What's that spectral triple

Plan for this talk:

- What is a spectral triple?
- What is a fuzzy space?
- Paint me a picture

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universität wien

Geometry as a spectral triple

$(\mathcal{A}, \mathcal{H}, \mathcal{D})$

Axioms of spectral triples

- faithful action $\mathcal{A}^{op}$ on $\mathcal{H}^{\ni v}$ av
- bimodule $\mathcal{H}$ over $\mathcal{A}$ av va
- first order condition $[a, [b, \mathcal{D}]] = 0$

(A. Connes, Int.J.Geom.Meth.Mod.Phys. 5, 1215-1242 (2008))

(more detail e.g. A. Connes, Commun.Math.Phys. 182, 155-176 (1996))

Spectral triples as quantum geometry?

I prefer my space-time discrete/ finite. Two options:

Fuzzy spaces:

- very symmetric
- use finite $\mathcal{A}$
- D respects first order

Truncations of spectral triples:

(PAP, PHP, PDP)

Operator system spectral triple

breaks first order

Fuzzy space (p, q)

(p, q)

$(s, \mathcal{H}, \mathcal{A}, \Gamma, J, \mathcal{D})$

- The algebra are matrices:
  $*$-algebra $M(n, \mathbb{C})$
- Acting on a Hilbert space:
  $V \otimes M(n, \mathbb{C})$ V - Clifford module

Extra ingredients for a real spectral triple

- KO-dimension; $s = (q - p) \% 8$
- Chirality; $\Gamma(v \otimes m) = \gamma v \otimes m$
- Real structure; $J(v \otimes m) = Cv \otimes m^*$

(as stated in J.W. Barrett J.Math.Phys. 56, 082301 (2015))

$\langle J\psi, J\phi \rangle = \langle \psi, \phi \rangle$

$a v$

$J^{-1} a J v$

Dirac operator : Form

Conditions on $\mathcal{D}$ for a real spectral triple

$$\mathcal{D} = \mathcal{D}^* \qquad \mathcal{D}\Gamma = \pm \Gamma \mathcal{D}$$

$$\mathcal{D}J = \pm J\mathcal{D} \qquad [[\mathcal{D}, a], b] = 0$$

right

Can be translated for a fuzzy space to:

left

$$\mathcal{D}(v \otimes m) = \sum_i \omega^i v \otimes \left(K_i m + \varepsilon\, m K_i^* \right)$$

(1,3)

$$\mathcal{D} = \sum_{j<k}^{3} \gamma^0 \gamma^j \gamma^k v \otimes [L_{jk}, \cdot\,] + \gamma^1 \gamma^2 \gamma^3 v \otimes \{H_{123}, \cdot\,\} + \gamma^0 \cancel{v} \otimes \{H_0, \cdot\,\} + \sum_{i=1}^{3} \gamma^i v \otimes [L_i, \cdot\,]$$

(J.W. Barrett, J.Math.Phys. 56, 082301 (2015).)

Fuzzy sphere

$(1,3)$

The continuum sphere

$$\left(\mathcal{A} = Su(2),\ \mathcal{H} = L^2(S^2, S),\ D = \sigma^\mu(\partial_\mu + \omega_\mu)\right)$$

with σ^μ the Pauli matrices and ω_μ a spin connection.

The fuzzy sphere is a finite spectral triple that approximates this.

- $\mathcal{A} =$ $M(n, \mathbb{C})$ irred. reps. $Su(2)$ $j = \frac{1}{2}(n-1)$
- $\mathcal{H} =$ $\mathbb{C}^4 \otimes M(n, \mathbb{C})$
- $\mathcal{D} =$ $\gamma^0 \nu \otimes m + \sum_{j<4}^{3} \gamma^0 \gamma^j \gamma^4 \otimes [L_{j4}, \cdot\,]$

L_{j4} $so(3)$

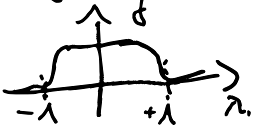
Explore path integral over fuzzy spaces

$$\langle f \rangle = \frac{\int f(D)\, e^{-S(D)}\, dD}{\int e^{-S(D)}\, dD}$$

$$= \frac{\int f(D(K_i))\, e^{-S(D(K_i))} \prod_i dK_i}{Z}$$

The simplest action

$S = \mathrm{Tr}\left(f\left(\frac{D}{\Lambda}\right)\right)$



$$\mathcal{S} = g_2 \mathrm{Tr}\left(\mathcal{D}^2\right) + \mathrm{Tr}\left(\mathcal{D}^4\right)$$

(J. Barrett, LG J.Phys. A49, 245001 (2016))

What do we want from an action?

- physical motivation

  $\Rightarrow$ lowest order expanding Heat kernel
- bounded from below
- rises fast to infinity

The simplest action

$$\mathcal{S} = g_2 \mathrm{Tr}\left(\mathcal{D}^2\right) + \mathrm{Tr}\left(\mathcal{D}^4\right)$$

(J. Barrett, LG J.Phys. A49, 245001 (2016))

$(2,0)$ geometry

$$\mathcal{D} = \gamma^1 \otimes \{H_1, \cdot\} + \gamma^2 \otimes \{H_2, \cdot\}$$

Matrix model

$$\mathrm{tr}\,\mathcal{D}^2 = 4n(\mathrm{tr}\,H_1^2 + \mathrm{tr}\,H_2^2) + 4\left((\mathrm{tr}\,H_1)^2 + (\mathrm{tr}\,H_2)^2\right)$$

$$\begin{aligned}\mathrm{tr}\,\mathcal{D}^4 = 4n\Bigg(&\mathrm{tr}\,H_1^4 + \mathrm{tr}\,H_2^4 + 4\,\mathrm{tr}\,H_1^2H_2^2 - 2\,\mathrm{tr}\,H_1H_2H_1H_2\Bigg)\\ &+ 16\Bigg(\mathrm{tr}\,H_1\left(\mathrm{tr}\,H_1^3 + \mathrm{tr}\,H_2^2H_1\right) + \mathrm{tr}\,H_2\left(\mathrm{tr}\,H_1^2H_2 + \mathrm{tr}\,H_2^3\right) + (\mathrm{tr}\,H_1H_2)^2\Bigg) +\\ &12\Bigg((\mathrm{tr}\,H_1^2)^2 + (\mathrm{tr}\,H_2^2)^2\Bigg) + 8\,\mathrm{tr}\,H_1^2\,\mathrm{tr}\,H_2^2\end{aligned}$$

Look for phase transitions

Phase transition

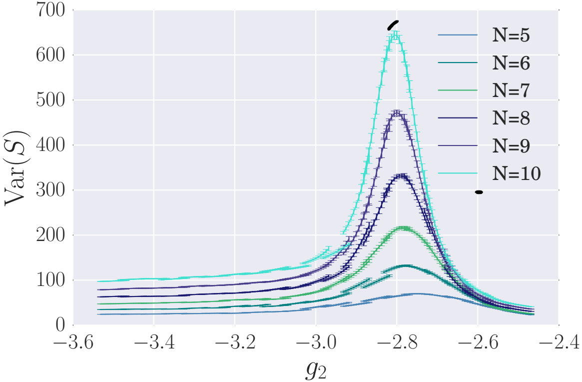
- qualitative change in behavior
- Phase transition marked by peak in Variance

$$Var(\mathcal{S}) = \left\langle \mathcal{S}^2 - \langle \mathcal{S} \rangle^2 \right\rangle$$

- Gets sharper in larger systems
- Higher order phase transitions show signs of correlation

(LG J.Phys.A50, 275201 (2017))

Look for phase transitions

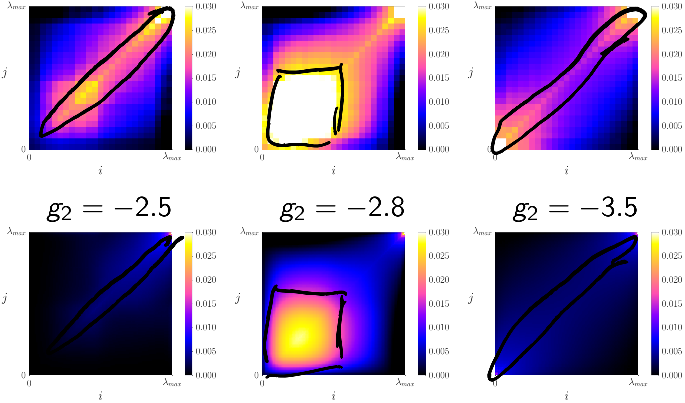


$$g_c^{(2,0)} = -2.781 \pm 0.289$$

(LG J.Phys.A50, 275201 (2017))

$\text{Cov}(\lambda_i^2, \lambda_j^2)$ Type $(2, 0)$

$N = 5$



$N = 10$

(LG J.Phys.A50, 275201 (2017))

The plan

- What is a spectral triple?
- What is a fuzzy space?
- Paint me a picture

$$d(\omega_1, \omega_2) = \sup_{a \in \mathcal{A}} \left\{ \left| \omega_1(a) - \omega_2(a) \right| : \| [D, a] \| \leq 1 \right\}$$

coherent states

$$\begin{pmatrix} d_{11} & d_{12} & d_{13} & \cdots \\ \vdots & & & \\ & & & \end{pmatrix}$$

Distance:

(A. Connes, Noncommutative Geometry. (Academic Press, 1994))

Coherent states:

(L. Schneiderbauer, H. Steinacker 2016, J.Phys. A49 285301)

Implementing the distance calculation

$$d(\omega_1, \omega_2) = \sup_{a \in \mathcal{A}} \{ |\omega_1(a) - \omega_2(a)| : ||[D, a]|| \leq 1 \}$$

- Parametrise algebra elements

- Minimize dispersion w constraint

What is the dispersion?

Using the algebraic data we could estimate the dispersion as

$$\eta(\omega) =$$

Problems:

-
-

How do we define states?

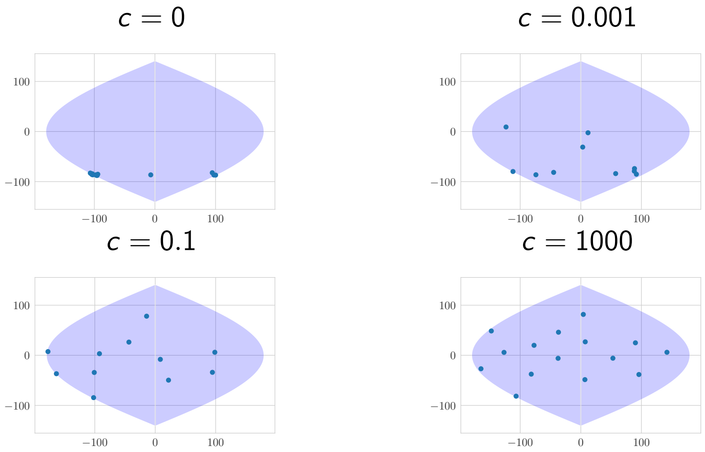
Solutions

- Use embedding maps
- Add repulsive potential

$$\eta(\omega_k) = \sum_i \langle\omega|Y_i^2|\omega\rangle - \langle\omega|Y_i|\omega\rangle^2 + \sum_{j<k} \frac{c}{d(\omega_j, \omega_k)}$$

Now find a set of coherent states ω that minimizes this and plug them into distance equation.

Effect of the repulsive potential

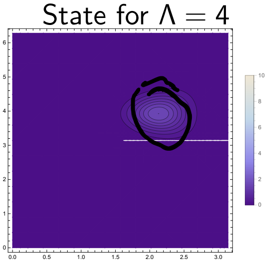


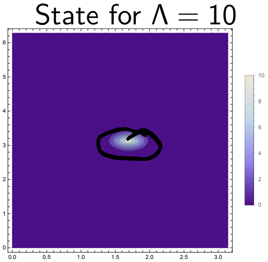
Sketch the proof

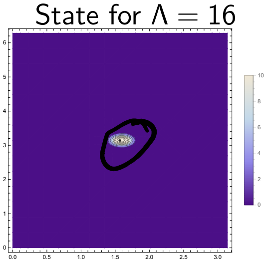
states $\longleftrightarrow$ point

detailed version

How does the dispersion change with Λ?



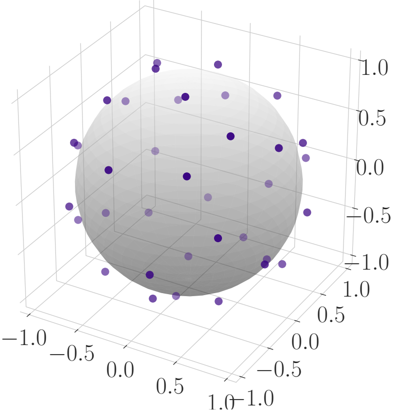




The algorithm for state generation

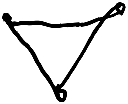
Find a vector v_0 (globally) minimizing η. Set $V = \{v_0\}$.
while $\sqrt{\eta(v)} + \sqrt{\eta(w)} \leq \alpha d(v, w)$ for $v \neq w \in V$, **do**
    Find a vector w (locally) minimizing $e(w; V)$.
    Append w to V.
    for $v \in V$, **do**
        Set $d(v, w) = \min\{|\langle v, av\rangle - \langle w, aw\rangle| : \|[D, a]\| \leq 1\}$.
    end for
end while

A picture of geometry



⇐ The truncated sphere at $\Lambda = 5$

- run algorithm → generate states and their distance matrix
- use graph embedding algorithm → find a locally isometric embedding
- wonder why the analytic solution is smaller



smacof

Deformed Fuzzy sphere

$$D = \gamma_0 \quad + \sum c_{jk} \gamma_0 \gamma_5 \gamma_k \otimes [L_{jk}, \cdot]$$

$$c_{jk} = \{c, 1, 1\}$$

Eigenvalues of the deformed fuzzy sphere

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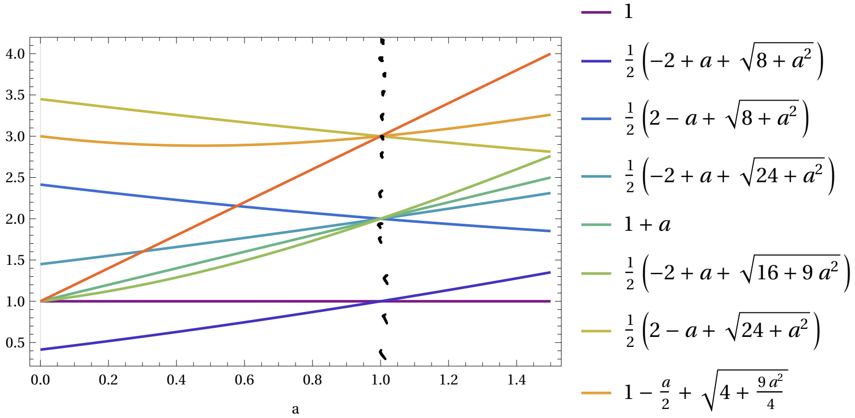
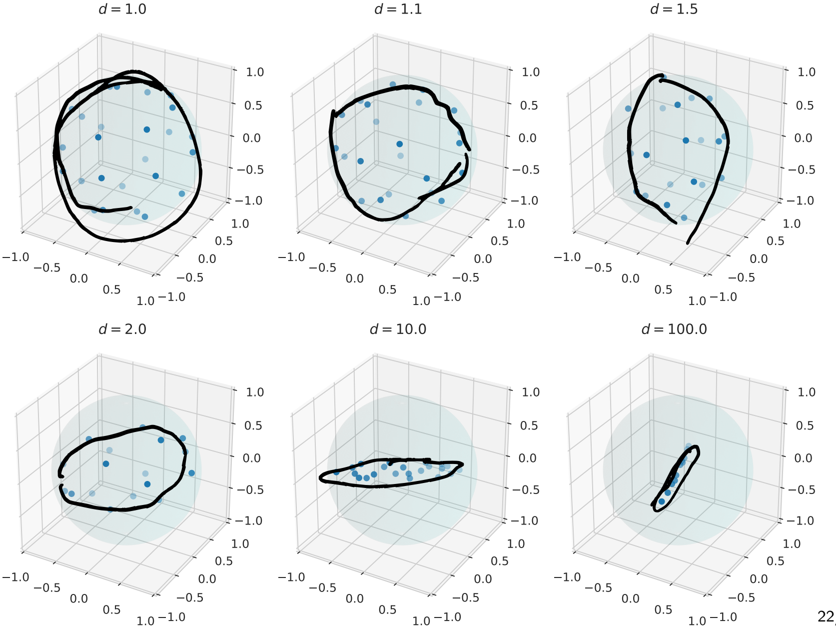
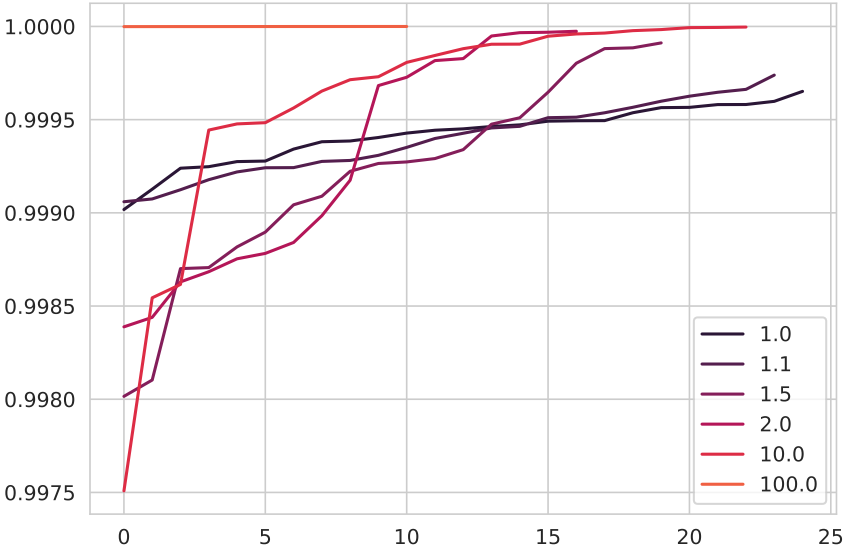


Visualisation of the deformed fuzzy sphere

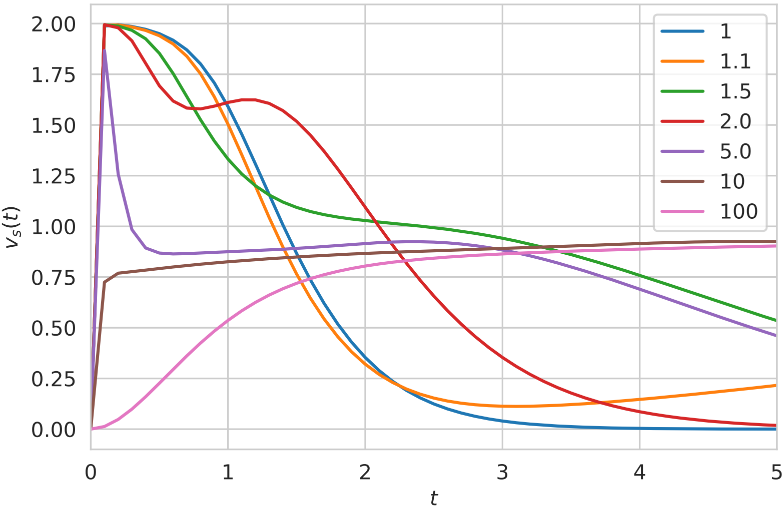


can I add a QFT
on the localized states?

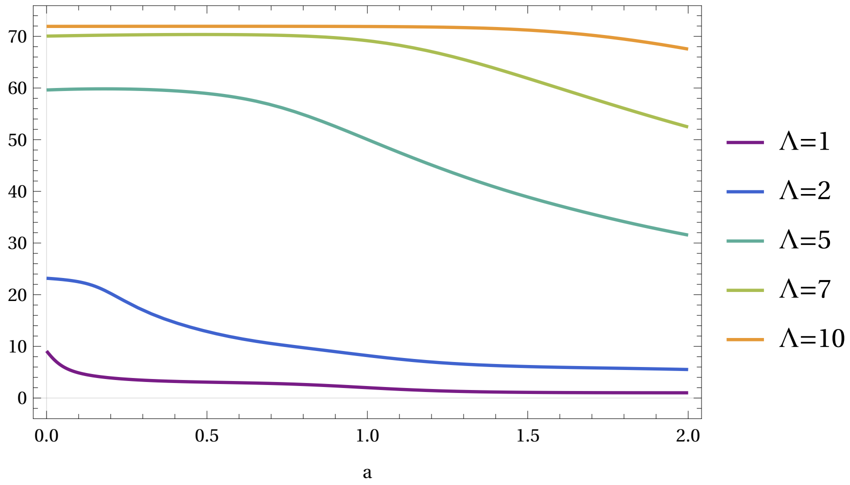
Accuracy of the embedding



Spectral dimension of the def. fuzzy sphere



Spectral action of the def. fuzzy sphere



Summary

Todays story:

- Exploring spectral triples using computer simulations
- a multi matrix model from finite spectral triples
- a picture of a spectral triple using Connes distance function

Future plans:

- More visualisations:
  - fuzzy torus, something about the different topology messes w. my algorithm
- Analytic results on the deformed fuzzy sphere
- Can we create 'bespoke' fuzzy spaces

Advertisement break

Quantum Gravity Across Approaches

- Next talk, next week: 26.5.22 at 18.00
- Event on: Black Holes and Gravitational Waves
- by Steve Giddings (UC Santa Barbara) and Hal Haggard (Bard College)
- last talk of “Season 2 - Observation", but starting next autumn will be “Season 3 - Measure and topology change"

`https://sites.google.com/view/qg-aa`

Thank you!

Contact l.glaser@univie.ac.at
or Twitter:@GravityWithHat

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Monte Carlo Simulations

- Simulate a Path integral, use Monte Carlo Markov Chain to calculate averages
- Use Markov Chain to probe space of solutions to find an optimum. Only examine the solution with minimal value of something.

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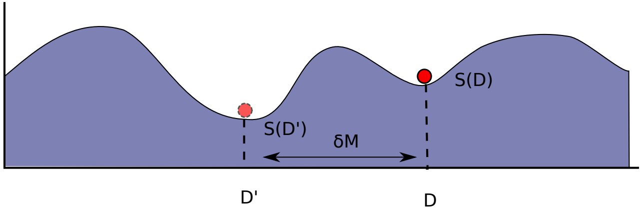
Note:

It is proven that the Metropolis algorithm will find the global optimum if sampled long enough.

Markov Chain Methods in one slide

The Metropolis Hastings algorithm

- propose new operator D'
  $D \to D' = D + \delta M$ with δM some small matrix
- if $S(D') < S(D)$ accept D' and add to the chain
- otherwise calculate $\exp\{-S(D') + S(D)\}$ & generate random uniform $p \in [0, 1]$
  - if $p < \exp\{-S(D') + S(D)\}$ accept D'
  - else add D to the chain again



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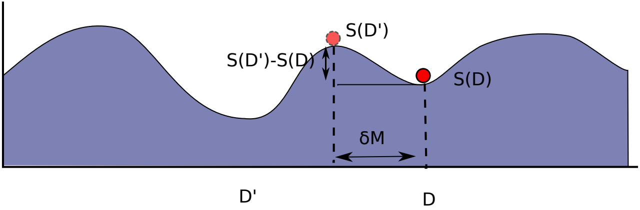

States are points

An element $\mathbf{v}$ of $\mathbb{P}(H_\Lambda)$ that is considered to be localized should be localized *somewhere*, that is, around some 'barycenter' $b(\mathbf{v}) \in M$.
We can prove:

Proposition

There exists a map $b\colon \mathbb{P}(H_\Lambda) \to M$ such that

$$|d_\Lambda(\mathbf{v}, \mathbf{w}) - d_M(b(\mathbf{v}), b(\mathbf{w}))| = O(\sqrt{\eta(\mu_v)} + \sqrt{\eta(\mu_w)})$$

as $\eta(\mu_v), \eta(\mu_w) \to 0$, uniformly in $\mathbf{v}$, $\mathbf{w}$.

Points are states

The converse: Each point x in M can be approximated through a state $\mathbf{v}$ with small dispersion and with barycenter $b(\mathbf{v})$ close to x

Proposition

Let M be equipped with a Dirac-type operator D on a Hermitian vector bundle $\pi\colon \mathbf{S} \to M$, and let $\widetilde{\pi}\colon \mathbb{P}(\mathbf{S}) \to M$ be its projectivized bundle. Then, there exists a family $\{\Phi_\Lambda\}_\Lambda$ of maps $\Phi_\Lambda : \mathbb{P}(\mathbf{S}) \to \mathbb{P}(H_\Lambda)$ such that for all $\epsilon > 0$,

- $d_\Lambda(\Phi_\Lambda(v), \Phi_\Lambda(w)) = d_M(\widetilde{\pi}(v), \widetilde{\pi}(w)) + \widetilde{O}\left(\Lambda^{-1}\right)$ *uniformly.*
- *The dispersion $\eta(\mu)$ of the measure μ associated to $\Phi_\Lambda(v)$ is $\widetilde{O}\left(\Lambda^{-2}\right)$ uniformly.*
- *The maps Φ_Λ asymptotically invert b, in the sense that $d_M(\widetilde{\pi}(v), b(\Phi_\Lambda(v))) = \widetilde{O}\left(\Lambda^{-1}\right)$ uniformly and $d_\Lambda(\Phi_\Lambda(v)), \mathbf{v}) = \widetilde{O}\left(\sqrt{\eta(\mu_v)} + \Lambda^{-2}\right)$ uniformly whenever $b(\mathbf{v}) = \widetilde{\pi}(v)$.*