A family of triangulations of the 3-sphere constructed from trees

Luca Lionni
Universität Heidelberg - Germany

Random geometry in Heidelberg – 19/05/22

Joint work with Timothy Budd - 2203.16105

1 – Motivations and background

2 – Three points of view on triple-tree triangulations

3 – Simulations

1 – Motivations and background

Motivations

Within the asymptotic safety scenario, the non-perturbative renormalization of gravity formulated as a field theory would end up on a UV fixed point at short scales, which would correspond to a scale-invariant probability distribution for the spacetime metric.

Results from many other approaches to quantum gravity agree that at short scales, space-time should develop fractal properties.

The *random geometry approach to quantum gravity* aims at building scale-invariant, fractal, random geometries, that could serve as models for quantum space-times.

→ Search such random geometries as continuum scaling limits of sequences of random triangulations, or more general random graphs.

→ Want background independence and intrinsec geometry (unlike e.g. Brownian motion),

→ Also want a universality classes (sequences of random graphs that differ only by small changes in the local properties should converge to the same limit).

Motivations

N.B: as far as I understand, we don't really know what topological or geometrical properties to expect for a random geometry to describe the UV fixed point of the renormalization flow.

What we know, is what we should recover locally at our scales, after some coarse graining procedure, yet to be defined…

For instance:
Must this random geometry have a well-defined topological dimension??
Or could the locally flat topology emerge at large scales? Please let me know your insights on the matter.

Well-defined topology or not for the application to quantum gravity, knowing how to build Brownian manifolds in topological dimensions 3, 4, would definitely help, starting with the n-sphere.

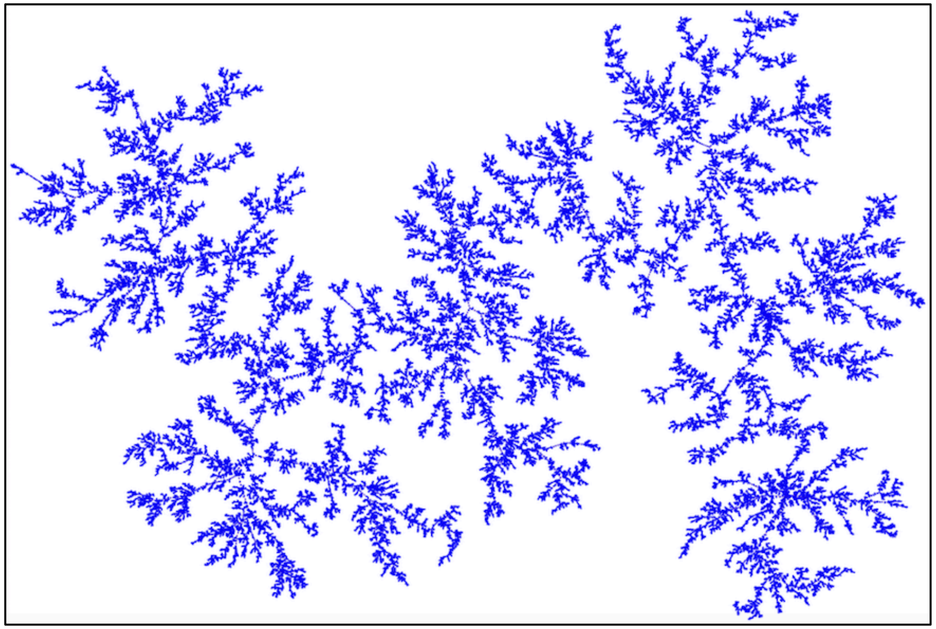
Scale-invariant, background independent, fractal random geometries…? >> Dimension 1

Brownian motion:

Scale invariant continuum limit but background dependent, and distances are infinite… *But we know very well the asymptotic properties of walks and their limits*

Brownian trees:

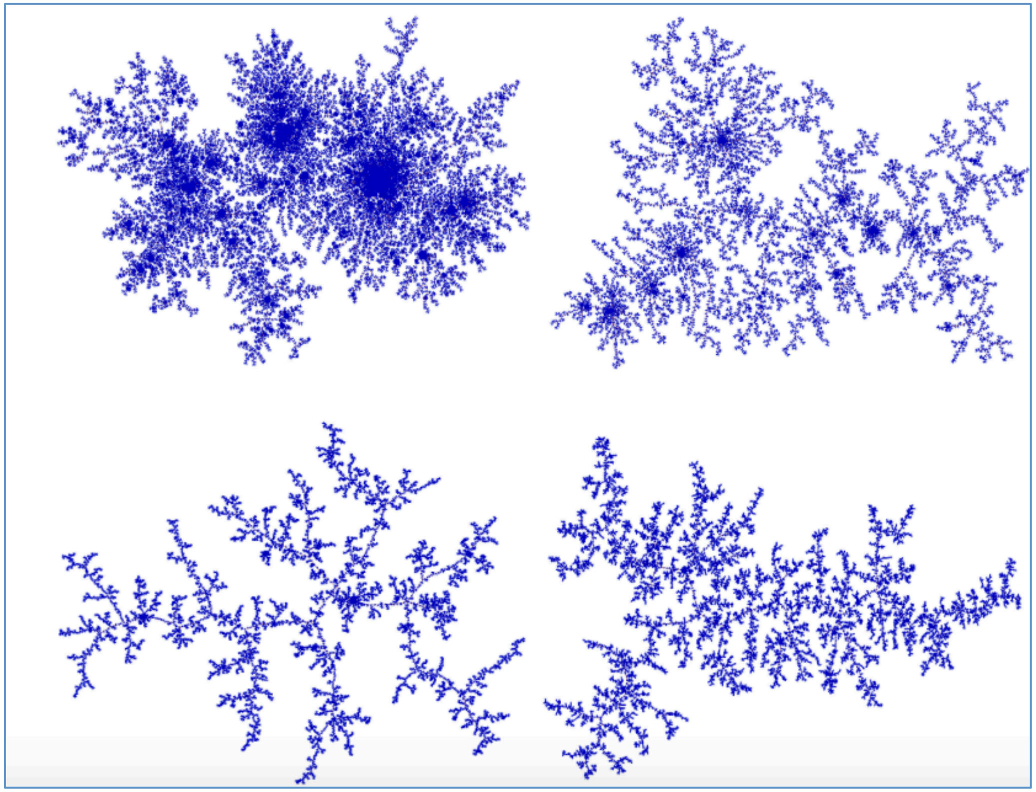
Simplest such object which is background independent and has intrinsec geometry is the continuum random tree (CRT) [Aldous…], which can be obtained as a limit of random trees with a uniform distribution (but many others… universality class)



Picture: L. Ménard 100000 edges

Scale-invariant, background independent, fractal random geometries…? >> Dimension 1

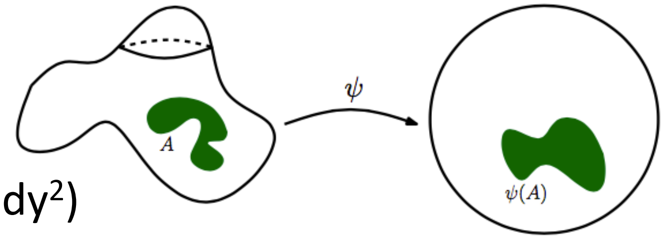
Other universality classes of Brownian trees exist, such as stable trees, but plenty of others (e.g. stable trees)



Pictures: Kortchemski.

Brownian trees of all kinds have been studied a LOT, and we have access to their geometric properties through ***encoding by walks in the plane***.

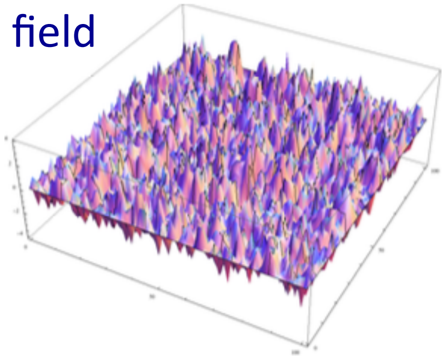
Scale-invariant, background independent, fractal random geometries…? >> Dimension 2



Our aim is now to build Brownian surfaces

◆ Any metric on the 2-sphere can be written as $e^{\rho(z)}(dx^2+dy^2)$

→ Make it random by taking $\rho(z) = \gamma\, h(z)$, where $h(z)$: Gaussian free field
→ Get Liouville quantum gravity with parameter γ (= γ-LQG), a random area measure on a fixed background.



[Physics : Polyakov, Knizhnik, Zamolodchikov, David, Distler, H. Kawai…]
[Mathématics : Duplantier, Sheffield, Miller, David, Rhodes, Vargas, Berestycki, Sun…]

Scale-invariant, background independent, fractal random geometries…? >> Dimension 2

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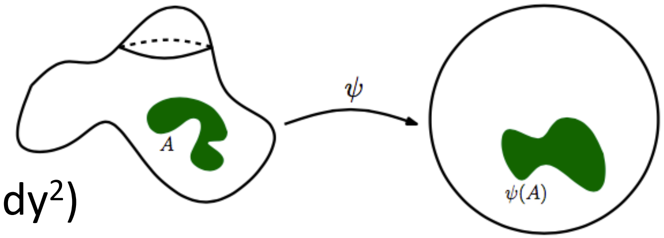

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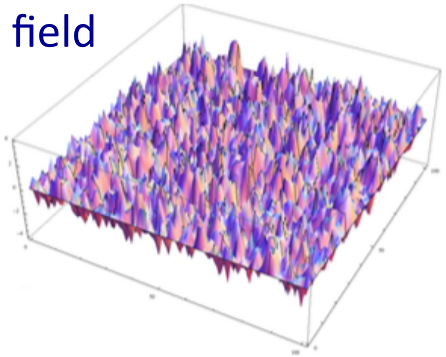
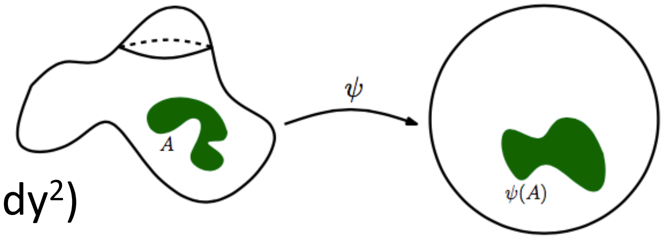


◆ Other way to produce a random 'typical' surface is to take uniform gluings of triangles and a continuum scaling limit.
This limit – the Brownian sphere - can be constructed rigorously and its topological/geometrical properties can be studied analytically:

- Random continuum surface with a distance
- A.s. 2-spherical
- Fractal: Hausdorff dimension 4, self-similar properties
- Universality class
- Currently, study of the properties of geodesics e.g., etc.

[Physique >80: Weingarten, David, Fröhlich, Kazakov, Ambjorn, Durhuus, …]
[Mathématiques > 04 : Marckert, Mokkadem, LeGall, Miermont, Albenque, Curien, Addario-Berry, …]

Scale-invariant, background independent, fractal random geometries…? >> Dimension 2



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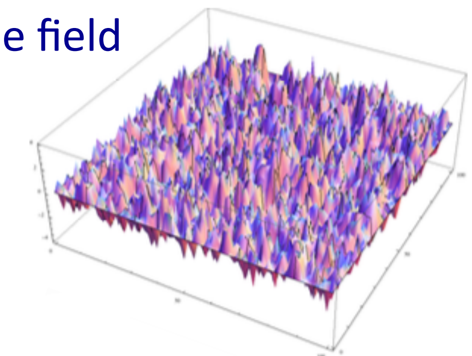

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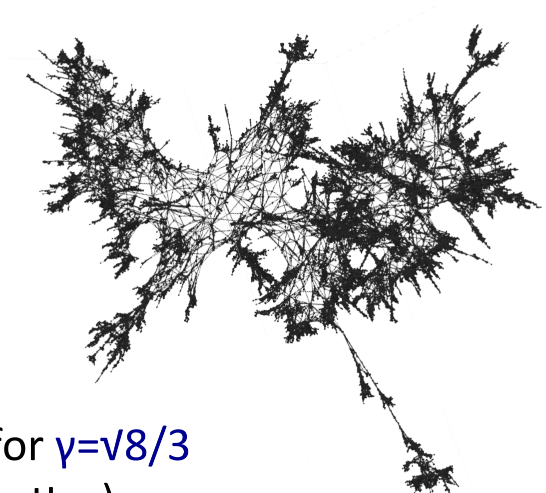
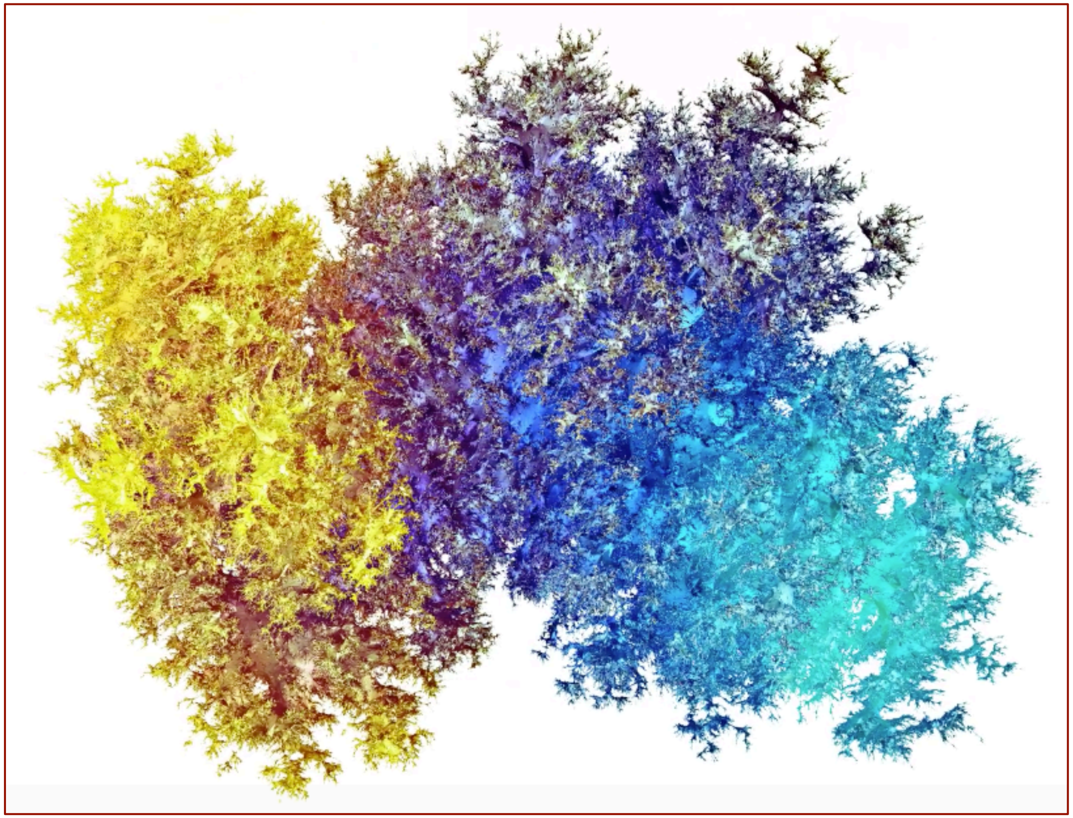


Shown rigorously that the Brownian map is equivalent to γ-LQG for $\gamma=\sqrt{8/3}$ [Miller, Sheffield, Holden, Sun]. Corresponds to ''pure gravity'' (no matter).

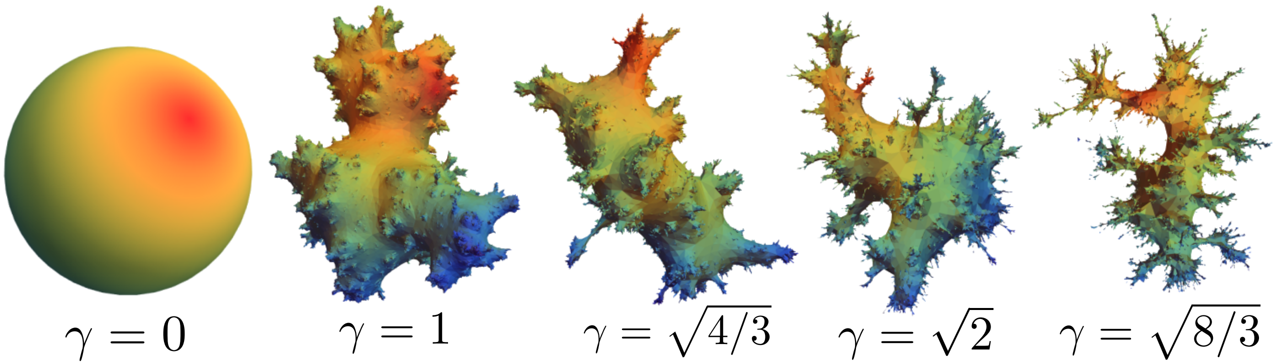
Scale-invariant, background independent, fractal random geometries…? >> Dimension 2



Movies by Benedikt Stufler

Scale-invariant, background independent, fractal random geometries…? >> Dimension 2

Other values of γ obtained for random triangulations coupled with statistical models (*e.g. Ising model* $\gamma = \sqrt{3}$, *number of spanning trees* $\gamma = \sqrt{2}$…), and correspond to LQG 'coupled to matter', but less strong convergence results.



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T. Budd

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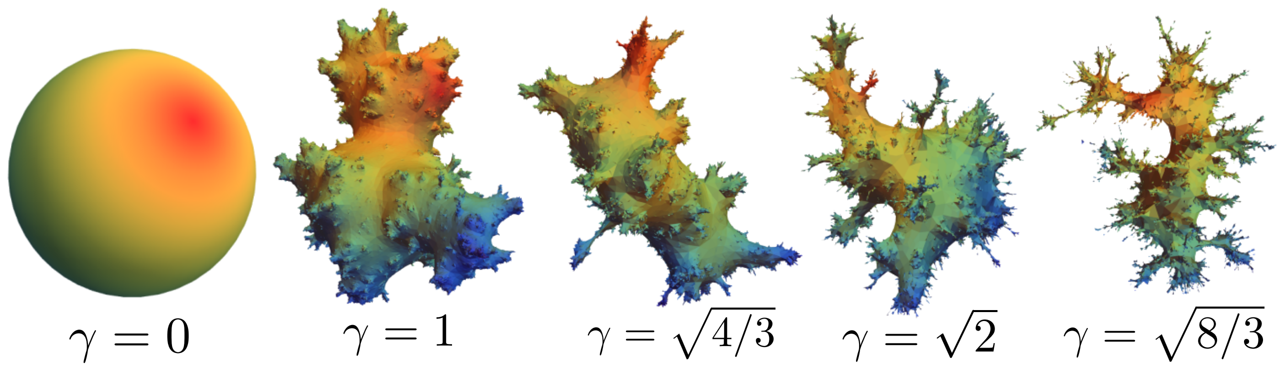

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In general, we get **stronger convergence results, and have better access to analytical results (about e.g. geometry) when there are bijective encodings with (pairs of) trees.**

For instance:

"Metric bijections": Cori-Vauquelin-Schaeffer; Bouttier-Guitter-DiFrancesco…

"Mating of trees bijections": Mullin; Bernardi; Li-Sun-Watson, Kenyon-Miller-Sheffield-Wilson, Biane…

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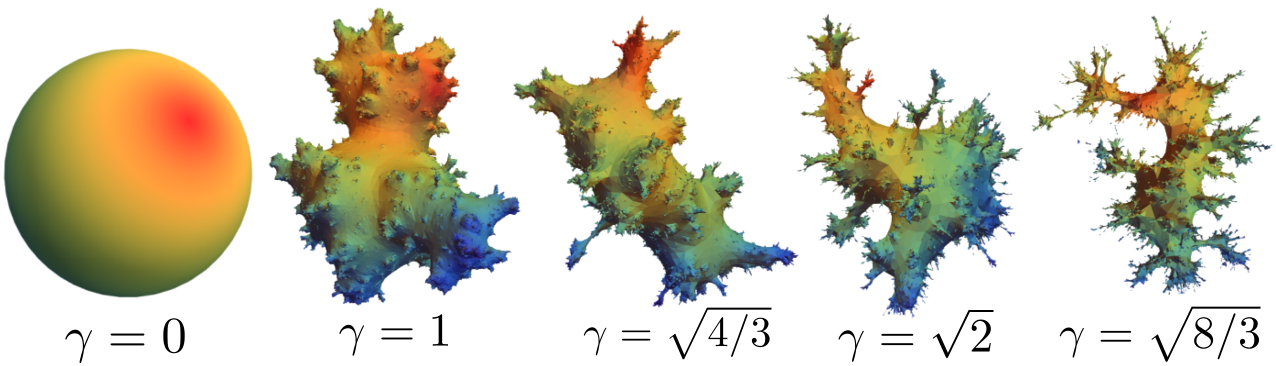

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There are also non-LQG Brownian spheres, higher genus Brownian surfaces…

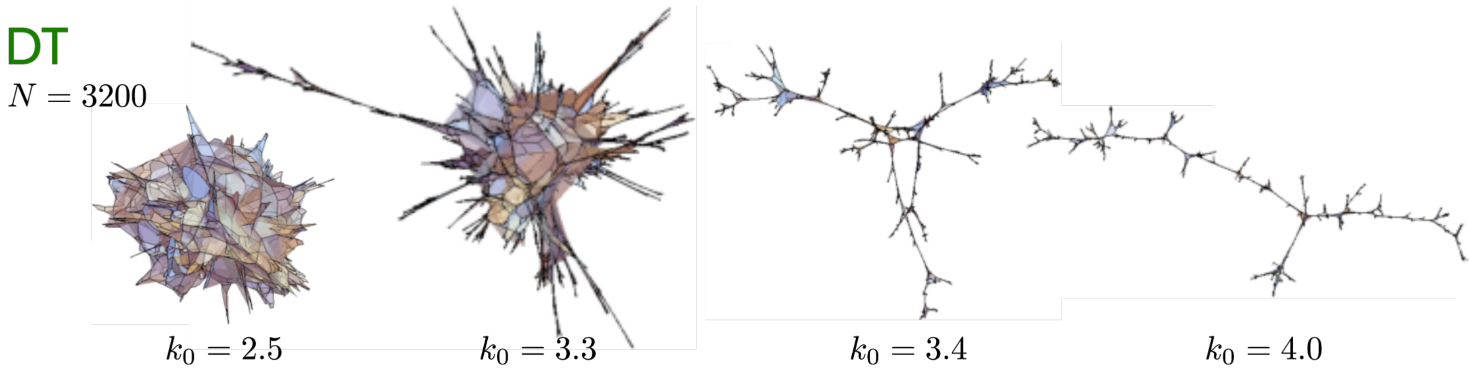
Scale-invariant, background independent, fractal random geometries…? >> Dimension 3

Consider triangulations of the 3-sphere with n tetrahedra and a weight x^V, where x is a variable and V is the number of vertices.

[Boulatov, Krzywicki, Ambjorn, Varsted, Hagura, Tsuda, Yukawa, Thorleifson…]

Scale-invariant, background independent, fractal random geometries…? >> Dimension 3

Consider triangulations of the 3-sphere with n tetrahedra and a weight x^V, where x is a variable and V is the number of vertices.
By varying x, simulations indicate two regimes, on both sides of a phase transition x_C:



- For $x > x_c$, class of "branched polymers": large scale geometry reminiscent of trees, conjectured continuum scaling limit is the continuum random tree
- For $x < x_c$, ''crumpled phase'': highly connected and degenerated triangulations, diameter grows very slowly or might be asymptotically constant, continuum scaling limit might be trivial. Includes the uniform triangulations of the 3-sphere
- Phase transition $x=x_c$ is where we could then hope to find the interesting critical behavior (e.g. Ising model), but the transition seems discontinuous, meaning we can't find anything more at the transition.

Scale-invariant, background independent, fractal random geometries…? >> Dimension 3

→ Numerically, seems like nothing interesting by taking *this* distribution on the *full set* of triangulations of the 3-sphere.

→ But don't know much about this crumpled phase really. Numerically not actually possible to see anything for a diameter growing this slow, can't differentiate with a trivial space with constant diameter for instance, for which scaling limit is not continuous.

→ **All analytical results completely out of reach**, even in the uniform case: no polynomial time algorithm to recognize the 3-sphere topology, and the enumeration of triangulations of the 3-sphere is completely out of reach (even whether exponentially bounded), so no hope for finer results about the geometry (e.g. asymptotic diameter).

Do not know of **any** continuum background-independent universality class of random geometry that cannot be obtained from random trees or surfaces (in contrast with the very well populated zoo of random trees and surfaces…)

I would argue that:

- Worrying about what topological and geometrical properties such a space should have to qualify as a model of quantum space-time is premature. It would require understanding how to take a classical limit ('coarse graining'-'smoothing') and studying how this and that property evolve through this process.

- We need to **produce examples** of such new spaces, for which it is possible to prove convergence and **we have analytical access to the geometrical properties of the continuum limit**. What makes a difference is having **bijective encodings using trees.**

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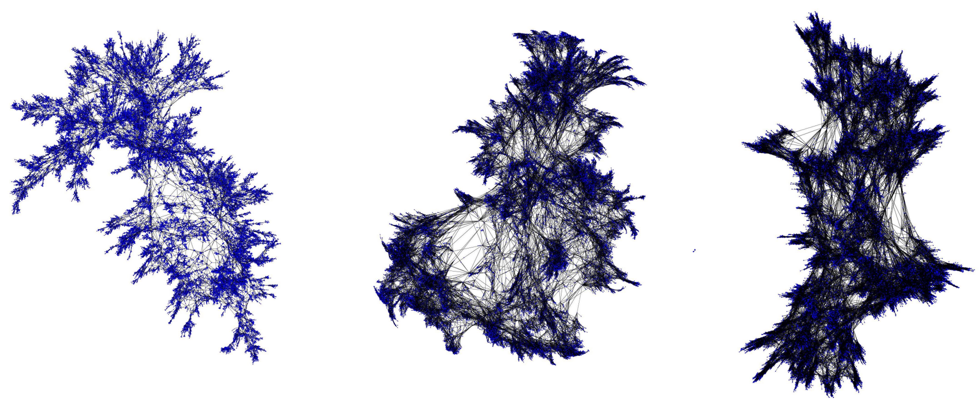
→ with J.F. Marckert, we produced the first candidates without worrying about any topological realization. Existence of new non-trivial scaling limit expected, not trees nor surfaces…

- Hausdorff dim.exp. 8, Str.susc −3/2.
- Bijective encoding using 3 trees.



→ Model with Budd, triangulations of the 3-sphere encoded by 3 trees: rest of the talk.

2 – Three points of view on triple-tree triangulations

2.1 – Little detour on planar triangulations

Little detour on planar triangulations

In comparison, verifying the topology of 2-dimensional triangulation is trivial (Euler characteristics)…

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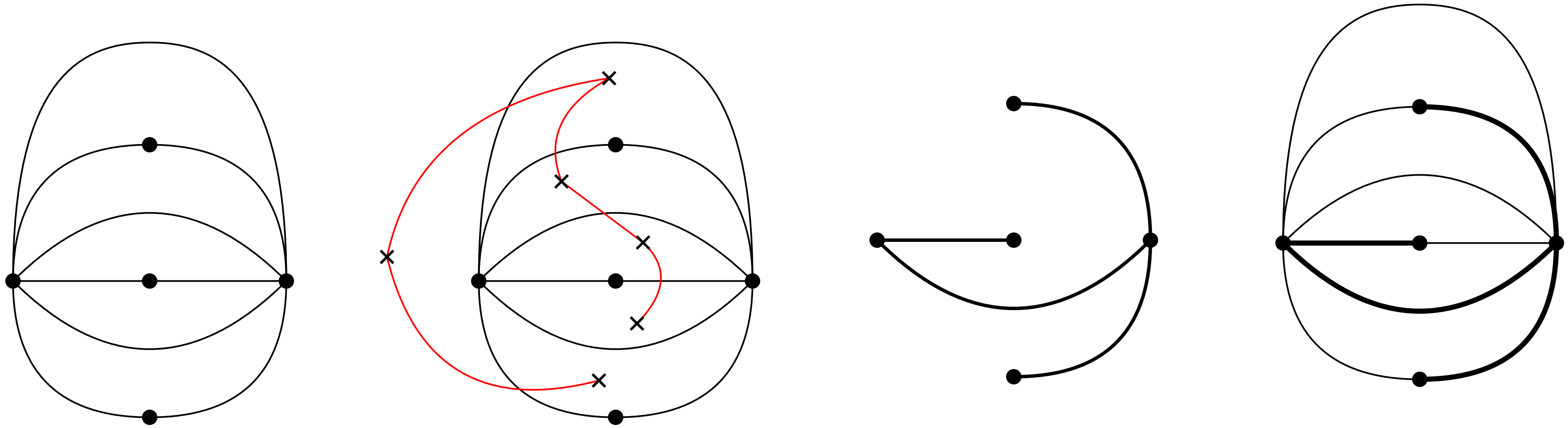
But also some characterizations: for any spanning tree of triangles, the complement is a plane spanning tree (of edges):



N.B. ”spanning tree of edges“ → reaches all the vertices

“spanning tree of triangles” → reaches all the vertices of the dual graph
→ visits all the triangles

Little detour on planar triangulations

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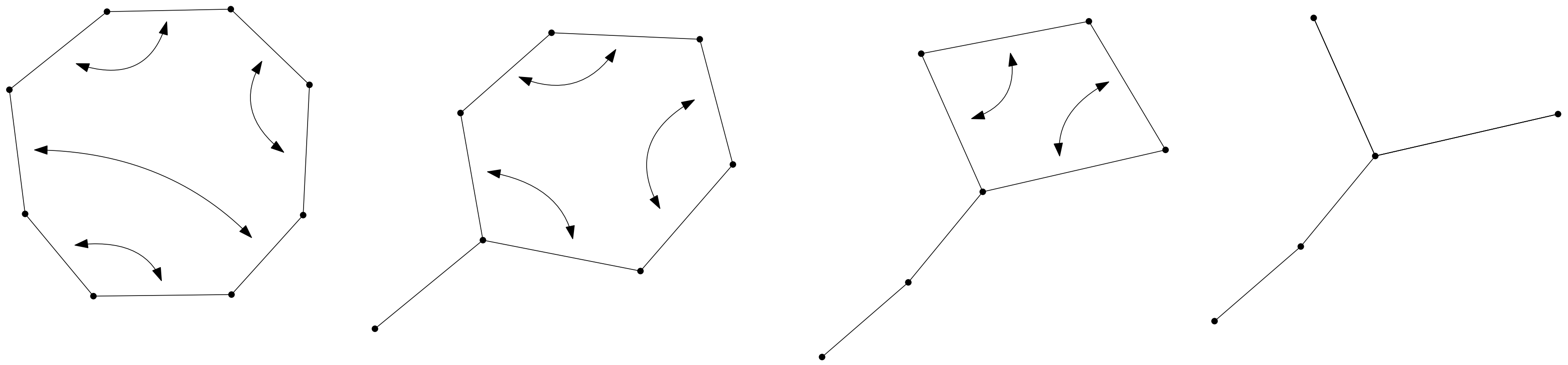
A plane tree is a non-crossing pairing of the edges around a circle:



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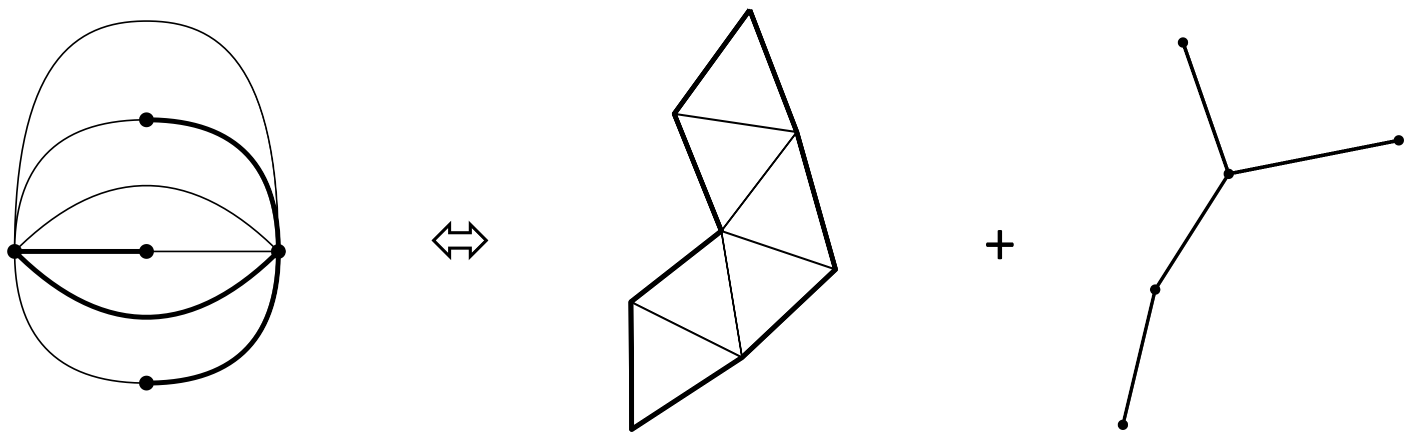
We see that a tree-decorated triangulation is encoded by two trees:



It's a bijection for rooted planar tree-decorated triangulations, which are therefore counted by $C_{2n}C_{n+1}$ for 2n triangles…

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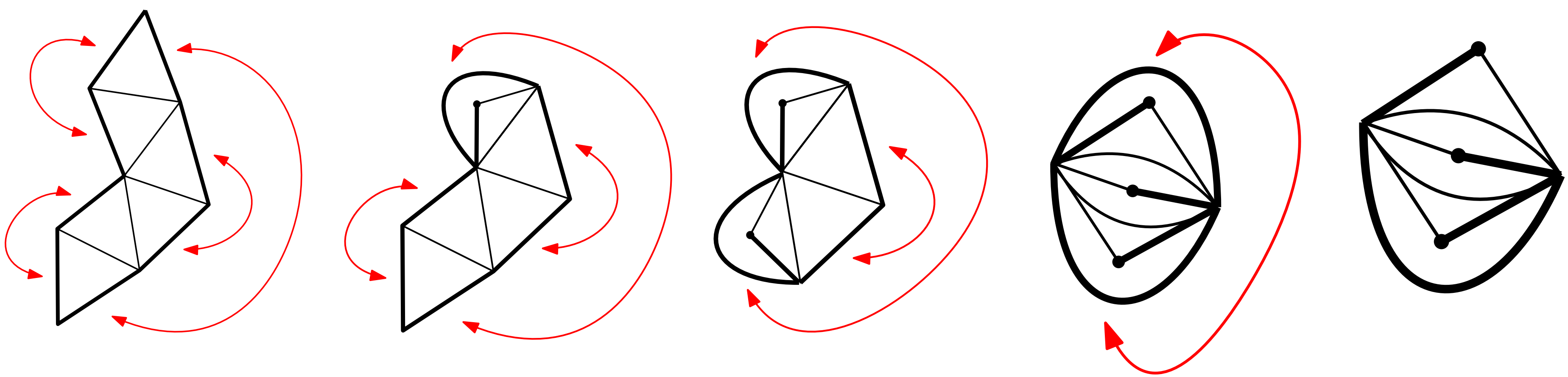
Each dynamical construction of the tree of edges from the non-crossing pairing gives a local construction of the tree-decorated triangulation:



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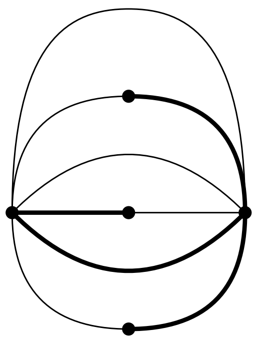


In particular we have seen that:

Triangulation is planar ⇔ Complement of any spanning tree of the dual graph is a tree of edges

⇔ There exists a local construction of the triangulation starting from a plane tree of triangles

And also that a rooted planar triangulation with a distinguished spanning tree of edges (or triangles) is bijectively encoded by a pair of trees.

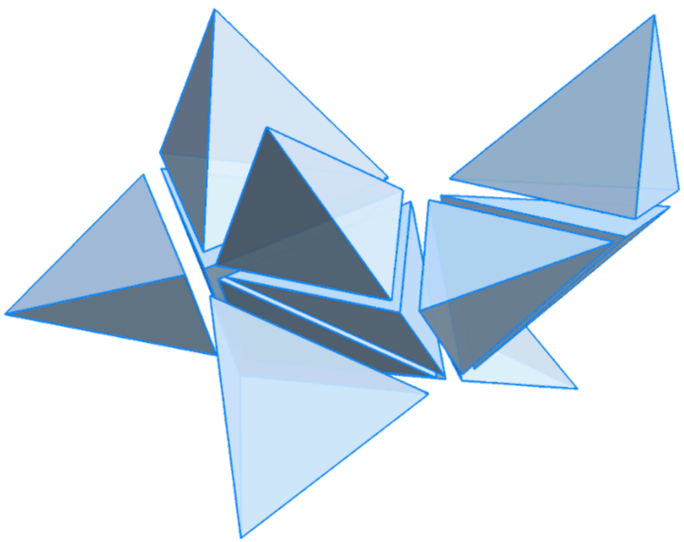
2.2 – Definition of triple-tree-triangulations

Triple-tree triangulations

The aim is to define a subset of triangulations of the 3-sphere, bijectively encoded by trees, for which we could use such tree encodings to study the asymptotic properties analytically

Triple-tree triangulations

- Spanning tree of tetrahedra T_0 : spanning tree of the dual graph
- Complement $T \backslash T_0$ of T_0 in T: replace T_0 by a 3-cell.
  → *embedded 2-complex*



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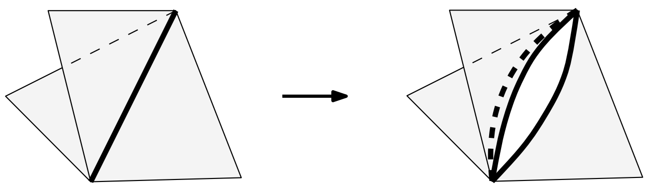
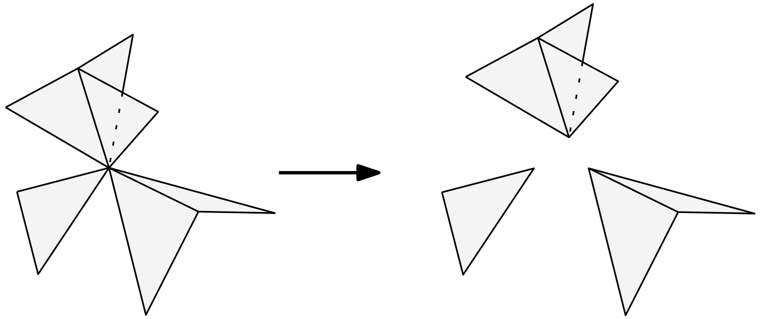
  Cut(E): set of duplicates of edges of E.





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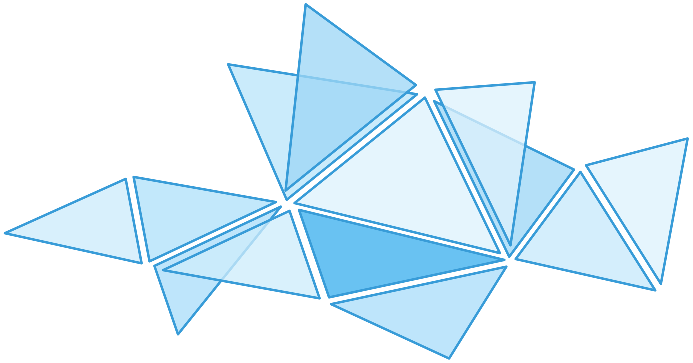

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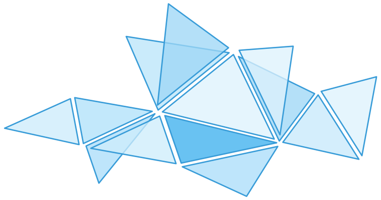
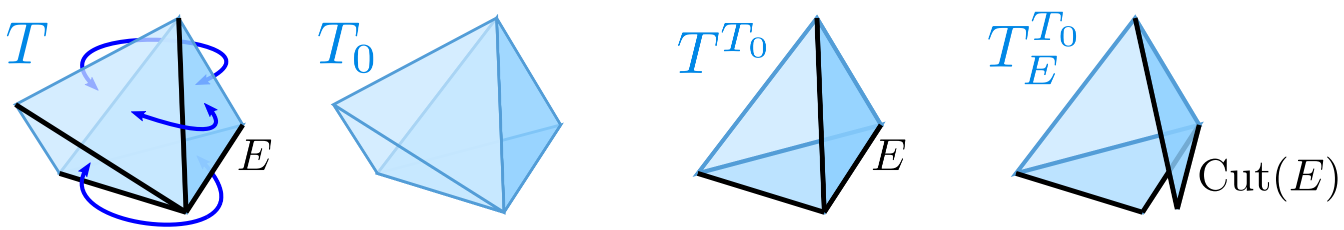


Starting from (T, T_0 , E), where T_0 , E spanning trees of tetrahedra and edges of T, consider $(T \backslash T_0)^E$ obtained by taking the complement of T_0 and cutting along E.

<u>Definition</u>: T is a triple-tree triangulation if:

- $(T \backslash T_0)^E$ is a tree of triangles
- Cut(E) is a tree



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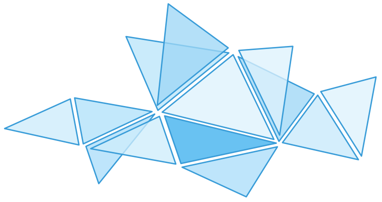

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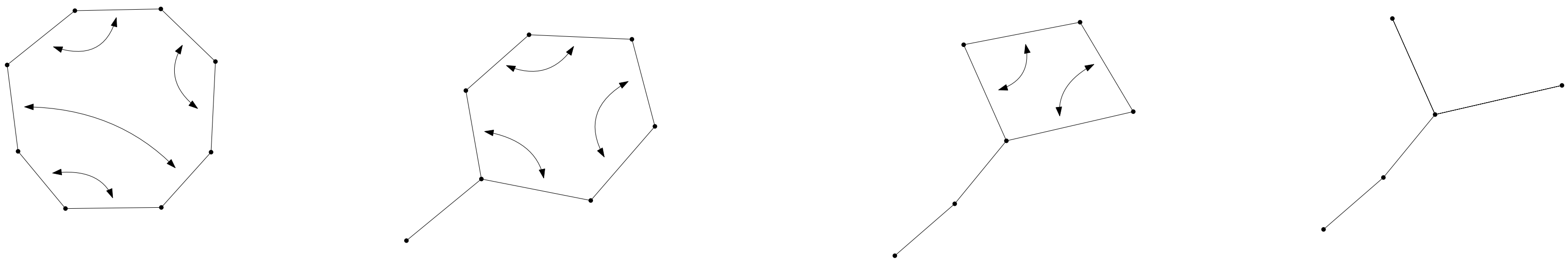
N.B. To be compared with:

"2d triangulation is planar ⇔ complement of a tree of triangles is a spanning tree of edges"

2.3 – Bijective encoding by plane trees

Bijective encoding by plane trees

Actually encoding by three non-crossing pairings (⬄ chord diagrams).



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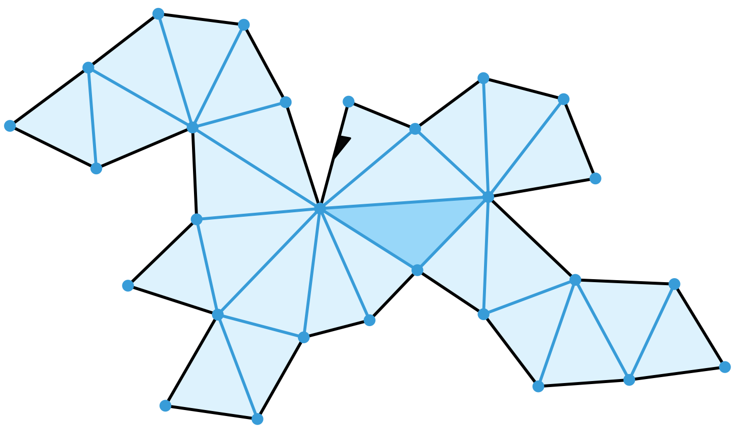

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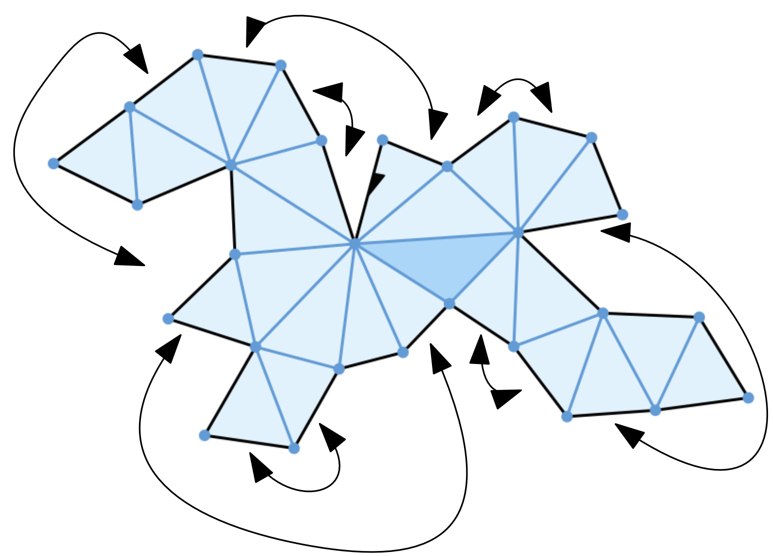

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  - *The planar triangulation obtained gluing τ using π_H is* Hierarchical (= melonic),
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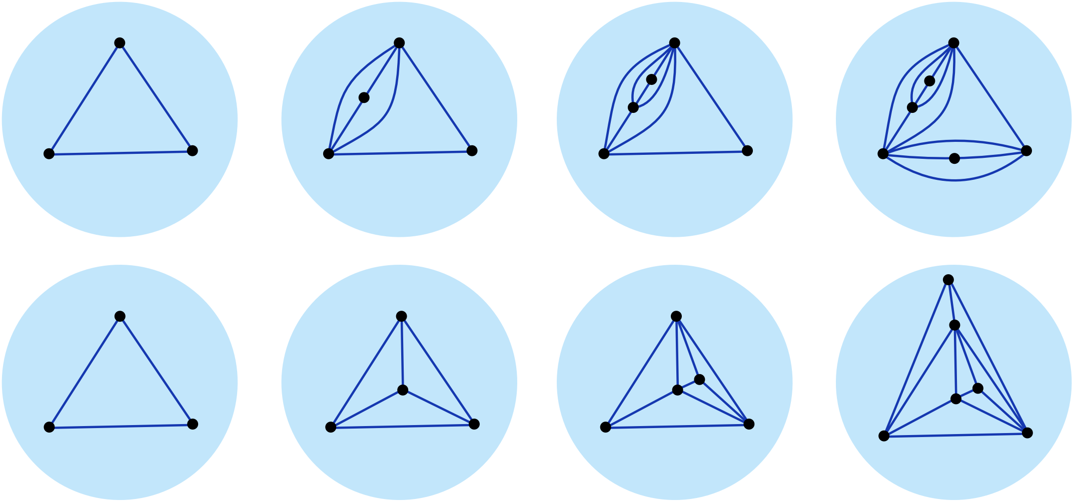

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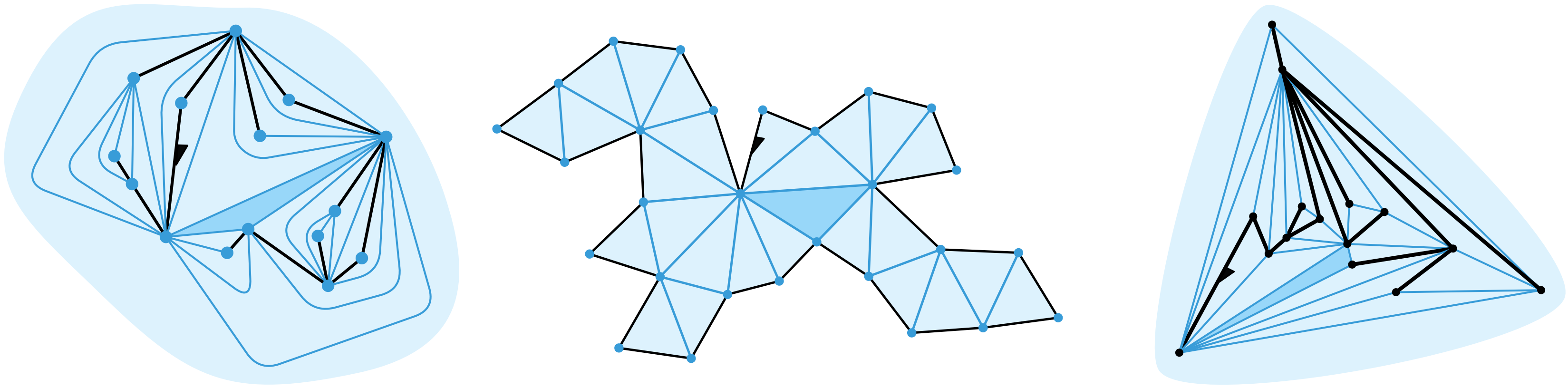

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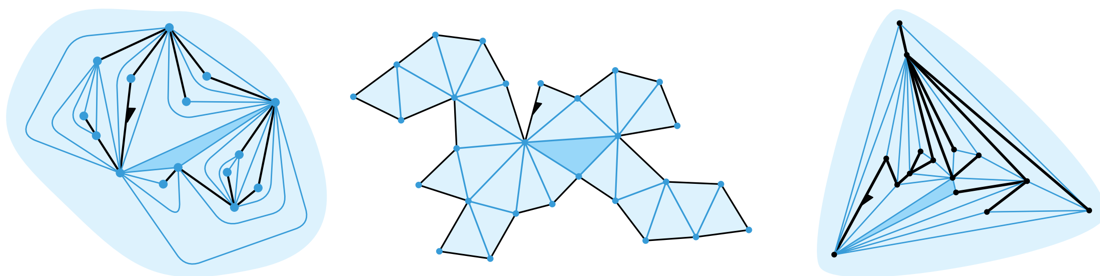


<u>Theorem</u> There is a bijection between:

- The set of non-crossing pairings (τ, $\pi_{H,}$ π_A) such that τ has 2n triangles
- The set of rooted triple-tree triangulations with n-1 tetrahedra

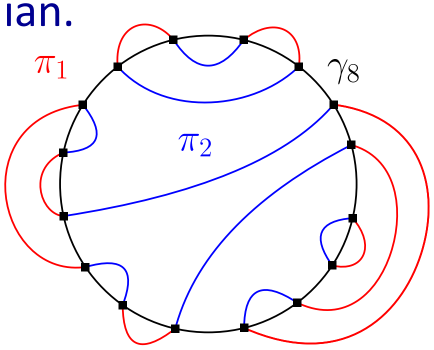
N.B. To be compared to the bijection between planar rooted tree-decorated 2d triangulations and the set of (τ, π)

Bijective encoding by plane trees

Actually encoding by three non-crossing pairings (⇔ chord diagrams).

Take:

- τ a rooted plane tree of triangles (= outerplanar triangulation of a polygon).
- Two non-crossing pairings π_A, π_H of the boundary edges of τ , such that:
  - *The planar triangulation obtained gluing τ using π_H is* Hierarchical (= melonic),
  - *The planar triangulation obtained gluing τ using π_A is* Apollonian.



<u>Theorem</u> There is a bijection between:

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The number of vertices of T minus 1 is given by the number of loops of the meander system [$\pi_{H,}$ π_A]

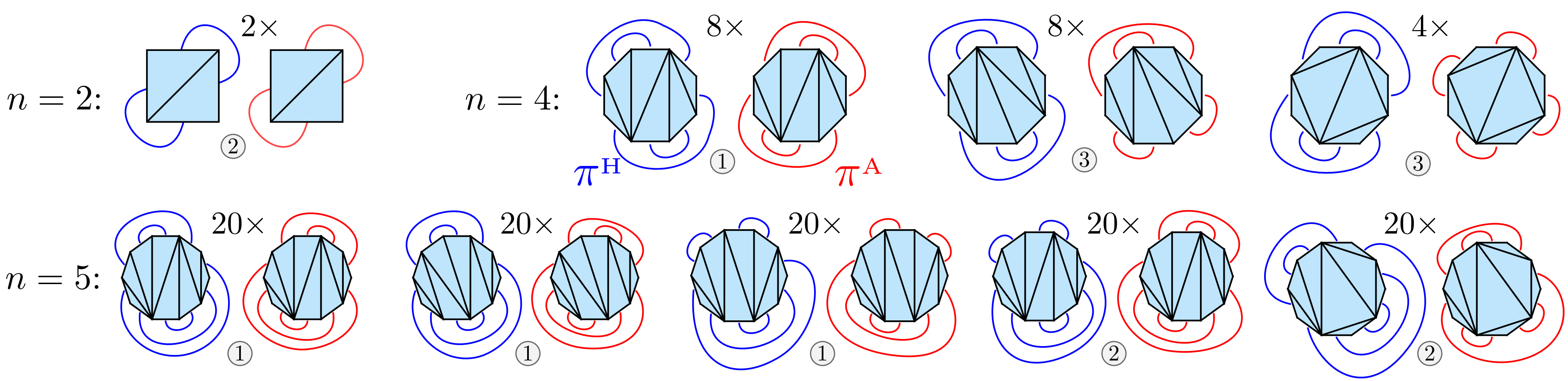
Bijective encoding by plane trees

Using this encoding, the partition function for random triple-tree triangulation distributed according to x^V can be expressed as:

$$M(z,x) = \sum_{n=1}^{\infty} z^n M_n(x), \qquad M_n(x) = \sum_{(\tau,\pi_{\mathrm{H}},\pi_{\mathrm{A}})\in\mathcal{T}_n} x^{N(\pi_{\mathrm{H}},\pi_{\mathrm{A}})},$$

Counting this triplets of trees is simpler than counting the 3-dimensional triangulations…

$$\begin{aligned} M(z,x) = {} & 2x^2z^2 + (8x+12x^3)z^4 + (60x+40x^2)z^5 + (336x+996x^2+420x^3+618x^4)z^6 \\ & + (5460x+10416x^2+6496x^3+1652x^4)z^7 \\ & + (63344x+135776x^2+150544x^3+75360x^4+46360x^5)z^8 + \cdots . \end{aligned}$$

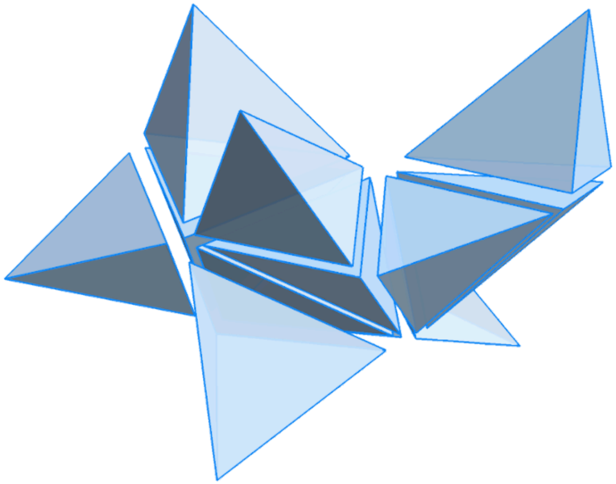


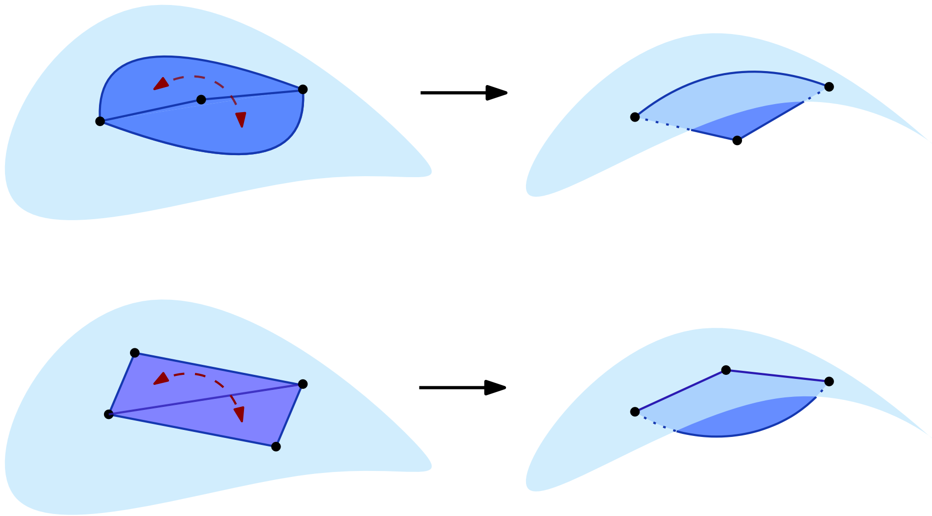
2.4 – Topology

Local constructibility

Topology: we show that triple-tree triangulations are locally constructible, which guarantees that they have the topology of the 3-sphere.

Locally constructible triangulations in dimension 3: Start from a tree of tetrahedra and recursively select an edge on the boundary and glue the triangles on both sides.





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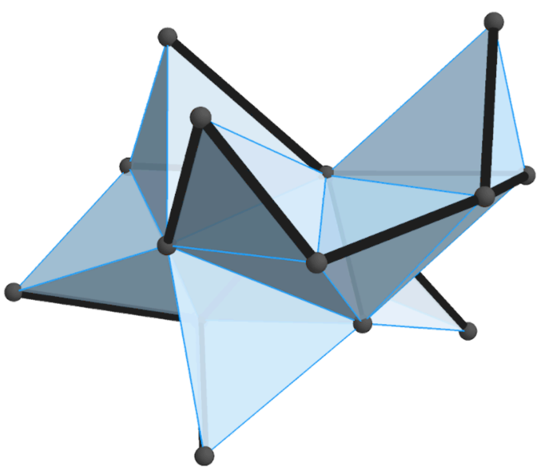
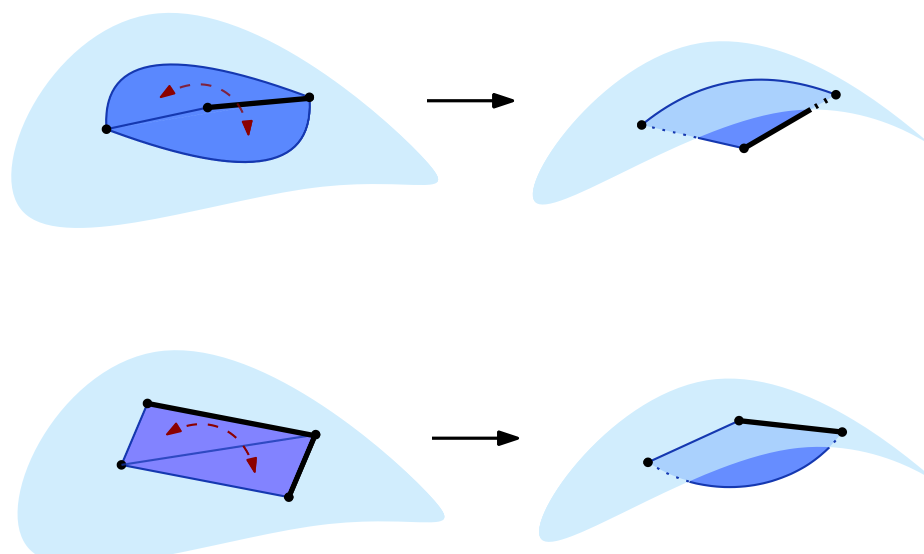
Theorem: Triple-tree triangulations admit a "tree-avoiding" local construction





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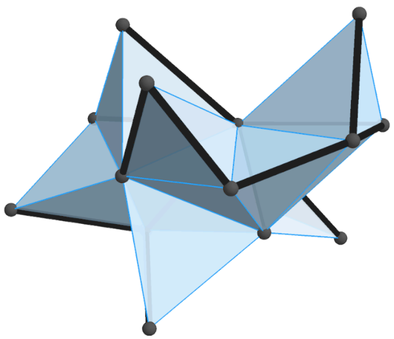
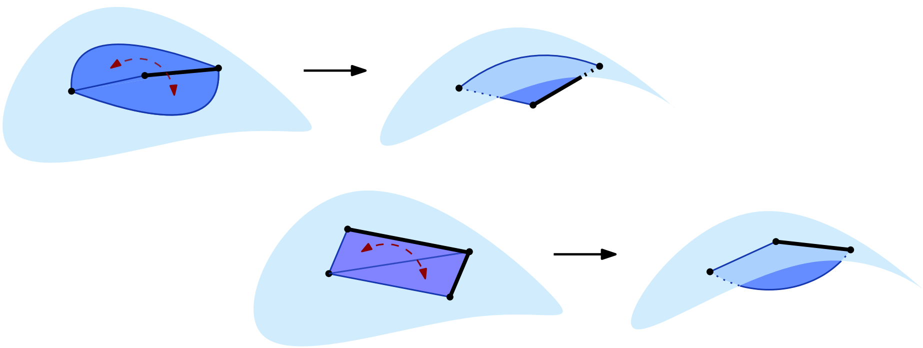
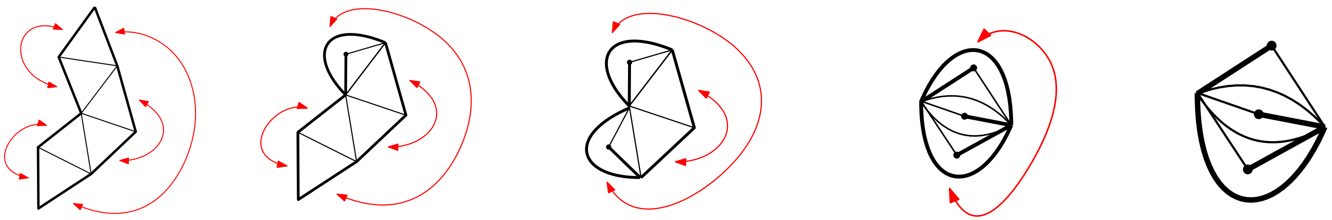




N.B. To be compared with: "2d triangulation is planar ⇔ it admits a local construction"



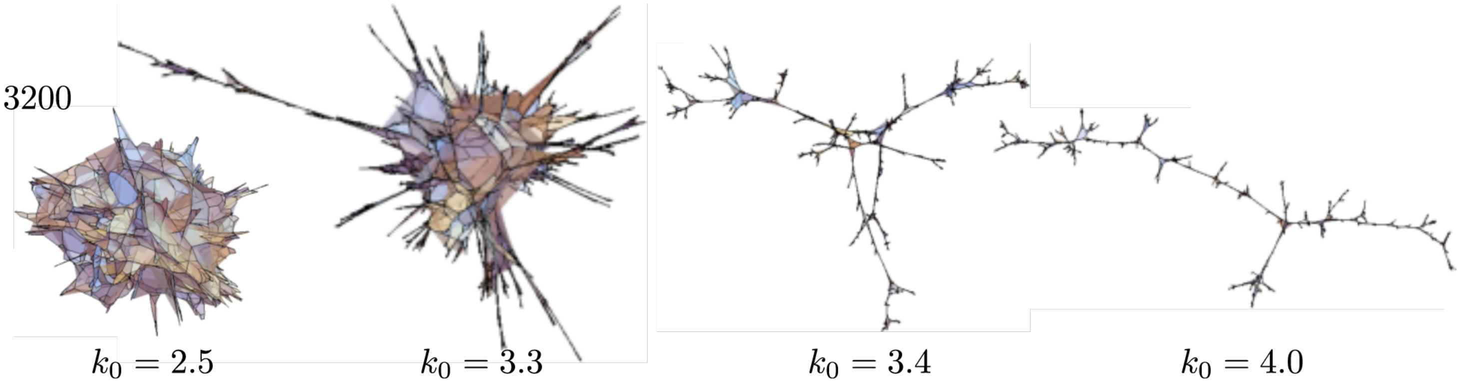
3 – Simulations

Simulations

We performed some Monte Carlo simulations for triple-tree triangulations distributed according to x^V. Dual graph looks like:

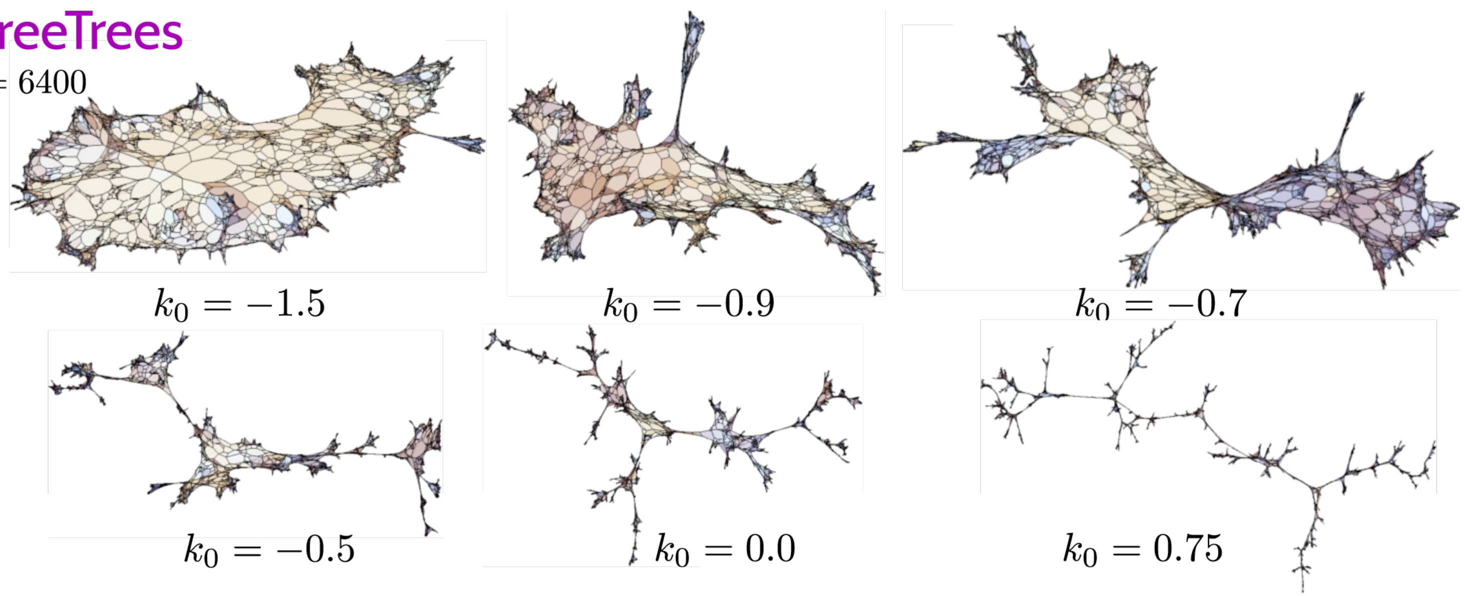
DT

$N = 3200$



ThreeTrees

$N = 6400$



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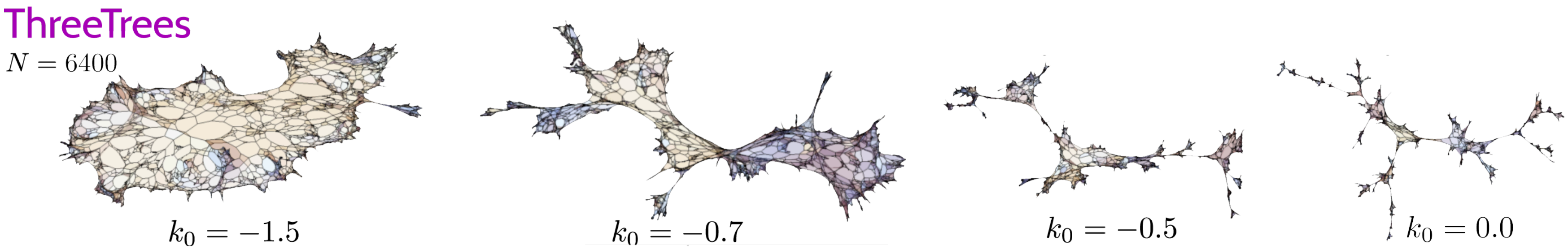
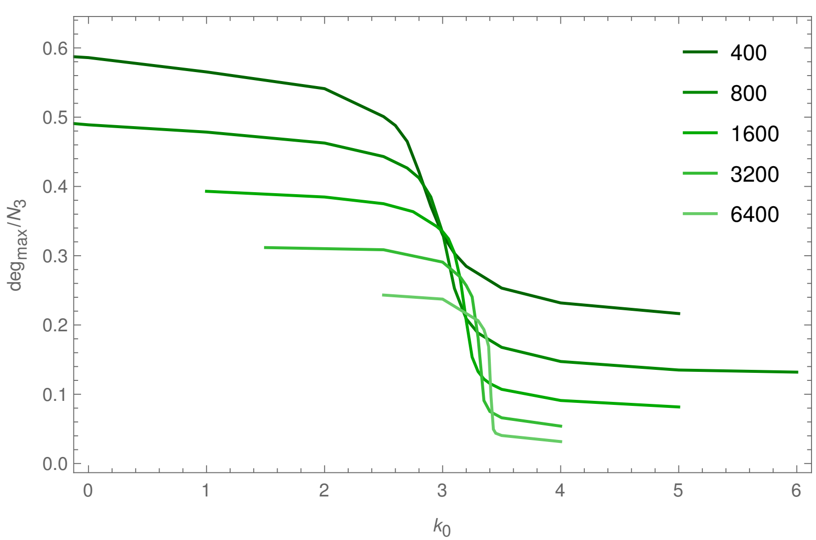
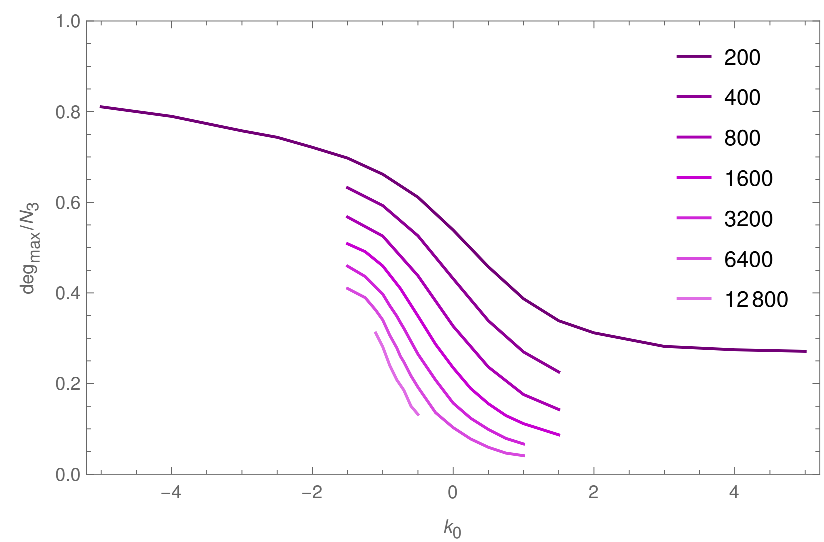


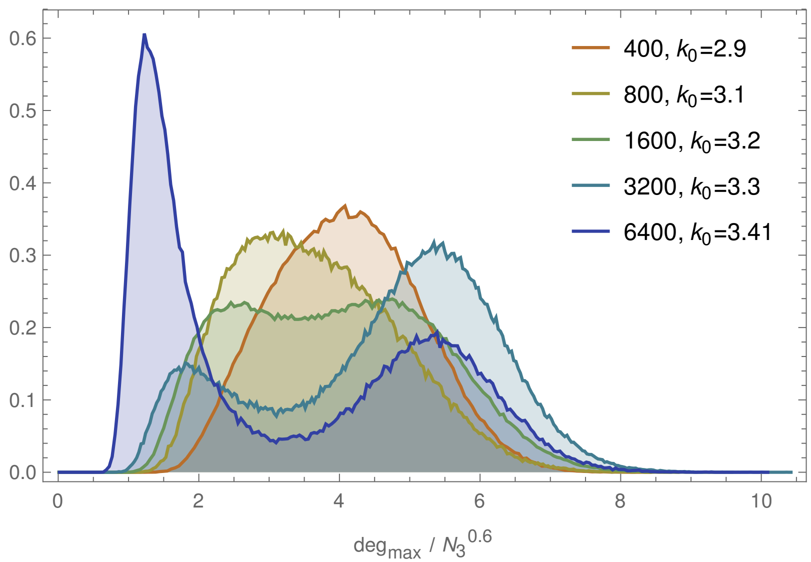
Looking at the max-vertex-degree as an order parameter, there appears to be a phase transition between crumpled-but-less-crumpled phase and a branched polymer phase.

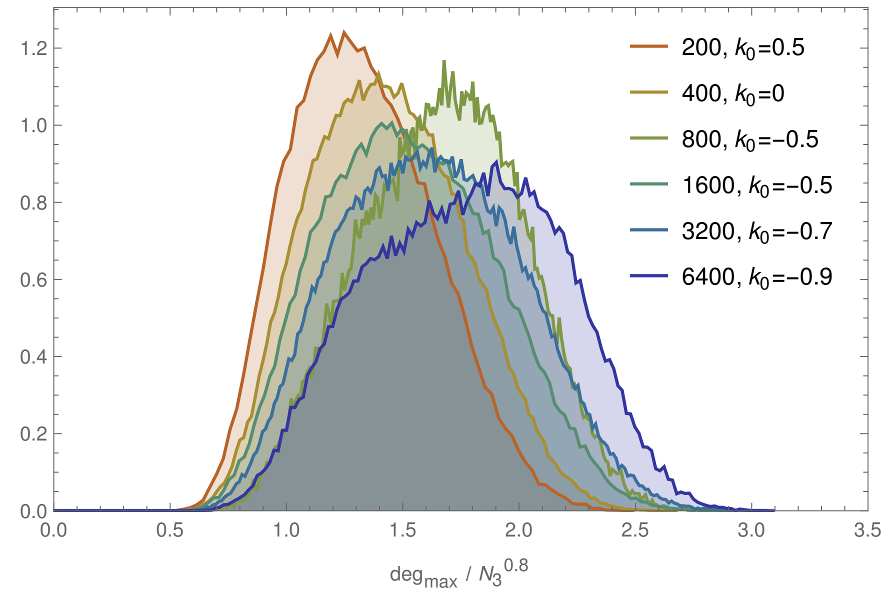


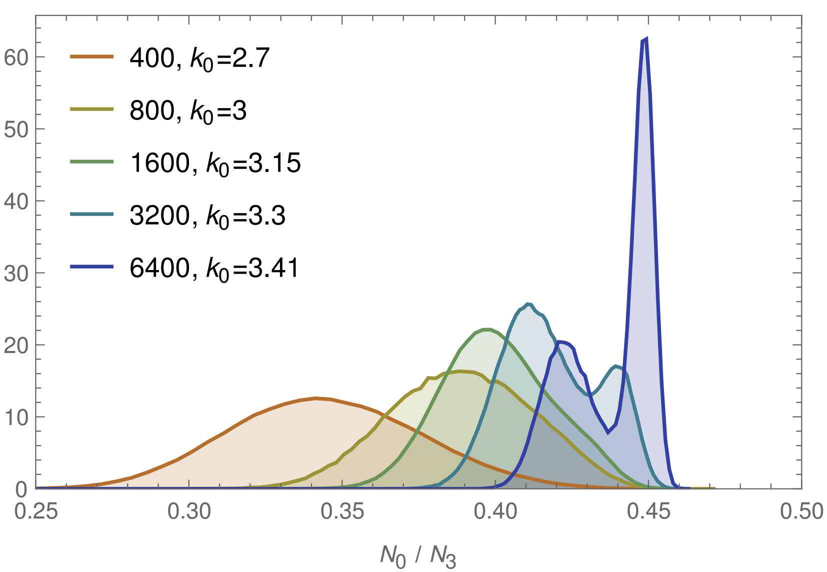


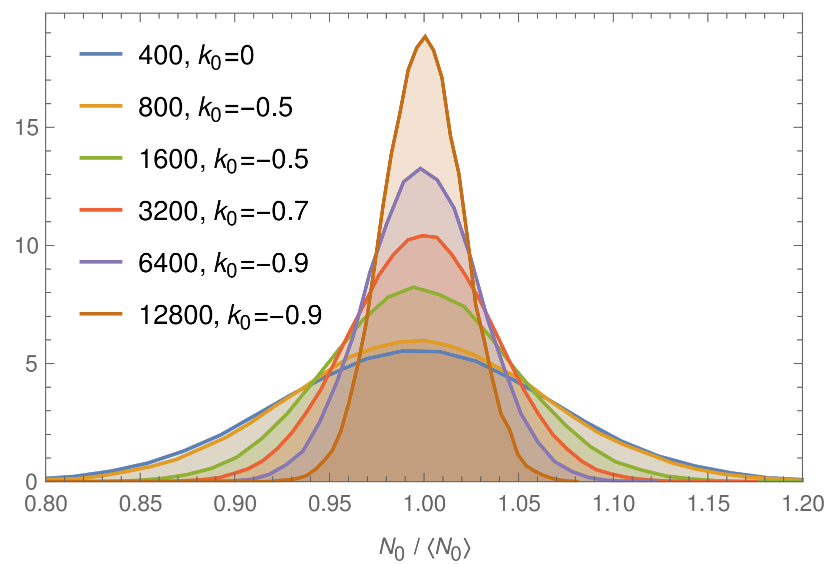
Simulations

The transition does not present the characteristics of a first order phase transition, as observed for the full set of triangulations of the 3-sphere:









Conclusions:

We defined a subset of triangulations of the 3-sphere, which should provide a good model to study the asymptotic properties analytically, because:

- Simple characterization (is $(T \setminus T0)^E$ a tree of triangles…?)

- Bijective encoding by trees → more hope to get some exact results on the asymptotic geometry

- Phase diagram seems more interesting at this preliminary stage

To Do:

- Better simulations?

- Exact enumeration?

- Encoding by trees that would contain information on the distances («metric bijection»)

Thank you for your attention!