filled ribbon' means contracted in the one-loop graph, and 'black ribbon' uncontracted.

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$$\mathrm{Hess}\, O_2 = \bar{g}_2 \begin{bmatrix} 0 & C\otimes 1_N & B \otimes 1_N \\ 1_N \otimes C & 0 & A\otimes 1_N \\ 1_N \otimes B & 1_N\otimes A & 0 \end{bmatrix} \Rightarrow [\mathrm{Hess}\, O_2]^{\star 2} = \bar{g}_2^2 \begin{bmatrix} \overbrace{C\otimes C} + \overbrace{B\otimes B} & B\otimes A & C\otimes A \\ A\otimes B & A\otimes A + C\otimes C & C\otimes B \\ A\otimes C & B\otimes C & B\otimes B + A\otimes A \end{bmatrix}$$

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Extracting coefficients

$$[\bar{g}_1\bar{g}_2]\,\mathrm{STr}\{\mathrm{Hess}\, O_1 \star [\mathrm{Hess}\, O_2]^{\star 2}\} = \mathrm{Tr}_N(A^2/2)\times\big[(\mathrm{Tr}_N\, C)^2 + (\mathrm{Tr}_N\, B)^2\big] + \mathrm{Tr}_N(ACAC + ABAB),$$

which are effective vertices of the four one-loop graphs that can be formed with the contractions of (the filled ribbon half-edges of) any of {, } with any of {, }. This is a toy example in [CP '21]; in [CP '20] 48 such operators run ⇒ (48 less friendly Hessians)$^3$. go to nc-Hessians examples ▷

# Conclusion

- spectral triple $\equiv$ spin without commutativity of the 'algebra of functions'
- spin $M \times \{$finite spectral triple$\} \equiv$ almost-commutative
  (reproduces classical Standard Model, but hard to quantize)
- *fuzzy* or *matrix* geometry $\approx$ finite spectral triple + $\mathbb{C}\ell$-action
  ($\mathcal{Z}_{\text{fuzzy}} = \int_{\text{Dirac}} \mathrm{e}^{-\frac{1}{\hbar} \mathrm{Tr}_H f(D)} \mathrm{d}D$ is a multimatrix model with multitraces)

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- $G_\ell \times F$= fuzzy $\times$ finite $=$ gauge matrix spectra triple
  is $\mathrm{PU}(n)$-Yang-Mills-Higgs-like if $F$ is over $M_n(\mathbb{C})$; partition func. is a $k$-matrix model, $k$ large. Small step towards [Connes Marcolli, *NCG, QFT and motives*, '07, next sceenshot]

« The far distant goal is to set up a functional integral evaluating spectral observables $\mathcal{S}$ as

$$\langle \mathcal{S} \rangle = \mathcal{N} \int \mathcal{S}\, e^{-\mathrm{Tr}(f(D/\Lambda)) - \frac{1}{2}\langle J\psi, D\psi\rangle - \rho(e,D)}\, \mathcal{D}[\psi]\mathcal{D}[D]\mathcal{D}[e], \tag{1.892}$$ »

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(the terms $\frac{1}{2}\langle J\psi, D\psi\rangle$, $\rho(e,D)$, $\mathcal{D}[\psi]$ and $\mathcal{D}[e]$ are struck through on the slide)

thank you!

References: [CP 1912.13288] [CP 2007.10914] [CP 2102.06999] [CP 2105.01025] [CP 2111.02858]

# The classical Dirac operator in Riemannian geometry

For us $M$ will be a Riemannian closed manifold, $\dim M = d$.
In physics, $M$ is a spacetime (for this first part, still deterministic).
Q: When do Dirac operators exist on $M$?
A: Only if the *obstruction* to a spin-structure $w_2(M) \in \check{H}^2(M, \mathbb{Z}_2)$ is trivial.

# The classical Dirac operator in Riemannian geometry

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$\mathbb{Z}_2$-ech cohomology in short,

- $U = \{U_i\}_i$ good open cover of $M$
- $j$-simplices are $\sigma = (k_0, \dots, k_j)$ such that $U_{k_0 k_1 \dots k_j} = U_{k_0} \cap U_{k_1} \cap \dots \cap U_{k_j} \neq \varnothing$
- $j$-cochains, maps $f : \{j\text{-simplices}\} \to \mathbb{Z}_2$ satisfying invariance $\tau^* f = f$ under $\tau \in \mathfrak{S}(j+1)$, form an abelian group $\check{C}^j(U, \mathbb{Z}_2)$
- coboundary maps $\delta^j : \check{C}^j(U, \mathbb{Z}_2) \to \check{C}^{j+1}(U, \mathbb{Z}_2)$ given by

$$(\delta^j f)(k_0, \dots, k_{j+1}) := f(k_1, \dots, k_{j+1}) f(k_0, k_1 \dots, k_{j+1}) \cdots f(k_0, \dots, k_j)$$

- $\check{H}^j(U, \mathbb{Z}_2) = \ker \delta^j / \mathrm{im}\, \delta^{j-1}$

A more familiar $\mathbb{Z}_2$-ech cohomology class is the orientability obstruction

- $\{U_i\}_i$ good open cover of $M$
- pick $s_i : U_i \to F(M)$ sections on the frame bundle $\mathrm{O}(d) \hookrightarrow F(M) \to M$
- on a 1-simplex $(k_0, k_1)$, $s_{k_0} = s_{k_1} G_{k_0 k_1}$

$$\begin{aligned} f(k_0, k_1) &= \det(G_{k_0,k_1}) \\ &= \det(G_{k_1,k_0}) = f(k_1, k_0) \end{aligned}$$

- since $\{G_{j,l}\}_{j.l}$ are transition functions,

$$(\delta^1 f)(j, l, m) = G_{j,l} G_{l,m} G_{m,j} = 1$$

- other choice of sections $s'_k$ yields $G'_{kl} = g_k G_{kl} g_l^{-1}$ and $f' = (\delta^0 h) f$ where $h = \det g_k$
- other choice $\{V\}_j$ of a good open cover yields a cochain complex map $\check{C}^*(V, \mathbb{Z}_2) \to \check{C}^*(U, \mathbb{Z}_2)$,

# Low Stiefel-Whitney classes (first two floors of Whitehead tower)

**Theorem** $M$ is orientable iff the 1st Stiefel-Whitney class $w_1(M) := [f] = 1$.

*Proof of 'if'.* If $w_1(M) = 1$, $f(k_0, k_1) = \det(G_{k_0,k_1})$ is a 0-coboundary, $f = \delta^0 h$. We can pick sections $\{s_k : U_k \to F(M)\}_k$ and $g_k \in \mathrm{O}(d)$ with $\det g_k = h(k)$, so that the transition functions for $s'_k := s_k \cdot g_k$ satisfy

$$\begin{aligned} \det(G'_{k_0,k_1}) &= \det(g_{k_0}^{-1} G_{k_0,k_1} g_{k_1}) \\ &= [(\delta^0 h) \cdot f](k_0, k_1) = 1. \quad \square \end{aligned}$$

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In similar way, a *spin structure* $\lambda$

$$\begin{array}{ccccc} \mathrm{Spin}(d) \times P(M) & \longrightarrow & P(M) & & \\ \downarrow & & \downarrow \lambda & \searrow & \\ & & & & M \\ & & & \nearrow & \\ \mathrm{SO}(d) \times F_{\mathrm{SO}}(M) & \longrightarrow & F_{\mathrm{SO}}(M) & & \end{array}$$

exists when the SO-frame bundle can be lifted in compatible way with the double cover $\mathbb{Z}_2 \to \mathrm{Spin}(d) \xrightarrow{\rho} \mathrm{SO}(d)$. Transition functions $g_{ij} : U_{ij} \to \mathrm{SO}(d)$ can be lifted to $\mathrm{Spin}(d)$-valued $\tilde{g}_{ij}$. For $U_{ijk} \neq \varnothing$, let

$$\tilde{g}_{ij} \tilde{g}_{jk} \tilde{g}_{ki} =: z(i,j,k) \mathrm{id}_{\mathrm{Spin}(d)}$$

**Theorem** [A. Haefliger, '56] Orientable $M$ is spin iff its second Stiefel-Whitney class $w_2(M) := [z] = 1$.

# Spectral Action: $\mathrm{Tr}_H D^{2m}$ [C.P. 1912.13288] Chord diags. $d = 2$ geometries, $\eta = \mathrm{diag}(e_1, e_2)$ go to more CD's▻

$$\mathrm{Tr}_H(D^6) = 2N \sum_{\boldsymbol{\mu}} \overline{\mathrm{Tr}_{\mathbb{S}}(\gamma^{\mu_1} \cdots \gamma^{\mu_6})} \times$$

$$+(-1)^2\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_4}\eta^{\mu_3\mu_6} \quad +(-1)^1\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_6}\eta^{\mu_3\mu_4} \quad +(-1)^0\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_3}\eta^{\mu_4\mu_5} \quad +(-1)^1\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_4}\eta^{\mu_3\mu_5} \quad +(-1)^0\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_5}\eta^{\mu_3\mu_4}$$

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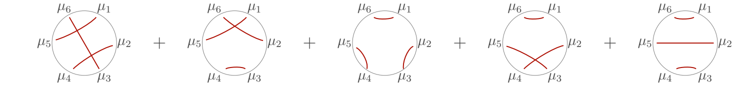
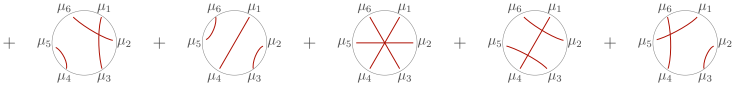
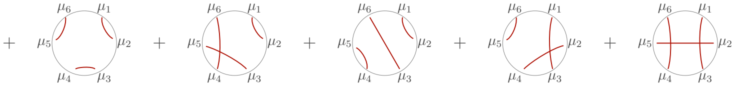
$$\mathrm{Tr}_H(D^6) = 2N \sum_{\boldsymbol{\mu}} \overbrace{\mathrm{Tr}_{\mathbb{S}}(\gamma^{\mu_1} \cdots \gamma^{\mu_6})}^{\text{solid circ.}} \times$$



$$+(-1)^2\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_4}\eta^{\mu_3\mu_6} +(-1)^1\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_6}\eta^{\mu_3\mu_4} +(-1)^0\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_3}\eta^{\mu_4\mu_5} +(-1)^1\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_4}\eta^{\mu_3\mu_5} +(-1)^0\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_5}\eta^{\mu_3\mu_4}$$



$$+(-1)^1\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_6}\eta^{\mu_4\mu_5} +(-1)^0\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}\eta^{\mu_5\mu_6} +(-1)^3\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_5}\eta^{\mu_3\mu_6} +(-1)^2\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_6}\eta^{\mu_3\mu_5} +(-1)^1\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_3}\eta^{\mu_4\mu_6}$$



$$+(-1)^0\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}\eta^{\mu_5\mu_6} +(-1)^1\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_5}\eta^{\mu_4\mu_6} +(-1)^0\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_6}\eta^{\mu_4\mu_5} +(-1)^1\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}\eta^{\mu_5\mu_6} +(-1)^2\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_5}\eta^{\mu_4\mu_6}$$

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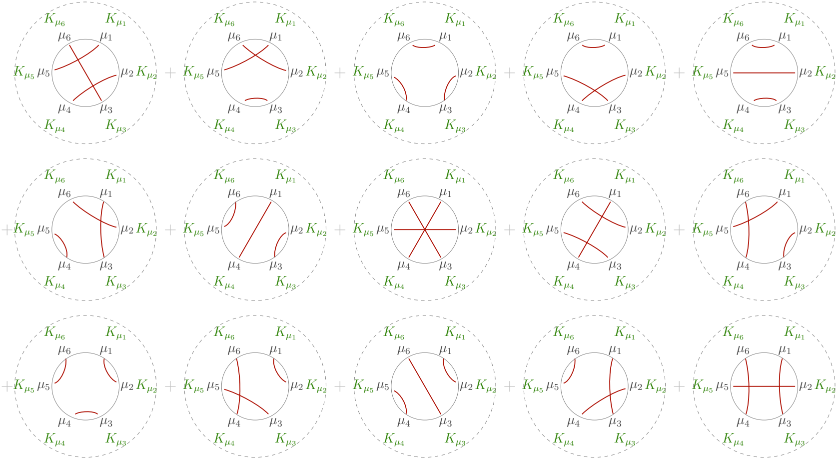

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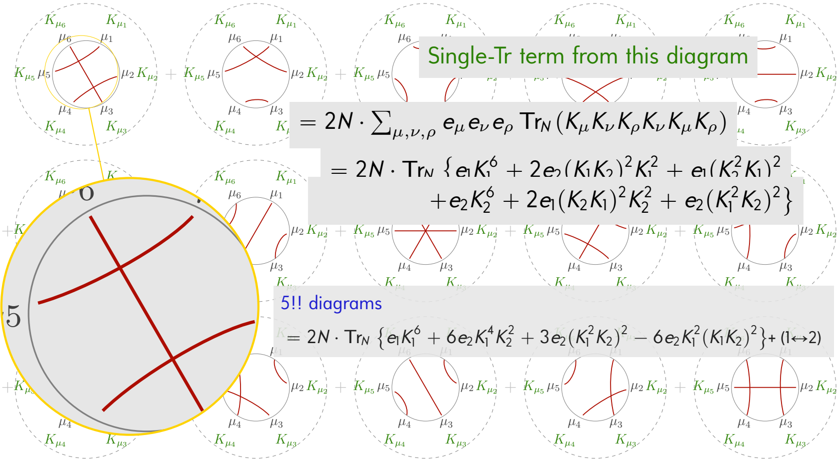

Single-Tr term from this diagram

$$= 2N \cdot \sum_{\mu,\nu,\rho} e_\mu e_\nu e_\rho \,\mathrm{Tr}_N(K_\mu K_\nu K_\rho K_\nu K_\mu K_\rho)$$

$$= 2N \cdot \mathrm{Tr}_N \{ e_1 K_1^6 + 2e_2 (K_1 K_2)^2 K_1^2 + e_1 (K_2^2 K_1)^2 + e_2 K_2^6 + 2e_1 (K_2 K_1)^2 K_2^2 + e_2 (K_1^2 K_2)^2 \}$$

5!! diagrams

$$= 2N \cdot \mathrm{Tr}_N \{ e_1 K_1^6 + 6e_2 K_1^4 K_2^2 + 3e_2 (K_1^2 K_2)^2 - 6e_2 K_1^2 (K_1 K_2)^2 \} + (1 \leftrightarrow 2)$$

# Classical Dirac operators (assume $d$ even)

- $M$ (space~~time~~) will be a closed, Riemannian manifold
- if $M$ is spin, there is a vector bundle $\mathbb{S}$ with fibers satisfying $\mathrm{End}(\mathbb{S}_x) \cong \mathbb{C}\ell(d)$ ($x \in M$). The sections $\Gamma(\mathbb{S})$ are spinors
- the Levi-Civita connection $\nabla^{\mathrm{lc}}$ can be also lifted to the *spin connection* $\nabla^s : \Gamma(\mathbb{S}) \to \Omega^1(M) \otimes \Gamma(\mathbb{S})$

$$\nabla^s c(\omega)\psi = c(\nabla^{\mathrm{lc}}\omega)\psi + c(\omega)\nabla^s\psi$$

$$\psi \in \Gamma(\mathbb{S}), \omega \in \Omega^1(M)$$

being $c$ Clifford multiplication, basically $c(\mathrm{d}x^\mu) = \gamma^\mu$

- on the space of square integrable spinors $L^2(M, \mathbb{S})$ there is an (ess.) self-adjoint operator, the *Dirac operator*,

$$D_M = -\mathrm{i}c \circ \nabla^s \overset{\text{loc.}}{=} -\mathrm{i}\sum_{\mu=1}^{d} \gamma^\mu(\partial_\mu + \omega_\mu)$$

and by Leibniz rule

$$[D_M, a] = -\mathrm{i}c(\mathrm{d}a) \quad a \in C^\infty(M)$$

which is bounded

back to 'spectral triples' ⋘

# Matrix or Fuzzy Geometries

Definition ("condensed" from [J. Barrett, *J. Math. Phys.* 2015]).
A *fuzzy geometry* of *signature* $(p, q) \in \mathbb{Z}^2_{\geqslant 0}$ is given by

- a simple matrix algebra $\mathcal{A}$ – we take always $\mathcal{A} = M_N(\mathbb{C})$
- a Hermitian $\mathbb{C}\ell(p, q)$-module $\mathbb{S}$ with a *chirality* $\gamma$. That is a linear map $\gamma : \mathbb{S} \to \mathbb{S}$ satisfying $\gamma^* = \gamma$ and $\gamma^2 = 1$
- a Hilbert space $\mathcal{H} = \mathbb{S} \otimes M_N(\mathbb{C})$ with inner product $\langle v \otimes R, w \otimes S \rangle = (v, w) \operatorname{Tr}_N (R^* S)$ for each $R, S \in M_N(\mathbb{C})$, being $(\cdot, \cdot)$ the inner product of $\mathbb{S}$
- a left-$\mathcal{A}$ representation $\rho(a)(v \otimes R) = v \otimes (aR)$ on $\mathcal{H}$, $a \in \mathcal{A}$ and $v \otimes R \in \mathcal{H}$

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- three signs $\epsilon, \epsilon', \epsilon'' \in \{-1, +1\}$ determined through $s := q - p$ by the following table:

| $s \equiv q - p \pmod 8$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\epsilon$ | + | + | − | − | − | − | + | + |
| $\epsilon'$ | + | − | + | + | + | − | + | + |
| $\epsilon''$ | + | + | − | + | + | + | − | + |

- a real structure $J = C \otimes *$, where $*$ is complex conjugation and $C$ is an anti-unitarity on $\mathbb{S}$ satisfying $C^2 = \epsilon$ and $C\gamma^\mu = \epsilon' \gamma^\mu C$ for all the gamma matrices $\mu = 1, \ldots, p + q$.
- a self-adjoint operator $D$ on $\mathcal{H}$ satisfying the *order-one condition*

$$[\,[D, \rho(a)]\,, J\rho(b)J^{-1}] = 0 \qquad \text{for all } a, b \in \mathcal{A}$$

- a chirality $\Gamma = \gamma \otimes 1_{\mathcal{A}}$ for $\mathcal{H}$, where $\gamma$ is the chirality of $\mathbb{S}$. The signs above impose:

# Two-matrix model from $\mathrm{Tr}_H D^6$, $\eta = \mathrm{diag}(e_1, e_2)$, $K_i^* = e_i K_i$ [CP '19]

$$\begin{aligned}\mathcal{S}_6[K_1, K_2] = 2 \cdot \mathrm{Tr}_N \big\{ & e_1 K_1^6 + 6 e_2 K_1^4 K_2^2 - 6 e_2 K_1^2 (K_1 K_2)^2 + 3 e_2 (K_1^2 K_2)^2 \\ & + e_2 K_2^6 + 6 e_1 K_2^4 K_1^2 - 6 e_1 K_2^2 (K_2 K_1)^2 + 3 e_1 (K_2^2 K_1)^2 \big\}\end{aligned}$$

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$$\left. + e_2 K_2^6 + 6 e_1 K_2^4 K_1^2 - 6 e_1 K_2^2 (K_2 K_1)^2 + 3 e_1 (K_2^2 K_1)^2 \right\}$$

and the double-trace part is

$$\begin{aligned}
\mathcal{B}_6[K_1, K_2] = {} & 6\,\mathrm{Tr}_N(K_1)\Big\{2\,\mathrm{Tr}_N(K_1^5) + 2\,\mathrm{Tr}_N(K_1 K_2^4) + 6 e_1 e_2\,\mathrm{Tr}_N(K_1^3 K_2^2) - 2 e_1 e_2\,\mathrm{Tr}_N(K_1^2 K_2 K_1 K_2)\Big]\Big\} \\
& + 6\,\mathrm{Tr}_N(K_2)\Big\{2\,\mathrm{Tr}_N(K_2^5) + 2\,\mathrm{Tr}_N(K_2 K_1^4) + 6 e_1 e_2\,\mathrm{Tr}_N(K_2^3 K_1^2) - 2 e_1 e_2\,\mathrm{Tr}_N(K_2^2 K_1 K_2 K_1)\Big]\Big\} \\
& + 48\,\mathrm{Tr}_N(K_1 K_2) \cdot \big[e_1\,\mathrm{Tr}_N(K_1^3 K_2) + e_2\,\mathrm{Tr}_N(K_2^3 K_1)\big] \\
& + 6\,\mathrm{Tr}_N(K_1^2) \cdot \Big\{e_2\big[8\,\mathrm{Tr}_N(K_1^2 K_2^2) - 2\,\mathrm{Tr}_N(K_2 K_1 K_2 K_1)\big] + e_1\big[5\,\mathrm{Tr}_N(K_1^4) + \mathrm{Tr}_N(K_2^4)\big]\Big\} \\
& + 6\,\mathrm{Tr}_N(K_2^2) \cdot \Big\{e_1\big[8\,\mathrm{Tr}_N(K_1^2 K_2^2) - 2\,\mathrm{Tr}_N(K_1 K_2 K_1 K_2)\big] + e_2\big[5\,\mathrm{Tr}_N(K_2^4) + \mathrm{Tr}_N(K_1^4)\big]\Big\} \\
& + 4\Big(5[\mathrm{Tr}_N(K_1^3)]^2 + 6 e_1 e_2\,\mathrm{Tr}_N(K_1 K_2^2)\,\mathrm{Tr}_N(K_1^3) + 9[\mathrm{Tr}_N\, K_1^2 K_2]^2 \\
& \quad + 5[\mathrm{Tr}_N(K_2^3)]^2 + 6 e_1 e_2\,\mathrm{Tr}_N(K_1^2 K_2)\,\mathrm{Tr}_N(K_2^3) + 9[\mathrm{Tr}_N\, K_1 K_2^2]^2\Big)\,.
\end{aligned}$$

# Sketch of the Standard Model derivation from NCG

One starts with the $M \times_{\text{s.t.}} F$ and $\mathcal{A}_{LR} = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C})$

- $F = (\mathcal{A}_{LR}, \mathcal{M}_F^{\#\text{generations}}, D_F)$, $\mathcal{M}_F$ an $\mathcal{A}_{LR}$-module
- $\mathcal{M}_F$ has to be of the form $\mathcal{M}_F = \mathcal{E} \otimes \mathcal{E}^o$, with

$$\mathcal{E} = (2_L \otimes 1^o) \oplus (2_R \otimes 1^o) \oplus (2_L \otimes 3^o) \oplus (2_L \otimes 3^o)\,, \quad \dim_{\mathbb{C}} \mathcal{E} = 16$$

- Thus the $\mathcal{H}_F \cong \mathbb{C}^{32\times 3}$. The $96 \times 96$ matrix $D_F$ can have off-diagonal elements only for the maximal subalgebra

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

- Lie group part of $\mathrm{SU}(\mathcal{A}_F) = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$

# Sketch of the Standard Model derivation from NCG

With $Q : \mathbb{C} \hookrightarrow \mathbb{H}$, $Q_\lambda = \mathrm{diag}(\lambda, \bar{\lambda})$ and $Q_\lambda \left|\pm\right\rangle = \pm\lambda \left|\pm\right\rangle$,

- Weak hypercharge:

| | $\nu$ | $e$ | $u$ | $d$ |
|---|---|---|---|---|
| $Y$ | $\left\|+\right\rangle \otimes 1^o$ | $\left\|-\right\rangle \otimes 1^o$ | $\left\|+\right\rangle \otimes 3^o$ | $\left\|-\right\rangle \otimes 3^o$ |
| L | $-1$ | $-1$ | $+1/3$ | $+1/3$ |
| R | $0$ | $-2$ | $+4/3$ | $-2/3$ |

- $\mathrm{SU}(2)$-adjoint action is 2 on $\mathcal{H}_L$ or trivial in the $\mathcal{H}_R$ sector
- $\mathrm{SU}(3)$-adjoint action is the color action on $\mathcal{H}_q$ and trivial on $\mathcal{H}_\ell$

$$\mathrm{Lie}(\mathrm{SU}(\mathcal{A}_F)) = \mathrm{U}(1)_Y \times \mathrm{SU}(2)_L \times \mathrm{SU}(3)_{\mathrm{color}}$$

- All $D_F$ such that $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ is a spectral triple are

$$D_F(\Upsilon_R, \Upsilon_\nu, \Upsilon_e, \Upsilon_u, \Upsilon_d)$$

the moduli of such Dirac operators has dimension 31 = num. Yukawa couplings in $\nu$MSM.

# Fermionic Spectral Action

- The fermionic part is not treated here but is essentially given by [not needed here]

$$S_f(D) = \frac{1}{2}\langle J\psi \mid D\psi \rangle$$

where $\psi$ are classical fermions, $J$ implements charge-conjugation ($J$ fixes the spin structure)

## Dirac $F_{\mathrm{SM}}$ operator

$$D_F = \left(\begin{array}{cccc|ccccc|cccc|ccccc}
0&0&\Upsilon_\nu^*&0&0&0&0&0&&\Upsilon_R^*&0&0&0&0&0&0&0&0\\
0&0&0&\Upsilon_e^*&0&0&0&0&&0&0&0&0&0&0&0&0&0\\
\Upsilon_\nu&0&0&0&0&0&0&0&&0&0&0&0&0&0&0&0&0\\
0&\Upsilon_e&0&0&0&0&0&0&&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&\Upsilon_u^*&0&&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\Upsilon_d^*&\otimes 1_3&0&0&0&0&0&0&0&0&0\\
0&0&0&0&\Upsilon_u&0&0&0&&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\Upsilon_d&0&0&&0&0&0&0&0&0&0&0&0\\
\hline
\Upsilon_R&0&0&0&0&0&0&0&0&0&0&\Upsilon_\nu^T&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&\Upsilon_e^T&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&\bar{\Upsilon}_\nu&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&\bar{\Upsilon}_e&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\Upsilon_u^T&0&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\Upsilon_d^T&\otimes 1_3\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\bar{\Upsilon}_u&0&0&0&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\bar{\Upsilon}_d&0&0&
\end{array}\right) \in M_{96}(\mathbb{C})_{\mathrm{s.a}}$$

$\sim 10^4$ zeroes from geometry.

back to spectral standard model ⋘

- Example 1: (Finite spectral triples) [Exercise from W. van Suijlekom's book]
  - $\mathcal{A} = \mathbb{C}^3$
  - $\mathcal{H} = \mathbb{C}^3 \leftharpoondown \mathcal{A}$, in defining representation
  - $D = \begin{pmatrix} 0 & 1/d & 0 \\ 1/d & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (0 \neq d \in \mathbb{R})$

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looks like


for
$$d_{ij} = \sup_{a \in \mathcal{A}} \{ |a(i) - a(j)| \ : \ \| [D, a] \| \leqslant 1 \}$$
$$= \sup_{a \in \mathcal{A}} \{ |a(i) - a(j)| \ : \ |a(1) - a(2)|^2 \leqslant d^2 \} \qquad \forall i, j = 1, 2, 3$$

- **Example 2:** (Finite spectral triples)
  - $\mathcal{A} = M_3(\mathbb{C})$
  - $\mathcal{H} = \mathbb{C}^3$
  - $D = 0$

back to main presentation ⋘

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back to main presentation ⋘

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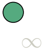


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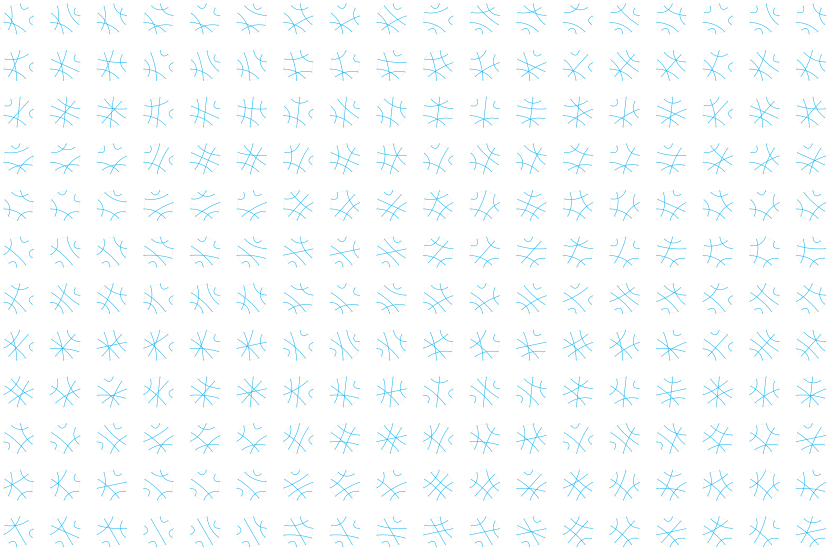
  - $\mathcal{A} = M_3(\mathbb{C}) = \mathbb{C}[\mathcal{G}_\sim]$
  - $\mathcal{H} = \mathbb{C}^3$
  - $D = 0$


back to main presentation ⋘

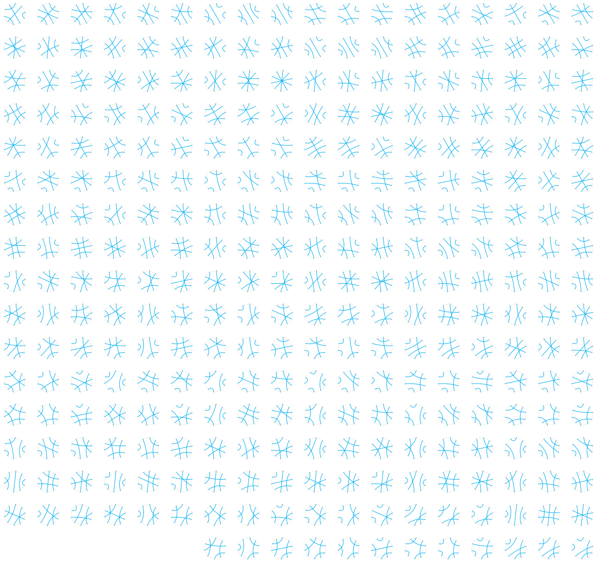
# Chord Diagrams of 10 points (1/4) (appear in $D^{10}|_{d=2}, D^4|_{d=4}, D^2|_{d=6}$ )

# Chord Diagrams of 10 points (2/4)

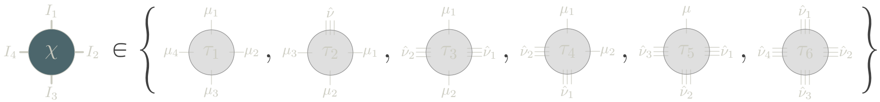


# Chord Diagrams of 10 points (3/4)

# Chord Diagrams of 10 points (4/4)



Non-vanishing ones... $\frac{1}{4}\operatorname{Tr} D^4 = N\mathcal{S}_4^{\mathrm{Riemann}} + \sum_i \operatorname{Tr}_N{}^{\otimes 2}(A_i \otimes B_i)$



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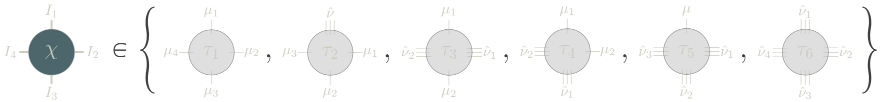


$$
\begin{aligned}
\mathcal{S}_4^{\text{Riemann}} = \mathrm{Tr}_N \Big\{ & 2\sum_{\mu}(L_\mu^4 + H_\mu^4) + 4\sum_{\mu<\nu}\left(2L_\mu^2L_\nu^2 + 2H_\mu^2H_\nu^2 - L_\mu L_\nu L_\mu L_\nu - H_\mu H_\nu H_\mu H_\nu\right) \\
& - \sum_{\mu\neq\nu}\left[2(L_\mu H_\nu)^2 + 4L_\mu^2 H_\nu^2\right] + \sum_\mu \left[2(L_\mu H_\mu)^2 - 4L_\mu^2 H_\mu^2\right] \\
& + 8\Big[H_1\big(L_2[L_3,L_4] + L_3[L_4,L_2] + L_4[L_2,L_3]\big) \\
& \quad + H_2\big(L_1[L_3,L_4] + L_3[L_4,L_1] + L_4[L_1,L_3]\big) \\
& \quad + H_3\big(L_1[L_2,L_4] + L_2[L_4,L_1] + L_4[L_1,L_2]\big) \\
& \quad + H_4\big(L_1[L_2,L_3] + L_2[L_3,L_1] + L_3[L_1,L_2]\big)\Big] + 8[H \leftrightarrow L]\Big\}
\end{aligned}
$$

[C.P. '19]

$L_\mu, H_\mu$ are random matrices!

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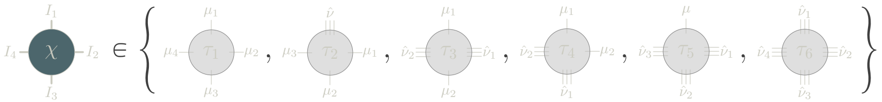


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& + 8\Big[ H_1\big(L_2[L_3,L_4] + L_3[L_4,L_2] + L_4[L_2,L_3]\big) \\
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\end{aligned}
$$

[C.P. '19]

$L_\mu, H_\mu$ are random matrices!

- Analogy $[L_\mu,\,\cdot\,] \to \partial_\mu \quad \{H_\mu,\,\cdot\,\} \to \omega_\mu$ [J. Barrett, L. Glaser, *J. Phys. A* 2016]
- Obtained for any signature, also the $A_i, B_i$ noncommutative-polynomials [C.P. '19]

| Operator | Its noncommutative Hessian |
|---|---|
| $\operatorname{Tr}(A)\operatorname{Tr}(A^3)$ | $3\cdot\begin{pmatrix} \operatorname{Tr}A\cdot(A\otimes 1+1\otimes A) + 1\boxtimes A^2 + A^2\boxtimes 1 & 0 \\ 0 & 0 \end{pmatrix}$ |
| $\operatorname{Tr}(ABAB)$ | $2\cdot\begin{pmatrix} B\otimes B & (1\otimes BA + AB\otimes 1) \\ (1\otimes AB + BA\otimes 1) & A\otimes A \end{pmatrix}$ |
| $\operatorname{Tr}A\operatorname{Tr}(AAABB)$ | $\begin{pmatrix} \begin{array}{c} (\operatorname{Tr}(A)1\otimes(ABB) + \operatorname{Tr}(A)1\otimes(BBA) + \\ \operatorname{Tr}(A)(ABB)\otimes 1 + \operatorname{Tr}(A)(BBA)\otimes 1 + 1\boxtimes \\ (AABB) + 1\boxtimes(ABBA) + 1\boxtimes(BBAA) + \\ (AABB)\boxtimes 1 + (ABBA)\boxtimes 1 + (BBAA)\boxtimes \\ 1 + \operatorname{Tr}(A)A\otimes B^2 + \operatorname{Tr}(A)B^2\otimes A) \end{array} & \begin{array}{c} \operatorname{Tr}(A)(1\otimes(AAB) + (BAA)\otimes 1 + A\otimes \\ AB + B\otimes A^2 + A^2\otimes B + BA\otimes A) + \\ 1\boxtimes(A^3B) + 1\boxtimes(BA^3) \end{array} \\ \begin{array}{c} \operatorname{Tr}(A)(1\otimes(BAA) + (AAB)\otimes 1 + A\otimes \\ BA + B\otimes A^2 + A^2\otimes B + AB\otimes A) + \\ (A^3B)\boxtimes 1 + (BA^3)\boxtimes 1 \end{array} & \operatorname{Tr}(A)(1\otimes(A^3) + (A^3)\otimes 1) \end{pmatrix}$ |

Table: Some Hessians order operators. Here $\operatorname{Tr} = \operatorname{Tr}_N$.

| Operator | Its noncommutative Hessian |
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| $\mathrm{Tr}(A)\,\mathrm{Tr}(A^3)$ | $3 \cdot \begin{pmatrix} \mathrm{Tr}\,A \cdot (A \otimes 1 + 1 \otimes A) + 1 \boxtimes A^2 + A^2 \boxtimes 1 & 0 \\ 0 & 0 \end{pmatrix}$ |
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| $\mathrm{Tr}\,A\,\mathrm{Tr}(AAABB)$ | $\begin{pmatrix} (\mathrm{Tr}(A)1 \otimes (ABB) + \mathrm{Tr}(A)1 \otimes (BBA) + \mathrm{Tr}(A)(ABB) \otimes 1 + \mathrm{Tr}(A)(BBA) \otimes 1 + 1 \boxtimes (AABB) + 1 \boxtimes (ABBA) + 1 \boxtimes (BBAA) + (AABB) \boxtimes 1 + (ABBA) \boxtimes 1 + (BBAA) \boxtimes 1 + \mathrm{Tr}(A) A \otimes B^2 + \mathrm{Tr}(A) B^2 \otimes A) & \mathrm{Tr}(A)(1 \otimes (AAB) + (BAA) \otimes 1 + A \otimes AB + B \otimes A^2 + A^2 \otimes B + BA \otimes A) + 1 \boxtimes (A^3 B) + 1 \boxtimes (BA^3) \\ \mathrm{Tr}(A)(1 \otimes (BAA) + (AAB) \otimes 1 + A \otimes BA + B \otimes A^2 + A^2 \otimes B + AB \otimes A) + (A^3 B) \boxtimes 1 + (BA^3) \boxtimes 1 & \mathrm{Tr}(A)(1 \otimes (A^3) + (A^3) \otimes 1) \end{pmatrix}$ |

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# $\beta$-functions of NCG two-matrix models, signature $\eta = \mathrm{diag}(e_1, e_2)$

$$2h_1(a_4 + c_{22} + 2d_{2|02} + 6d_{2|2}) = \eta_a$$
$$2h_1(b_4 + c_{22} + 6d_{02|02} + 2d_{2|02}) = \eta_b$$
$$-h_1[e_a(a_4 - c_{1111}) + 2d_{1|12} + 6d_{1|3}] + d_{1|1}(\eta + 1) = \beta(d_{1|1})$$
$$-h_1[e_b(b_4 - c_{1111}) + 6d_{01|03} + 2d_{01|21}] + d_{01|01}(\eta + 1) = \beta(d_{01|01})$$

The next block encompasses the connected quartic couplings:

$$h_2(4a_4^2 + 4c_{22}^2) + a_4(2\eta + 1) - h_1(24a_6 e_a + 4c_{42} e_b + 4d_{02|4} e_b + 4d_{2|4} e_a) = \beta(a_4)$$

$$h_2(4b_4^2 + 4c_{22}^2) + b_4(2\eta + 1) - h_1(24b_6 e_b + 4c_{24} e_a + 4d_{02|04} e_b + 4d_{2|04} e_a) = \beta(b_4)$$

$$-h_1(2e_a c_{1212} + e_b 2c_{2121} + 3e_a c_{24} + 3e_b c_{42} + e_a d_{02|22} + e_b d_{2|22}) + h_2(2a_4 c_{22} + 2b_4 c_{22} + 2e_a e_b c_{1111}^2 + 2e_a e_b c_{22}^2) + c_{22}(2\eta + 1) = \beta(c_{22})$$

$$8e_a e_b c_{1111} c_{22} h_2 + c_{1111}(2\eta + 1) + h_1(4e_a c_{1311} + 4e_b c_{3111} + 2e_a d_{02|1111} + 2e_b d_{2|1111}) = \beta(c_{1111})$$

back to main presentation ⋘

$$2h_2(6a_4a_6 + e_a e_b c_{22}c_{42}) + a_6(3\eta + 2) = \beta(a_6)$$

$$2h_2(6b_4b_6 + e_a e_b c_{22}c_{24}) + b_6(3\eta + 2) = \beta(b_6)$$

$$\begin{aligned} 4h_2\{a_4c_{3111} + e_a e_b[c_{22}(c_{1311} + 2c_{3111}) \\ -c_{1111}(2c_{2121} + c_{42})]\} + c_{3111}(3\eta + 2) = \beta(c_{3111}) \end{aligned}$$

$$\begin{aligned} 2h_2[2a_4c_{2121} + e_a e_b(-2c_{1111}c_{3111} \\ +4c_{2121}c_{22} + c_{22}c_{24})] + c_{2121}(3\eta + 2) = \beta(c_{2121}) \end{aligned}$$

$$\begin{aligned} 2h_2[a_4c_{24} + 3b_4c_{24} + 2e_a e_b(c_{22}(3b_6 + c_{2121} + c_{24} + c_{42}) \\ -c_{1111}c_{1311})] + c_{24}(3\eta + 2) = \beta(c_{24}) \end{aligned}$$

$$\begin{aligned} 4h_2\{b_4c_{1311} + e_a e_b[c_{22}(2c_{1311} + c_{3111}) \\ -c_{1111}(2c_{1212} + c_{24})]\} + c_{1311}(3\eta + 2) = \beta(c_{1311}) \end{aligned}$$

$$\begin{aligned} 2h_2[2b_4c_{1212} + e_a e_b(c_{22}(4c_{1212} + c_{42}) \\ -2c_{1111}c_{1311})] + c_{1212}(3\eta + 2) = \beta(c_{1212}) \end{aligned}$$

$$\begin{aligned} 2h_2[3a_4c_{42} + 2e_a e_b(3a_6c_{22} - c_{1111}c_{3111} + c_{1212}c_{22} \\ +c_{22}c_{24} + c_{22}c_{42}) + b_4c_{42}] + c_{42}(3\eta + 2) = \beta(c_{42}) \end{aligned}$$