The F-theorem in the melonic limit

Davide Lettera

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Introduction

Monotonicity theorems state that there exist a monotonically decreasing quantity along the RG flow.

- **d=2: c-theorem (Zamolodchikov 1986)**
- **d=4: a-theorem (Cardy 1988; Komargodski and Schwimmer 2011)**
- **d=3: F-theorem (Casini and Huerta 2012; Jafferis et al. 2011)**

Generalized F-theorem: $\tilde{F} = \sin(\frac{\pi d}{2}) \log(Z_{S^d})$ (Giombi and Klebanov 2015)

F-theorem

F is the finite part of the free energy on the sphere:

$\rightarrow$**Weak version:** $F_{IR} < F_{UV}$
$\rightarrow$**Strong version:** F **is monotonically decreasing along the RG flow.**

- the sphere regularizes IR divergencies.
- There are UV divergences which need to be regularized.
- The proof relies on **unitarity** and **locality**.

F-theorem on $O(N)^3$

The $O(N)^3$ is a non trivial test for the F-theorem:

- We know four (lines of) fixed points.
- One of them is an IR stable.
- There is am RG trajectory connecting the UV and the IR fixed point.
- Hints of unitarity at Large N.
- At finite N the unitarity is broken.

Mapping on the sphere

- In stereographic coordinates: $ds^2 = \frac{4a^2}{(1+x^2)^2} \sum_{i=1}^{d} dx_i{}^2$

$$\Rightarrow g_{\mu\nu}(x) = \Omega(x)^2 \delta_{\mu\nu}$$

- Given a *CFT* on flat space, we can map it on the sphere via

$$\mathcal{O}(x) \to \Omega(x)^{-\Delta_{\mathcal{O}}} \mathcal{O}(x)$$

- On the two point function:

$$\frac{1}{|x_1 - x_2|^{2\Delta_{\mathcal{O}}}} \to \frac{\Omega(x_1)^{-\Delta_{\mathcal{O}}} \Omega(x_2)^{-\Delta_{\mathcal{O}}}}{|x_1 - x_2|^{2\Delta_{\mathcal{O}}}}$$

which is equivalent to:

$$|x_1 - x_2| \to \Omega(x_1)^{1/2} \Omega(x_2)^{1/2} |x_1 - x_2| = s(x_1, x_2)$$

Laplacian on the sphere

- **Short range:** $\mathcal{L}_{free} = \frac{1}{2}\phi(-\partial^2 + b)\phi$

$$b = b_W \equiv \frac{d-2}{4(d-1)} R = \frac{d(d-2)}{4a^2} \;\Rightarrow\; \text{Weyl invariant}$$

  - $-\partial^2$ is diagonalized by spherical harmonics
  - Know spectrum

$$\omega_n = n(n+d-1)/a^2, \quad D_n = \frac{(n+d-2)!\,(2n+d-1)}{n!(d-1)!}$$

- **Long range** $(-\partial^2 + b_w) \;\Rightarrow\; \mathcal{D}_\zeta \neq (-\partial^2 + b_w)^\zeta$
  - Known spectrum (Branson 1995)

$$\omega_n^{(\zeta)} = a^{-2\zeta} \frac{\Gamma(n+\frac{d}{2}+\zeta)}{\Gamma(n+\frac{d}{2}-\zeta)}, \quad D_n = \frac{(n+d-2)!\,(2n+d-1)}{n!(d-1)!}$$

Flow between free theories

$$S_{\text{Gauss}}[\phi] = \frac{1}{2}\int \mathrm{d}^d x\, \phi(x)(-\partial^2)^\zeta \phi(x) + \frac{\lambda}{2}\int \mathrm{d}^d x\, \phi(x)(-\partial^2)\phi(x)$$

Whith $\zeta < 1$

- The propagator is $\frac{1}{p^{2\zeta}+\lambda p^2}$
  - In the UV $p \to \infty$: $\frac{1}{p^2}$
  - In the IR $p \to 0$: $\frac{1}{p^{2\zeta}}$

The theory flows from UV short range free theory to a IR long range free theory.

Flow between free theories

The free theory of a Gaussian theory is

$$F = \log Z = \frac{1}{2}\mathrm{Tr}\log\left[C^{-1}\right] \to \frac{1}{2}\sum_{n=0}^{\infty} D_n \log \omega_n^{(\zeta)}$$

- It can be computed analytically continuing from $d < 0$
- Take a derivative $\frac{dF}{d\zeta}$
- Use $\sum_{n=0}^{\infty} D_n = 0$ in $d < 0$

In the end we get

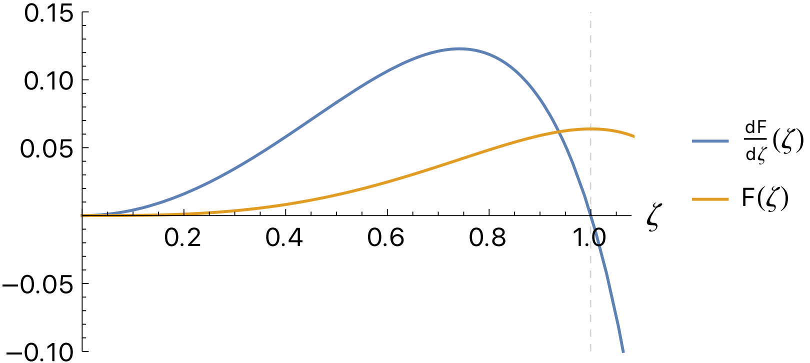
$$\frac{dF}{d\zeta} = -\zeta \frac{\sin(\pi\zeta)}{\sin(\pi d/2)} \frac{\Gamma(d/2-\zeta)\Gamma(d/2+\zeta)}{\Gamma(1+d)}$$

(Poles in d even as expected)

Flow between free theories

When $d = 3$ it simplifies a bit

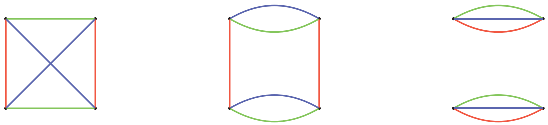
$$\frac{dF}{d\zeta} = \frac{1}{24}\pi\zeta\left(1 - 4\zeta^2\right)\tan(\pi\zeta)$$



The $O(N)^3$ model.

$$\mathcal{L} = \frac{1}{2}\phi_{abc}(-\partial^2)^{\zeta}\phi_{abc} + \mathcal{L}_{int}$$

- Symmetry group $O(N)^3$.
- Tune ζ such that ϕ^4 is marginal.
- 3 possible quartic interactions (Carrozza and Tanasa 2016; Klebanov and Tarnopolsky 2017)



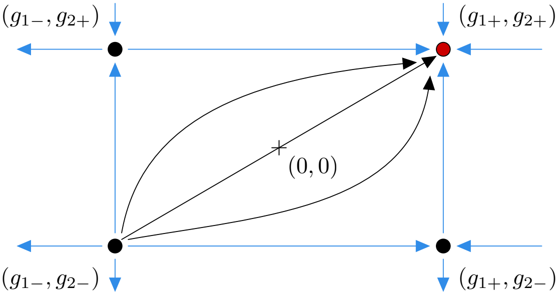
General features

- Manageble $1/N$ expansion.
- The coupling of the thetraedron g is precisely marginal.
- There are four lines of fixed points (parametrized by g) (Benedetti, Gurau, and Harribey 2019).
  - The fixed points are conformally invariant.
  - Exists trajectory connecting UV and IR fixed points.
- Strong indication of unitarity at large N.
- Non unitary at NLO.

note that putting the theory on the sphere does not change the structure of fixed points.

Fixed point structure

Flow diagram:

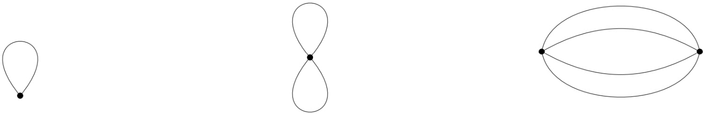


$$g_{1\pm} = \pm\sqrt{g^2} + \mathcal{O}(g^2)\,, \qquad g_{2\pm} = \pm\sqrt{3g^2} + \mathcal{O}(g^2)$$

SD equation on the sphere

$$\Gamma[G] = \frac{N^3}{2}\mathrm{Tr}\left[\underbrace{C^{-1}G + \ln(G^{-1})}_{\text{generic}} + \underbrace{m^{2\zeta}G + \frac{\lambda_2}{2}\mathcal{B} + \frac{\lambda^2}{4}\mathcal{B}^2}_{\text{2PI vacuum diagrams}}\right]$$

where G is the 2pt function and $\mathcal{B}(x,y) = G(x,y)^2$



The SD equation is:

$$\frac{\delta\Gamma[G]}{\delta G(x,y)} = 0$$

SD equation on the sphere

$$G^{-1}(x,y) = C^{-1}(x,y) + \left(m^{d/2} + \lambda_2 G(x,x)\right) \frac{\delta(x-y)}{\sqrt{g(x)}} + \lambda^2 G(x,y)^3$$

- $\sqrt{g(x)} = \Omega(x)^d = \left(\frac{2a}{(1+x^2)}\right)^d$
- $C(x,y) = \frac{c(\Delta)}{s(x,y)^{2\Delta}}\,, \quad c(\Delta) = \frac{\Gamma(\Delta)}{2^{d-2\Delta}\pi^{d/2}\Gamma(\frac{d}{2}-\Delta)}, \quad \Delta = \frac{d}{4}.$

Tune m^2 to cancel local divergences. $\Rightarrow$ like on flat space.

$$G_\star(x,y) = \mathcal{Z}\, C(x,y) \Rightarrow \underbrace{\mathcal{Z} = 1 + \lambda^2 \mathcal{Z}^4 \frac{4\Gamma(1-d/4)}{d(4\pi)^d \Gamma(3d/4)}}_{\text{4-Catalan numbers}}$$

$\mathcal{Z}$ **finite constant, square root singularity at** $\lambda = \lambda_c \sim 7$.

We finite $\mathrm{g} = \mathcal{Z}^2\lambda$ $\rightarrow$ resum melonic insertions

LO free energy

The LO free energy is $\Gamma_{LO}[G = \mathcal{Z} C]$

$$\Gamma_{LO}[G] = N^3 \left(\underbrace{\frac{1}{2}\mathcal{Z}\mathrm{Tr}[C^{-1}C]} + \frac{1}{2}\mathrm{Tr}[\ln(\mathcal{Z}^{-1}C^{-1})] + \frac{m^{2\zeta}}{2}\mathcal{Z}\int_x C(x,x) + \frac{\lambda_2 \mathcal{Z}^2}{4}\int_x C(x,x)^2 + \frac{\lambda^2 \mathcal{Z}^4}{8}\int_{x,y} C(x,y)^4 \right) .$$

- $\mathrm{Tr}[1] = 0$ in dimensional regularization.
- Gives N^3 times the free enrgy of a free scalar.
- $C(x,x) = 0$ in $d < 0 \ \rightarrow \ 0$ by analytical continuation
- melon integral M_ϵ

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LO free energy

M_ϵ **is divergent. We shift** $\Delta = \frac{d}{4} \to \frac{d-\epsilon}{4}$

$$M_\epsilon = c(\Delta)^4 \int_{x,y} \frac{1}{s(x,y)^{2(d-\epsilon)}}$$

Alternative point of view: on shell $G(x,y) \propto C(x,y)$ and $M_\epsilon = \mathrm{Tr}\left[C \star C^3\right]$

$$\text{For } \Delta = \frac{d}{4} \;\to\; C(x,y)^3 = \langle \phi_{\tilde{\Delta}}(x) \phi_{\tilde{\Delta}}(y) \rangle$$

We can think: $M_\epsilon = \int_{x,y} \langle \phi_\Delta(x)\phi_\Delta(y)\rangle \langle \phi_{\tilde{\Delta}-\epsilon}(x)\phi_{\tilde{\Delta}-\epsilon}(y)\rangle$

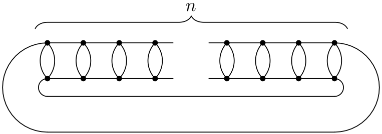
$$\epsilon > \frac{d}{2} > 0, \quad M_\epsilon = \frac{a^{2\epsilon}\,\Gamma\left(\frac{d+\epsilon}{4}\right)^4 \Gamma\left(-\frac{d}{2} + \epsilon\right)}{2^{3d-1}\,\pi^{d-1/2}\,\Gamma\left(\frac{d-\epsilon}{4}\right)^4 \Gamma\left(\frac{d+1}{2}\right)\Gamma(\epsilon)} \to 0$$

F-theorem is trivial

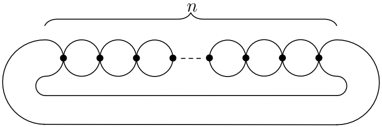
$\frac{1}{N}$ corrections

- The Free energy has a $\frac{1}{\sqrt{N}}$ expansion
- NLO and NNLO (Bonzom, Nador, and Tanasa 2019)
  - NLO: 'eight' diagram with tetrahedral coupling $\rightarrow 0$
  - NNLO: four types of diagrams **non trivial result**

1. Chain of ladders with tetrahedral vertices



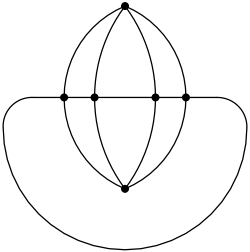
2. Chain of bubbles with pillow vertices



3. Mixed chain with ladders and bubbles

$\frac{1}{N}$ corrections

The last type of diagram is an isolated contribution



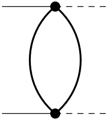
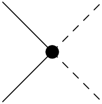
- Gives a finite contribution to the free energy
- Only tetrahedral vertices
- The tetrahedral coupling does not flow

For the F-theorem we can ignore this term!

NNLO free energy

$$\Gamma_{NNLO}[G] = \frac{N^2}{2}\left(\mathrm{Tr}[\ln(\mathbb{I} - K_1)] + \underbrace{\mathrm{Tr}[K_1]}_{\text{2PI condition}}\right)$$

Where the kernel K_1 is:

$K_1 = -\lambda^2$  $- \lambda_1$ 

Conformal Partial Wave (CPW) technology for resummation

CPW

Known that 3pt function diagonalize conformal kernels

$$\int_{x_3 x_4} K(x_1,x_2,x_3,x_4)\,|\psi^J_\Delta(x_3,x_4,x_0)\rangle = k(h,J)\,|\psi^J_\Delta(x_1,x_2,x_0)\rangle$$

- $|\psi\rangle$ is a basis of an appropriate space of bilocal functions
- Completeness relation: $\mathbb{I} = \sum_n a_n |\psi_n\rangle\langle\psi_n|$

$$\delta(x_1-x_3)\delta(x_2-x_4)$$

$$= \underbrace{\sum_{J\in\mathbb{N}_0}\int_{\frac{d}{2}}^{\frac{d}{2}+\mathrm{i}\infty}\frac{\mathrm{d}h}{2\pi\,\mathrm{i}}\rho(h,J)}_{\sum_n}\ \underbrace{\mathcal{N}^{\Delta}_{h,J}\mathcal{N}^{\tilde{\Delta}}_{\tilde{h},J}}_{a_n}\ \underbrace{\Psi^{\Delta,\Delta,\tilde{\Delta},\tilde{\Delta}}_{h,J}(x_1,x_2,x_3,x_4)}_{|\psi_n\rangle\langle\psi_n|},$$

Integral on principal series $\mathcal{P}_+ = \left\{h \,\middle|\, h = \frac{d}{2} + \mathrm{i}\,r,\ r \in \mathbb{R}_+\right\}$

Non normalizable contributions

When one primary has $\Delta < \frac{d}{2}$ $\rightarrow$ does not work

$\Rightarrow$ need to add a term to the completeness relation

$$\mathbb{I} = \sum_{J \in \mathbb{N}_0^{\text{even}}} \int_{\frac{d}{2}}^{\frac{d}{2}+\mathrm{i}\,\infty} \frac{\mathrm{d}h}{2\pi\,\mathrm{i}} \rho(h, J)\, \mathcal{N}_{h,J}^{\Delta} \mathcal{N}_{\tilde{h},J}^{\tilde{\Delta}}\, \Psi_{h,J}^{\Delta,\Delta,\tilde{\Delta},\tilde{\Delta}}(x_1, x_2, x_3, x_4)$$
$$- \sum_i \operatorname{Res}\left[\rho(h, J)\, \mathcal{N}_{h,J}^{\Delta} \mathcal{N}_{\tilde{h},J}^{\tilde{\Delta}}\, \Psi_{h,J}^{\Delta,\Delta,\tilde{\Delta},\tilde{\Delta}}(x_1, x_2, x_3, x_4)\right]_{h=h_i<\frac{d}{2}}$$

The extra term will be needed in the $O(N)^3$ model

NNLO free energy

$$F_{\mathrm{NNLO}} = \frac{N^2}{2} \sum_{J\in\mathbb{N}_0} \int_{\frac{d}{2}}^{\frac{d}{2}+\mathrm{i}\infty} \frac{\mathrm{d}h}{2\pi\,\mathrm{i}} \rho(h,J) \left(\ln(1-k(h,J)) + k(h,J) \right)$$
$$\times \mathcal{N}^{\Delta}_{h,J} \mathcal{N}^{\widetilde{\Delta}}_{\tilde{h},J} \, \mathrm{Tr}[\Psi^{\Delta,\Delta,\widetilde{\Delta},\widetilde{\Delta}}_{h,J}] + \dots$$

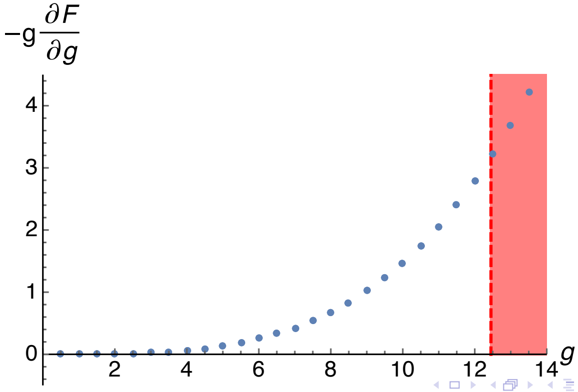
- $\rho(h,J)$ and $\mathcal{N}^{\Delta}_{h,J}$ known conformal quantities
- $k(h,J) = -\frac{g^2}{(4\pi)^d} \frac{\Gamma(-\frac{d}{4}+\frac{h+J}{2})\Gamma(\frac{d}{4}-\frac{h-J}{2})}{\Gamma(\frac{3d}{4}-\frac{h-J}{2})\Gamma(\frac{d}{4}+\frac{h+J}{2})}$
- Take a derivative with respect to g
- regularize $\mathrm{Tr}[\Psi^{\Delta,\Delta,\widetilde{\Delta},\widetilde{\Delta}}_{h,J}]$ shifting $\tilde{\Delta} \to \tilde{\Delta} - \epsilon$

NNLO free energy

We are able to evaluate numerically $-g\frac{\partial}{\partial g}F_{\text{NNLO}}$. For $g=1$, $a=1$ and $d=3$

$$-g\frac{\partial}{\partial g}F_{\text{NNLO}} = 7.57\times 10^{-4}\,N^2$$

$d=3$ and $a=1$ fixed:



the non-normalizable contribution

At $J = 0$ there are 2 solutions to $k(h,0) = 0$

$$h_\pm = \frac{d}{2} \pm \frac{2\sqrt{g^2}}{\Gamma(d/2)(4\pi)^{d/2}} + \mathcal{O}(|g|^3)$$

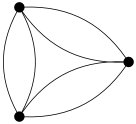
- h_+ **in the IR**
- h_- **in the UV** $\Rightarrow$ **left of the principal series!**

$$g\frac{\partial}{\partial g}\left(F^{UV}_{\mathrm{NNLO}} - F^{IR}_{\mathrm{NNLO}}\right) = N^2 \,\mathrm{Res}\left[\rho(h,0)\,\frac{k(h,0)^2}{1-k(h,0)}\,\mathcal{N}^{\Delta}_{h,0}\mathcal{N}^{\tilde{\Delta}}_{\tilde{h},0}\,\underbrace{\mathcal{I}(0)}_{\mathrm{Tr}\Psi}\right]_{h_-}$$

$$= 16\frac{\Gamma(-d/2)|g|^3}{2^{3d}\pi^{3d/2}\Gamma(d)}N^2 + \mathcal{O}(|g|^5) > 0$$

Perturbative check

- **One contribution to order g^3:**



- **one** g **two** g_1 or **three** g_1
- g_1 **jumps** $g_1 = -\sqrt{g^2}$ **UV** $\rightarrow g_1 = +\sqrt{g^2}$

$$g\frac{\partial}{\partial g}(F^{UV}_{\mathrm{NNLO}} - F^{IR}_{\mathrm{NNLO}}) = 2|g|^3 N^2 \mathcal{A} + \mathcal{O}(|g|^5)$$

$$= 16\frac{\Gamma(-d/2)|g|^3}{2^{3d}\pi^{3d/2}\Gamma(d)}N^2 + \mathcal{O}(|g|^5)$$

Conclusions

- We tested the F-theorem in a non trivial case:
    - Interacting theory
    - Long-range
    - Hopefully unitary
- **The F-theorem holds $\rightarrow$ further hint of unitarity**
- **We applied the CPW technology to resum an infinite family of diagrams**
- Computed numerically the value of regularized free energy

Thank you!

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