

WHAT'S THAT SPECTRAL TRIPLE

PLAN FOR THIS TALK:

- WHAT IS A SPECTRAL TRIPLE?
- WHAT IS A FUZZY SPACE?
- PAINT ME A PICTURE

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GEOMETRY AS A SPECTRAL TRIPLE

$$(A, \mathcal{H}, \mathcal{D})$$

AXIOMS OF SPECTRAL TRIPLES

▶ faithful action

$$A^{\text{op}} \text{ on } \mathcal{H} \xrightarrow{a \cdot v}$$

▶ bimodule

$$\mathcal{H} \text{ over } A \quad a \cdot v \quad v \cdot a$$

▶ first order condition

$$[a, [\mathcal{D}, b]] = 0$$

(A. Connes, Int.J.Geom.Meth.Mod.Phys. 5, 1215-1242 (2008))

(more detail e.g. A. Connes, Commun.Math.Phys. 182, 155-176 (1996))

SPECTRAL TRIPLES AS QUANTUM GEOMETRY?

I prefer my space-time discrete/ finite. Two options:

Fuzzy spaces:

- very symmetric
- use finite Δ
- D respects first order

Truncations of spectral triples:

(PAP, PHP, PDP)

operator system
spectral triple

breaks first order

FUZZY SPACE (p, q)

(p, q)

$(s, \mathcal{H}, \mathcal{A}, \Gamma, J, \mathcal{D})$

- ▶ The algebra are matrices:

\ast -algebra $M(n, \mathbb{C})$

- ▶ Acting on a Hilbert space:

$V \otimes M(n, \mathbb{C})$ $V \subset$ Clifford module

EXTRA INGREDIENTS FOR A REAL SPECTRAL TRIPLE

- ▶ KO-dimension; $s = (q - p) \% 8$
- ▶ Chirality; $\Gamma(v \otimes m) = \gamma v \otimes m$
- ▶ Real structure; $\jmath(v \otimes m) = C v \otimes m^*$

(as stated in J.W. Barrett J.Math.Phys. 56, 082301 (2015))

$$\langle \jmath \psi, \jmath \phi \rangle = \langle \gamma, \phi \rangle$$

$$\begin{matrix} a v \\ \jmath^{-1} a \jmath v \end{matrix}$$

DIRAC OPERATOR : FORM

Conditions on \mathcal{D} for a real spectral triple

$$\mathcal{D} = \mathcal{D}^*$$

$$\mathcal{D}\Gamma = \pm \Gamma\mathcal{D}$$

$$\mathcal{D}\gamma = \pm \gamma\mathcal{D}$$

$$[[\mathcal{D}, a], b] = 0 \quad \text{right}$$

Can be translated for a fuzzy space to:

$$\mathcal{D}(v \otimes m) = \sum_i \omega^i v \otimes \left(\overset{\text{left}}{k_j^* m + \varepsilon} \quad \downarrow \quad m k_j^* \right)$$

(1,3)

$$\mathcal{D} = \sum_{j < k}^3 \gamma^0 \gamma^j \gamma^k v \otimes [L_{jk}, \cdot] + \gamma^1 \gamma^2 \gamma^3 v \otimes [H_{123}, \cdot] + \gamma^0 \otimes \{A_4, \cdot\} + \sum_{i=1}^3 \gamma^i v \otimes [L_i, \cdot]$$

(J.W. Barrett, J.Math.Phys. 56, 082301 (2015).)

THE CONTINUUM SPHERE

$$(A = \mathfrak{S}(2), \mathcal{H} = L^2(S^2, \mathbb{C}), D = \sigma^\mu (\partial_\mu + \omega_\mu))$$

with σ^μ the Pauli matrices and ω_μ a spin connection.

The fuzzy sphere is a finite spectral triple that approximates this.

- ▶ $A = M(n, \mathbb{C})$ irred. rep. $SU(2)$ $j = \frac{1}{2}(n-1)$
 - ▶ $\mathcal{H} = \mathbb{C}^4 \otimes M(n, \mathbb{C})$
 - ▶ $D = \gamma^0 \nu \otimes m + \sum_{j < k}^3 \gamma^j \gamma^k \otimes [L_{jk}, \cdot]$
- $L_{jk} \quad so(3)$

EXPLORE PATH INTEGRAL OVER FUZZY SPACES

$$\begin{aligned}
 \langle f \rangle &= \frac{\int f(\mathcal{D}) e^{-s(\mathcal{D})} d\mathcal{D}}{\int e^{-s(\mathcal{D})} d\mathcal{D}} \\
 &= \frac{\int f(\mathcal{D}(k_i)) e^{-s(\mathcal{D}(k_i))} \prod_i \pi d k_i}{Z}
 \end{aligned}$$

THE SIMPLEST ACTION

$$\mathcal{S} = g_2 \text{Tr}(\mathcal{D}^2) + \text{Tr}(\mathcal{D}^4)$$

(J. Barrett, LG J.Phys. A49, 245001 (2016))

$$\mathcal{S} = \text{Tr} \left(\int \left(\frac{\mathcal{D}}{\lambda} \right) \right)$$

WHAT DO WE WANT FROM AN ACTION?

- ▶ physical motivation
 ⇒ lowest order expanding Heat kernel
- ▶ bounded from below
- ▶ rises fast to infinity



THE SIMPLEST ACTION

$$\mathcal{S} = g_2 \text{Tr}(\mathcal{D}^2) + \text{Tr}(\mathcal{D}^4)$$

(J. Barrett, LG J.Phys. A49, 245001 (2016))

(2, 0) GEOMETRY

$$\mathcal{D} = \gamma^1 \otimes \{H_1, \cdot\} + \gamma^2 \otimes \{H_2, \cdot\}$$

$$\text{tr} \mathcal{D}^2 = 4n(\text{tr} H_1^2 + \text{tr} H_2^2) + 4((\text{tr} H_1)^2 + (\text{tr} H_2)^2)$$

$$\text{tr} \mathcal{D}^4 = 4n \left(\text{tr} H_1^4 + \text{tr} H_2^4 + 4 \text{tr} H_1^2 H_2^2 - 2 \text{tr} H_1 H_2 H_1 H_2 \right)$$

$$+ 16 \left(\text{tr} H_1 (\text{tr} H_1^3 + \text{tr} H_2^2 H_1) + \text{tr} H_2 (\text{tr} H_1^2 H_2 + \text{tr} H_2^3) + (\text{tr} H_1 H_2)^2 \right) +$$

$$12 \left((\text{tr} H_1^2)^2 + (\text{tr} H_2^2)^2 \right) + 8 \text{tr} H_1^2 \text{tr} H_2^2$$

⇒ Matrix model

LOOK FOR PHASE TRANSITIONS

PHASE TRANSITION

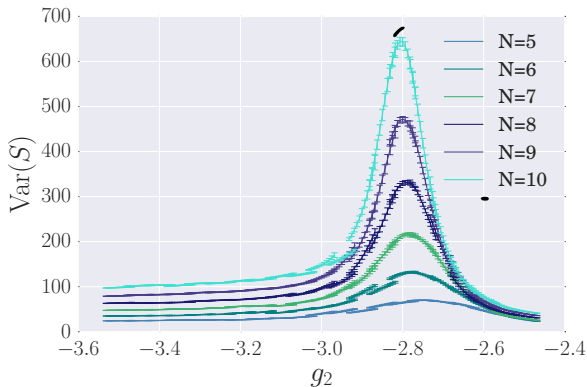
- ▶ qualitative change in behavior
- ▶ Phase transition marked by peak in Variance

$$\text{Var}(S) = \langle S^2 - \langle S \rangle^2 \rangle$$

- ▶ Gets sharper in larger systems
- ▶ Higher order phase transitions show signs of correlation

(LG J.Phys.A50, 275201 (2017))

LOOK FOR PHASE TRANSITIONS

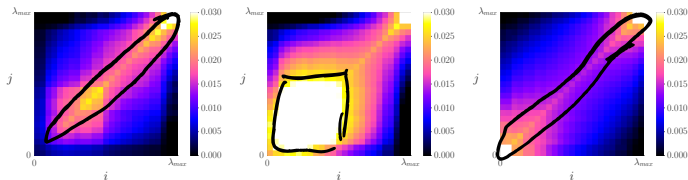


$$g_c^{(2,0)} = -2.781 \pm 0.289$$

(LG J.Phys.A50, 275201 (2017))

$\text{Cov}(\lambda_i^2, \lambda_j^2)$ TYPE (2, 0)

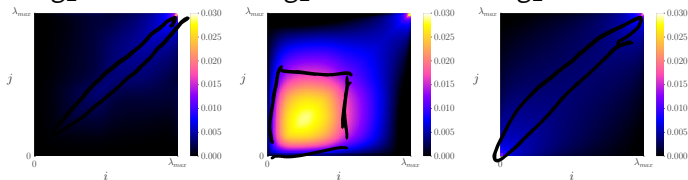
$N = 5$



$g_2 = -2.5$

$g_2 = -2.8$

$g_2 = -3.5$



$N = 10$

(LG J.Phys.A50, 275201 (2017))

THE PLAN

- What is a spectral triple?
- What is a fuzzy space?
- Paint me a picture

A DISTANCE MEASURE FOR SPECTRAL TRIPLES

$$d(\omega_1, \omega_2) = \sup_{\substack{a \in \mathcal{A} \\ \|a\| \leq 1}} \{ \|\omega_1(a) - \omega_2(a)\| \}$$

coherent states

$$\begin{pmatrix} d_{11} & d_{12} & d_{13} & \dots \\ \vdots & & & \end{pmatrix}$$

Distance:

(A. Connes, *Noncommutative Geometry*. (Academic Press, 1994))

Coherent states:

(L. Schneiderbauer, H. Steinacker 2016, *J.Phys.* A49 285301)

IMPLEMENTING THE DISTANCE CALCULATION

$$d(\omega_1, \omega_2) = \sup_{a \in \mathcal{A}} \{ |\omega_1(a) - \omega_2(a)| : \|[D, a]\| \leq 1 \}$$

- ▶ Parametrise algebra elements

- ▶ Minimize dispersion w constraint

WHAT IS THE DISPERSION?

Using the algebraic data we could estimate the dispersion as

$$\eta(\omega) =$$

PROBLEMS:



HOW DO WE DEFINE STATES?

SOLUTIONS

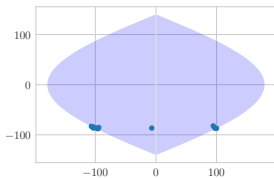
- ▶ Use embedding maps
- ▶ Add repulsive potential

$$\eta(\omega_k) = \sum_i \langle \omega | Y_i^2 | \omega \rangle - \langle \omega | Y_i | \omega \rangle^2 + \sum_{j < k} \frac{c}{d(\omega_j, \omega_k)}$$

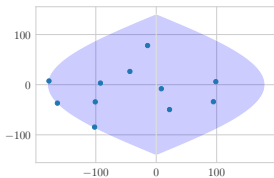
Now find a set of coherent states ω that minimizes this and plug them into distance equation.

EFFECT OF THE REPULSIVE POTENTIAL

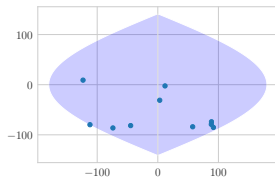
$c = 0$



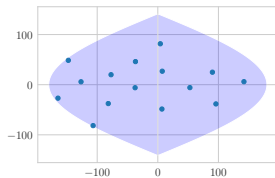
$c = 0.1$



$c = 0.001$



$c = 1000$

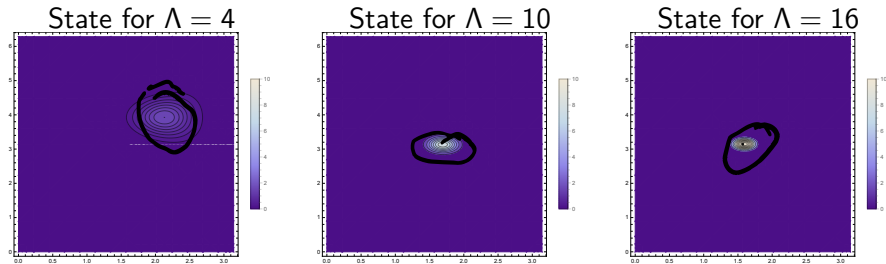


SKETCH THE PROOF

states \longleftrightarrow point

detailed version

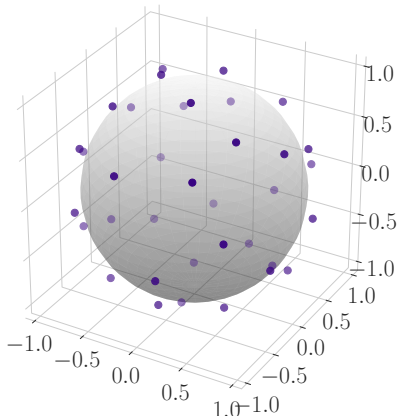
HOW DOES THE DISPERSION CHANGE WITH Λ ?



THE ALGORITHM FOR STATE GENERATION

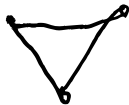
- 1: Find a vector v_0 (globally) minimizing η . Set $V = \{v_0\}$.
- 2: **while** $\sqrt{\eta(v)} + \sqrt{\eta(w)} \leq \alpha d(v, w)$ for $v \neq w \in V$, **do**
- 3: Find a vector w (locally) minimizing $e(w; V)$.
- 4: Append w to V .
- 5: **for** $v \in V$, **do**
- 6: Set $d(v, w) = \min\{|\langle v, av \rangle - \langle w, aw \rangle| : |[D, a]| \leq 1\}$.
- 7: **end for**
- 8: **end while**

A PICTURE OF GEOMETRY



⇐ The truncated sphere at $\Lambda = 5$

- ▶ run algorithm → generate states and their distance matrix
- ▶ use graph embedding algorithm → find a locally isometric embedding
- ▶ wonder why the analytic solution is smaller



Samalob

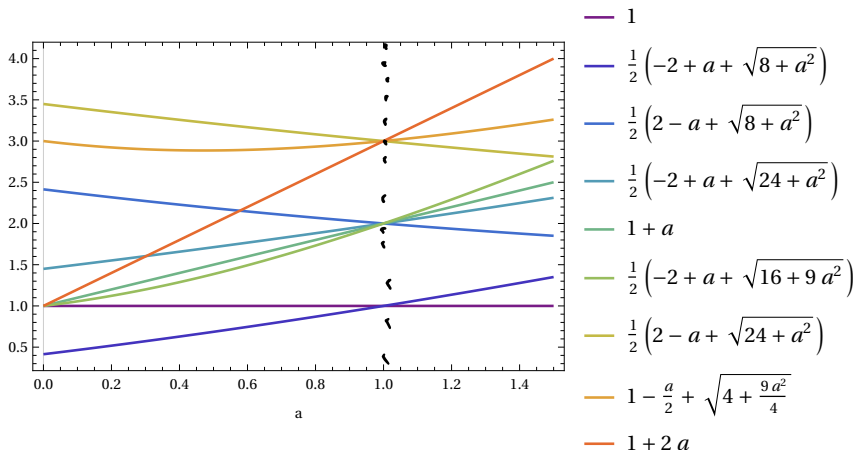
DEFORMED FUZZY SPHERE

$$\mathcal{D} = \gamma_0 + \sum c_{jk} \gamma_0 \gamma_s \gamma_k \otimes [L_{jk, i}]$$

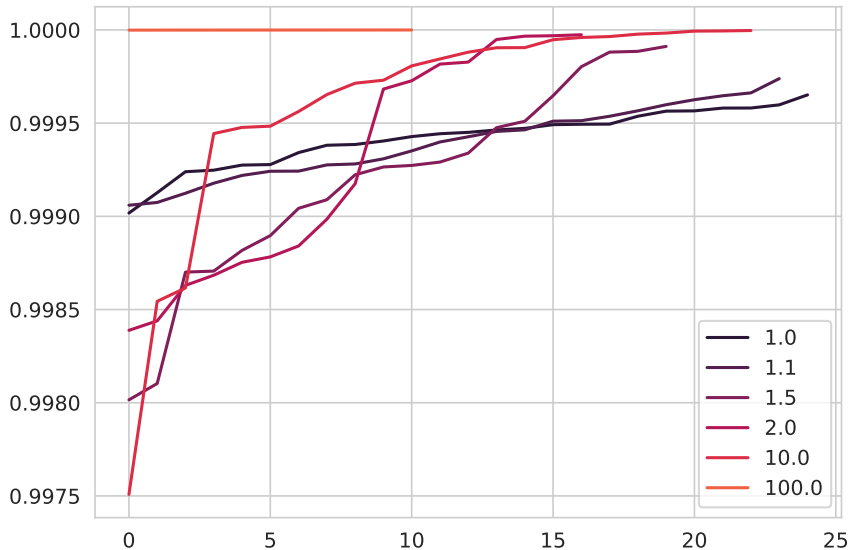
$$c_{jk} = \{c, 1, 1\}$$

EIGENVALUES OF THE DEFORMED FUZZY SPHERE

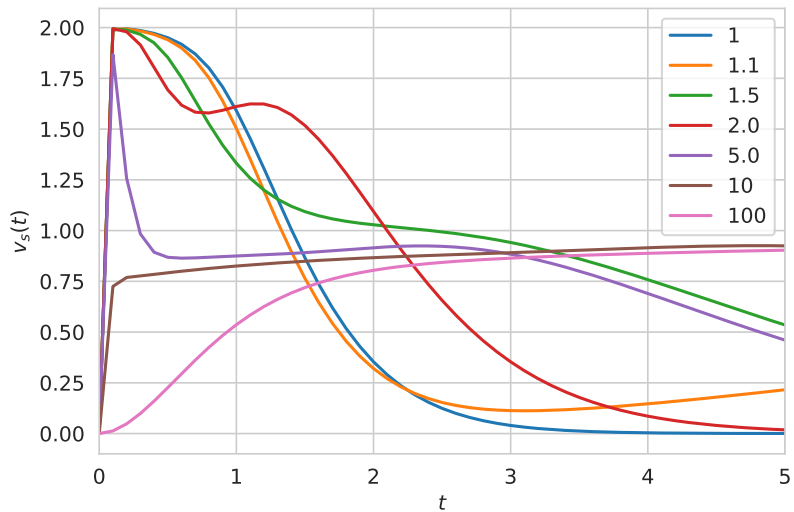
EIGENVALUES OF THE DEFORMED FUZZY SPHERE



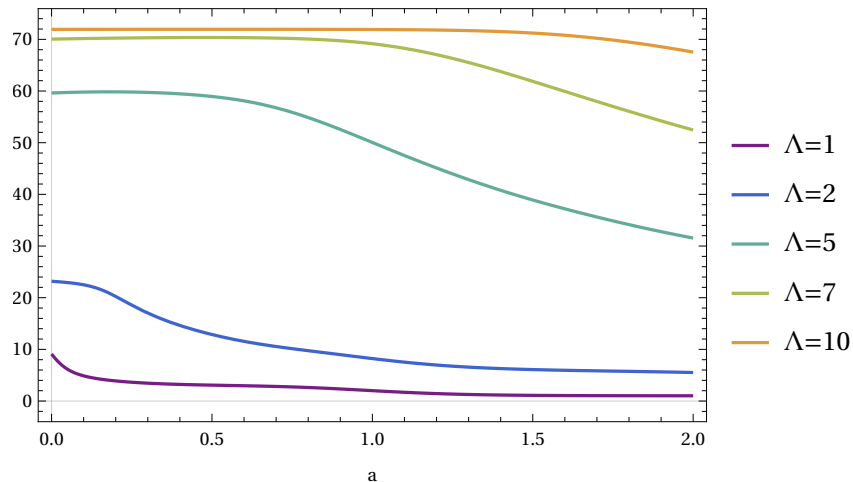
ACCURACY OF THE EMBEDDING



SPECTRAL DIMENSION OF THE DEF. FUZZY SPHERE



SPECTRAL ACTION OF THE DEF. FUZZY SPHERE



SUMMARY

TODAYS STORY:

- ▶ Exploring spectral triples using computer simulations
- ▶ a multi matrix model from finite spectral triples
- ▶ a picture of a spectral triple using Connes distance function

FUTURE PLANS:

- ▶ More visualisations:
 - ▶ fuzzy torus, something about the different topology messes w. my algorithm
- ▶ Analytic results on the deformed fuzzy sphere
- ▶ Can we create 'bespoke' fuzzy spaces

Quantum
Gravity Across
Approaches

- ▶ Next talk, next week: 26.5.22 at 18.00
- ▶ Event on: Black Holes and Gravitational Waves
- ▶ by Steve Giddings (UC Santa Barbara) and Hal Haggard (Bard College)
- ▶ last talk of "Season 2 - Observation", but starting next autumn will be "Season 3 - Measure and topology change"

<https://sites.google.com/view/qg-aa>

THANK YOU!

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OR TWITTER: @GRAVITYWITHHAT

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MONTE CARLO SIMULATIONS

- ▶ Simulate a Path integral, use Monte Carlo Markov Chain to calculate averages
- ▶ Use Markov Chain to probe space of solutions to find an optimum. Only examine the solution with minimal value of something.

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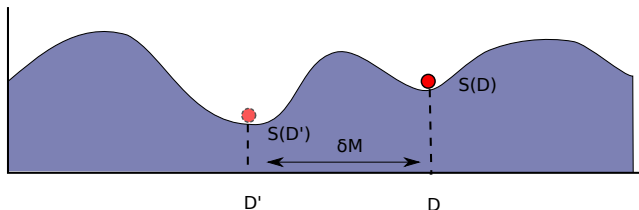
NOTE:

It is proven that the Metropolis algorithm will find the global optimum if sampled long enough.

MARKOV CHAIN METHODS IN ONE SLIDE

THE METROPOLIS HASTINGS ALGORITHM

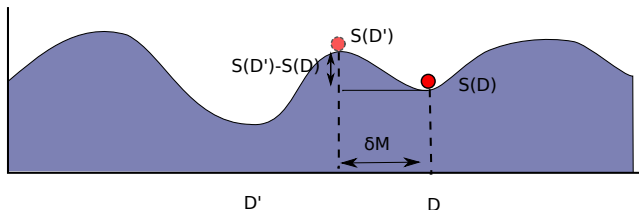
- ▶ propose new operator D'
 $D \rightarrow D' = D + \delta M$ with δM some small matrix
- ▶ if $S(D') < S(D)$ accept D' and add to the chain
- ▶ otherwise calculate $\exp\{-S(D') + S(D)\}$ & generate random uniform $p \in [0, 1]$
 - ▶ if $p < \exp\{-S(D') + S(D)\}$ accept D'
 - ▶ else add D to the chain again



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STATES ARE POINTS

An element \mathbf{v} of $\mathbb{P}(H_\Lambda)$ that is considered to be localized should be localized *somewhere*, that is, around some ‘barycenter’ $b(\mathbf{v}) \in M$.

We can prove:

PROPOSITION

There exists a map $b: \mathbb{P}(H_\Lambda) \rightarrow M$ such that

$$|d_\Lambda(\mathbf{v}, \mathbf{w}) - d_M(b(\mathbf{v}), b(\mathbf{w}))| = O(\sqrt{\eta(\mu_\mathbf{v})} + \sqrt{\eta(\mu_\mathbf{w})})$$

as $\eta(\mu_\mathbf{v}), \eta(\mu_\mathbf{w}) \rightarrow 0$, uniformly in \mathbf{v}, \mathbf{w} .

POINTS ARE STATES

The converse: Each point x in M can be approximated through a state \mathbf{v} with small dispersion and with barycenter $b(\mathbf{v})$ close to x

PROPOSITION

Let M be equipped with a Dirac-type operator D on a Hermitian vector bundle $\pi: \mathbf{S} \rightarrow M$, and let $\tilde{\pi}: \mathbb{P}(\mathbf{S}) \rightarrow M$ be its projectivized bundle. Then, there exists a family $\{\Phi_\Lambda\}_\Lambda$ of maps $\Phi_\Lambda: \mathbb{P}(\mathbf{S}) \rightarrow \mathbb{P}(H_\Lambda)$ such that for all $\epsilon > 0$,

- ▶ $d_\Lambda(\Phi_\Lambda(v), \Phi_\Lambda(w)) = d_M(\tilde{\pi}(v), \tilde{\pi}(w)) + \tilde{O}(\Lambda^{-1})$ uniformly.
- ▶ The dispersion $\eta(\mu)$ of the measure μ associated to $\Phi_\Lambda(v)$ is $\tilde{O}(\Lambda^{-2})$ uniformly.
- ▶ The maps Φ_Λ asymptotically invert b , in the sense that $d_M(\tilde{\pi}(v), b(\Phi_\Lambda(v))) = \tilde{O}(\Lambda^{-1})$ uniformly and $d_\Lambda(\Phi_\Lambda(v), \mathbf{v}) = \tilde{O}(\sqrt{\eta(\mu_v)} + \Lambda^{-2})$ uniformly whenever $b(\mathbf{v}) = \tilde{\pi}(v)$.