Conformal Symmetry in Momentum Space and Anomaly Actions in Gravity

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STRUCTURES
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Introduction

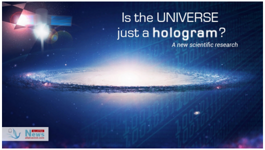
- Conformal invariance imposes strong constraints on correlation functions
- It determines 2- and 3-point functions of scalars, conserved vectors and the stress-energy tensor
- It determines the form of the higher functions up to functions of cross-ratios
- These results were obtained in position space and this is in contrast with general QFT were Feynman diagrams are typically computed in momentum space
- While position space methods are powerful, typically they can not be Fourier transformed easily in momentum space

Why study CFT directly in momentum space?

Momentum space results are needed in several recent applications

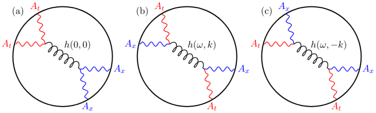
Holographic Cosmology

CMB primordial power spectra $\langle T_{\mu\nu}(p) T_{\rho\sigma}(-p)\rangle$ and Non-Gaussianities $\langle T_{ij} T_{kl} T_{mn}\rangle$



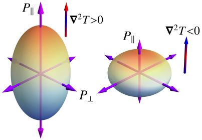
Studies of 3d critical phenomena

AdS/CFT correspondence for non-Fermi liquid metallic system. Chemical potential to a strongly-coupled CFT.



Anomalies

Breaking of conformal invariance by quantum effects.



Overview

- CFT in coordinate space
- Weyl invariance implies conformal invariance
- Weyl anomaly and effective action
- CFT in momentum space and explicit solution of the 4–point function imposing dual conformal symmetry
- Anomalous contribution to correlation functions involving stress energy tensors in momentum space

Conformal invariance in Coordinate Space

Conformal transformations $x_\mu \to x'_\mu(x)$ preserve the infiniesimal length up to a local factor

$$dx_\mu dx^\mu \to dx'_\mu dx'^\mu = \Omega(x)^{-2} dx_\mu dx^\mu$$

and in the infinitesimal form $x'_\mu(x) = x_\mu + a_\mu + \omega_{\mu\nu} x^\nu + \lambda x_\mu + b_\mu x^2 - 2b \cdot x\, x_\mu$ with $\Omega(x) = 1 - \sigma(x) = 1 - \lambda + 2b \cdot x$.

Translation	$\delta x^\mu = a_\mu,$		$P_\mu = -i\partial_\mu,$
Rotation	$\delta x^\mu = \omega_{\mu\nu} x^\nu$	$\Longrightarrow$	$L_{\mu\nu} = -i(x_\nu \partial_\mu - x_\mu \partial_\nu),$
Dilatations	$\delta x^\mu = \lambda x^\mu,$		$D = -ix^\mu \partial_\mu,$
SCT	$\delta x^\mu = b^\mu x^2 - 2x^\mu b \cdot x.$		$K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu).$

Defining the generators J_{ab}, $J_{ab} = -J_{ba}$ with $a, b = -1, 0, 1, \dots, d$ as

$$J_{\mu\nu} = L_{\mu\nu}, \quad J_{-1,\mu} = \frac{1}{2}\left(P_\mu - K_\mu\right), \quad J_{-1,0} = D, \quad J_{0,\mu} = \frac{1}{2}\left(P_\mu + K_\mu\right)$$
$$[J_{ab}, J_{cd}] = -i(\eta_{ac} J_{bd} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc} - \eta_{bc} J_{ad})$$

with $\eta_{ab} = \mathrm{diag}(-1, 1, \dots, 1)$. Then $Conf(\mathbb{R}^d) \cong SO(d+1, 1) \Rightarrow \frac{(d+2)(d+1)}{2}$ parameters.

Conformal Symmetry in Coordinate Space

Primary scalar field

$$\phi(x^\mu) \to \phi'(x'^\mu) = \left|\frac{\partial x'^\mu}{\partial x^\nu}\right|^{-\Delta/d} \phi(x^\mu)$$

Consider the two-point function

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{1}{Z}\int [d\Phi]\phi_1(x_1)\phi_2(x_2)\, e^{-S[\Phi]}$$

Assuming $S[\Phi]$ to be conformally invariant

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \left|\frac{\partial x'^\mu}{\partial x^\nu}\right|_{x=x_1}^{\Delta_1/d} \left|\frac{\partial x'^\mu}{\partial x^\nu}\right|_{x=x_2}^{\Delta_2/d} \langle \phi_1(x_1')\phi_2(x_2')\rangle$$

Dil. $\quad \langle \phi_1(x_1)\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2} \langle \phi_1(\lambda x_1)\phi_2(\lambda x_2)\rangle$

Rot. and Transl. $\quad \langle \phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1 - x_2|) = \lambda^{\Delta_1+\Delta_2} f(|\lambda x_1 - \lambda x_2|)$

$$\Longrightarrow \langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}$$

Conformal Symmetry in Coordinate Space

$$SCT = Inversion + translation + Inversion \qquad x^\mu \to \frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} - b^\mu \to \frac{\frac{x^\mu}{x^2} - b^\mu}{\left(\frac{x^\mu}{x^2} - b^\mu\right)^2}$$

We impose the invariance under SCT on the two point function

$$\text{SCT:} \qquad \left|\frac{\partial x_i'^\mu}{\partial x_i'^\nu}\right| = \gamma_i^{-d}, \qquad \gamma_i = (1 - 2b\cdot x_i + b^2\, x_i^2),$$

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{C_{12}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}\,\frac{(\gamma_1\gamma_2)^{(\Delta_1+\Delta_2)/2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}$$

and in general

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle = \frac{C_{12}\,\delta_{\Delta_i\Delta_j}}{(x-y)^{\Delta_i+\Delta_j}},$$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{\lambda_{123}}{|x_{12}|^{\Delta_{123}}|x_{13}|^{\Delta_{132}}|x_{23}|^{\Delta_{231}}}$$

$$\langle \mathcal{O}(x_1)\ldots\mathcal{O}(x_4)\rangle = \frac{f(u,v)}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}}, \quad u = \frac{x_{12}^2 x_{24}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}$$

Conformal Symmetry in Coordinate Space

Conformal Invariance on correlators (Conformal Ward Identities)

Dilatation

$$0 = \left[\sum_{j=1}^{n} \Delta_j + \sum_{j=1}^{n} x_j^\alpha \frac{\partial}{\partial x_j^\alpha} \right] \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle$$

SCT

$$0 = \left[\sum_{j=1}^{n} \left(2\Delta_j x_j^\kappa + 2 x_j^\kappa x_j^\alpha \frac{\partial}{\partial x_j^\alpha} - x_j^2 \frac{\partial}{\partial x_{j\,\kappa}} \right) \right] \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle$$

$$2 \sum_{j=1}^{n} \sum_{h=1}^{r_j} \left[(x_j)_{\alpha_{j_h}} \delta^{\kappa \mu_{j_h}} - x_j^{\mu_{j_h}} \delta^{\kappa}_{\alpha_{j_h}} \right] \langle \mathcal{O}^{\mu_{1_1} \mu_{1_2} \dots \mu_{1_{r_1}}}(x_1) \dots \mathcal{O}^{\mu_{j_1} \mu_{j_2} \dots \mu_{j_{r_j}}}(x_j) \dots \mathcal{O}^{\mu_{n_1} \mu_{n_2} \dots \mu_{n_{r_n}}}(x_n) \rangle$$

Embedding space

Taking a Euclidean CFT in d-dimensions with the conformal group acting non-linearly on $\mathbb{R}^d$, it is locally isomorphic to $SO(d+1,1)$, then it acts linearly on the space $\mathbb{R}^{d+1,1}$ (the embedding space).

The procedure consists in embedding the target space $\mathbb{R}^d$ into $\mathbb{R}^{d+1,1}$. We introduce the light-cone coordinates in $\mathbb{R}^{d+1,1}$ as

$$X^A = (X^+, X^-, X^a) \in \mathbb{R}^{d+1,1},$$

where $a = 1, \ldots, d$ with the inner product

$$X \cdot X = \eta_{AB} X^A X^B = -X^+ X^- + X_a X^a.$$

Using such coordinates, the condition $X^2 = 0$ defines a $SO(d+1,1)$ invariant subspace of dimension $d+1$, the null cone. Imposing the gauge condition $X^+ = 1$, that defines the Poincaré section of the embedding, we identify the projective null-cone with $\mathbb{R}^d$, and the null vectors take the form

$$X^A = (1, x^2, x^\mu).$$

Finally, taking two points X_i and X_j, setting $X_{ij} = X_i - X_j$, on the Poincaré section where $X^2 = 0$, one obtains

$$X_{ij}^2 = -2X_i \cdot X_j = (x_i - x_j)^2 = x_{ij}^2.$$

[P. A. M. Dirac (1936); G. Mack and A. Salam (1969); S. Ferrara, A.F. Grillo and R. Gatto, (1973); S. Weinberg (2010)]
[M. S. Costa, J. Penedones, D. Poland, S. Rychkov(2011); D. Simmons-Duffin (2014);...]

The first step is to consider a primary scalar field $\mathcal{O}_d(x)$ in $\mathbb{R}^d$ with dimension Δ, for which one can define a scalar on the entire null-cone, with the requirement

$$\mathcal{O}_{d+1,1}(\lambda X) = \lambda^{-\Delta}\mathcal{O}_{d+1,1}(X),$$

with the dimension of $\mathcal{O}_d$ reflected in the degree of $\mathcal{O}_{d+1}$. Conformal invariance requires that the correlators containing $\mathcal{O}_{d+1,1}(X)$ are invariant under $SO(d+1,1)$.

For example, the two-point function of operators of dimension Δ is fixed by conformal invariance, homogeneity, and the null condition $X_i^2 = 0$, to take the form

$$\langle \mathcal{O}(X_1)\mathcal{O}(X_2)\rangle = \frac{C_{12}}{(-2X_1 \cdot X_2)^{\Delta}}.$$

We notice that $-2X_1 \cdot X_2$ is the only Lorentz invariant that we can construct out of two points, since on the projective null-cone the condition $X_j^2 = 0$ is satisfied. We obtain

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{C_{12}}{(X_{12}^2)^{\Delta}} = \frac{C_{12}}{|x_1 - x_2|^{2\Delta}},$$

We are going to illustrate the method for totally symmetric and traceless tensors, since such tensors transform with the simplest irreducible representation of $SO(d+1,1)$.
Considering a tensor field of $SO(d+1,1)$, denoted as $\mathcal{O}_{A_1\dots A_n}(X)$, with the properties

- defined on the null-cone $X^2 = 0$,
- traceless and symmetric,
- homogeneous of degree $-\Delta$ in X, i.e., $\mathcal{O}_{A_1\dots A_n}(\lambda X) = \lambda^{-\Delta}\mathcal{O}_{A_1\dots A_n}(X)$,
- transverse $X^{A_i}\mathcal{O}_{A_1\dots A_i\dots A_n}(X) = 0$, with $i = 1, \dots, n$,

To find the corresponding tensor in $\mathbb{R}^d$, one has to restrict $\mathcal{O}_{A_1\dots A_n}(X)$ to the Poincaré section and project the indices as

$$\mathcal{O}_{\mu_1\dots\mu_n}(x) = \frac{\partial X^{A_1}}{\partial x_1^{\mu}} \dots \frac{\partial X^{A_n}}{\partial x_n^{\mu}} \mathcal{O}_{A_1\dots A_n}(X).$$

For example, the most general form of the two-point function of two operators with spin-1 and dimension Δ can be derived as

$$\langle \mathcal{O}^A(X_1)\mathcal{O}^B(X_2)\rangle = \frac{C_{12}}{(X_1 \cdot X_2)^{\Delta}}\left[\eta^{AB} + \alpha \frac{X_2^A X_1^B}{X_1 \cdot X_2}\right]$$

The transverse condition

$$X_1^A \langle \mathcal{O}^A(X_1)\mathcal{O}^B(X_2)\rangle = \frac{C_{12}}{(X_1 \cdot X_2)^\Delta}\left[X_1^B + \alpha X_1^B\right] = 0,$$

implies that $\alpha = -1$. The projection on $\mathbb{R}^d$ leads to define the two-point function in $\mathbb{R}^d$ as

$$\langle \mathcal{O}^\mu(x_1)\mathcal{O}^\nu(x_2)\rangle = \frac{\partial X_1^A}{\partial x_1^\mu}\frac{\partial X_2^B}{\partial x_2^\nu}\langle \mathcal{O}^A(X_1)\mathcal{O}^B(X_2)\rangle.$$

Therefore

$$\langle \mathcal{O}^\mu(x_1)\mathcal{O}^\nu(x_2)\rangle = \frac{C_{12}\, I^{\mu\nu}(x_{12})}{x_{12}^{2\Delta}},$$

where

$$I^{\mu\nu}(x) = \delta^{\mu\nu} - \frac{2x^\mu x^\nu}{x^2},$$

and C_{12} is an undetermined constant. One can easily construct correlation functions of higher spin, and for a traceless spin-2 operator of dimension Δ, we find

$$\langle \mathcal{O}^{\mu\nu}(x_1)\mathcal{O}^{\rho\sigma}(x_2)\rangle = \frac{C_{12}}{x_{12}^{2\Delta}}\left[I^{\mu\rho}(x_{12})I^{\nu\sigma}(x_{12}) + I^{\mu\sigma}(x_{12})I^{\nu\rho}(x_{12}) - \frac{2}{d}\delta^{\mu\nu}\delta^{\rho\sigma}\right].$$

The position space expressions are only valid at separated points and do not possess a Fourier transform prior to renormalization. For instance

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{C_{\mathcal{O}}}{|x|^{2\Delta}}$$

does not have a Fourier transform when

$$\Delta = \frac{d}{2} + k, \qquad k = 0, 1, 2, \ldots$$

because of short-distance singularities. In momentum space

$$\langle \mathcal{O}(p_1)\mathcal{O}(p_2)\rangle = (2\pi)^d \delta(p_1 + p_2) \frac{C_{\mathcal{O}}\, \pi^{d/2}\, 2^{d-2\Delta}\, \Gamma\left(\frac{d}{2} - \Delta\right)}{\Gamma(\Delta)} p_1^{2\Delta - d}, \quad \Delta \text{ generic}$$

$$\langle \mathcal{O}(p_1)\mathcal{O}(p_2)\rangle = (2\pi)^d \delta(p_1 + p_2) \left[\frac{C_{\mathcal{O}}\, \pi^{d/2}\, 2^{d-2\Delta}\, \Gamma\left(\frac{d}{2} - \Delta\right)}{\Gamma(\Delta)} \ln \frac{p_1^2}{\mu^2} + c_{\Delta'} \right] p_1^{2k}, \quad \Delta = \frac{d}{2} + k$$

with $c_{\Delta'}$ scheme dependent constant that can be absorbed by a redefinition of the scale μ.

[C. Coriano, L. Delle Rose, E. Mottola, and M. Serino(2012); A. Bzowski, P. McFadden and K. Skenderis (2016); C. Corianò and M.M.M. (2019)]

Weyl invariance

The infinitesimal transformation $x^\mu(x) \to x'^\mu(x) = x^\mu + \xi^\mu$ is classified as an isometry if leaves the metric $g_{\mu\nu}(x)$ invariant in form $g'_{\mu\nu}(x') = g_{\mu\nu}(x')$. This property leads to the Killing equation

$$\nabla_\nu \xi_\mu + \nabla_\nu \xi_\mu = 0.$$

The condition $g'_{\mu\nu}(x') = \Omega^{-2}(x) g_{\mu\nu}(x')$ instead brings to the conformal Killing equation

$$\nabla_\nu \xi_\mu + \nabla_\nu \xi_\mu = 2\sigma\, g_{\mu\nu}$$

In the flat spacetime limit the conformal Killing equation has the general solution

$$\xi^\mu(x) = a_\mu + \omega_{\mu\nu} x^\nu + \lambda\, x_\mu + b_\mu x^2 - 2x_\mu\, b \cdot x$$

Weyl transformations: $g_{\mu\nu} \to g'_{\mu\nu} = e^{2\sigma(x)} g_{\mu\nu} \quad \Longrightarrow \quad \delta^W_\sigma\, g_{\mu\nu} = 2\sigma\, g_{\mu\nu}$

Diffeomorphism : $x_\mu \to x_\mu + \xi_\mu \quad \Longrightarrow \quad \delta^E_\xi\, g_{\mu\nu} = \nabla_\nu \xi_\mu + \nabla_\nu \xi_\mu$

and the conformal killing equations are given by the invariance of Weyl rescaling and diffeomorphism

$$\delta^E_\xi\, g_{\mu\nu} + \delta^W_\sigma\, g_{\mu\nu} = 0$$

Weyl invariance

The transformations of the conformal group cause specific rescaling of the metric in curved space, i.e. $g_{\mu\nu} \to e^{2\sigma(x)} g_{\mu\nu}$ for specific $\sigma(x)$, but Weyl transformations in curved space allow for a completely general $\sigma(x)$.

Weyl invariance implies conformal invariance when explicitly choosing a fixed "background" metric manifold.

Weyl invariance + Diffeomorphism $\Longrightarrow$ Conformal invariance in flat space

Weyl invariance + Diffeomorphism $\not\Longleftarrow$ Conformal invariance in flat space

Let Δ be a linear covariant differential operator built with $g_{\mu\nu}$ and ∇_μ of rank s ($\Delta \sim \partial^s + \dots$) such that

$$\Delta \to \Delta' = e^{-\frac{d+s}{2}\sigma} \Delta \, e^{\frac{d-s}{2}\sigma}.$$

Introducing a field Φ of dimensions $\frac{d-s}{2}$,

$$\Phi \to \Phi' = e^{-\frac{d-s}{2}\sigma} \Phi,$$

and $\bar{\Phi}$ conjugate with the same dimension, we construct a Weyl invariant action in an arbitrary dimension

$$S[\Phi] = -\int d^d x \sqrt{-g}\, \bar{\Phi} \, \Delta \Phi.$$

There exist interesting families of operators Δ defined above (Grham - Jenne - Mason - Sparling). They are rank$2n$ operators $n \in \mathbb{N}_0$ which act on real scalar fields ϕ of dimension $\delta_{2n} = \frac{d-2n}{2}$

$$\Delta_{2n} = \Delta_{2n,1} + \frac{d-2n}{2} Q_{2n},$$

where $\Delta_{2n,1}$ is purely derivative part and Q_{2n} the curvature part.

[A. Stergiou, G. P. Vacca, O. Zanusso (2022)]

The first member is the Yamabe operator

$$\Delta_2 = -\nabla^2 + \frac{d-2}{4(d-1)} R.$$

giving the action of a scalar conformally coupled

$$S[\phi, g] = -\frac{1}{2} \int d^d x \sqrt{-g}\, \phi\, \Delta_2 \phi \quad \xrightarrow[g \to \delta]{} \quad S[\phi, g] = \frac{1}{2} \int d^d x\, \phi\, \nabla^2 \phi$$

The energy momentum tensor in a general background can be defined as

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

The variational EMT is traceless and conserved when evaluated on-shell using the equation of motion of ϕ

$$\nabla_\mu T^{\mu\nu}\Big|_{on-shell} = g_{\mu\nu} T^{\mu\nu}\Big|_{on-shell} = 0 \qquad (1)$$

Trace (Weyl) Anomaly

Let us consider a four-dimensional system of massless fields in interaction with an external gravitational field whose action is invariant under conformal transformations of the metric tensor with appropriate rescalings of the matter fields.

In absence of self-interaction of the matter fields, the effective action $\mathcal{W}$ is given exactly by its one-loop value and it is a non-local functional of the metric tensor and the fields. In dimensional regularization $\mathcal{W}$ will be a function of the dimensions d and exhibits a simple pole at $d = 4$

$$\mathcal{W}(d) = \frac{1}{d-4} \int d^d x \sqrt{-g}\, A(d)$$

with

$$A(d) \sim R\, \Box^{\frac{d-4}{2}} R\,,\, F^a_{\mu\nu}\, \Box^{\frac{d-4}{2}} F^{a\,\mu\nu}, \ldots$$

The infinities must be removed by a counterterm $\Delta\mathcal{W}$ which is d dependent and

$$\mathcal{W}_{reg} \equiv \mathcal{W}(d) + \delta\mathcal{W}(d)$$

is finite at $d = 4$.

Weyl anomaly

$\Delta\mathcal{W}$ must be a local functional and conformally invariant in $d=4$. On dimensional grounds the only possibility is

$$\Delta\mathcal{W} = \frac{\mu^{d-4}}{d-4}\int d^d x\sqrt{-g}\Big[bC^2 + b' E_4 + b'' e^2 F^2\Big]$$

$$C^2 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2, \qquad E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

The regularized stress tensor will be finite at $d=4$ and its trace is

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{W}_{reg}}{\delta g_{\mu\nu}},$$

$$T^{\mu}{}_{\mu} = \frac{2}{\sqrt{-g}}\left[g_{\mu\nu}\frac{\delta\mathcal{W}(d)}{\delta g_{\mu\nu}} + g_{\mu\nu}\frac{\delta\Delta\mathcal{W}(d)}{\delta g_{\mu\nu}}\right]_{d=4} = \frac{2}{\sqrt{-g}}\left[g_{\mu\nu}\frac{\delta\Delta\mathcal{W}(d)}{\delta g_{\mu\nu}}\right]_{d=4}$$

but

$$\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int d^d x\sqrt{-g}\Big\{C^2\ ;\ E_4\ ;\ F^2\Big\} = (d-4)\Big\{C^2 + \frac{2}{3}\Box R\ ;\ E_4\ ;\ F^2\Big\}$$

$$\langle T^{\mu}{}_{\mu}\rangle = b\left(C^2 + \frac{2}{3}\Box R\right) + b' E_4 + b'' e^2 F^2$$

[Deser Duff Isham (1976); Brown (1977); Fujikawa (1981); Brown and Cassidy(1977),. . .]

Anomaly effective action

The effective action that results from renormalization can be written as

$$\mathcal{S} = \mathcal{S}_{inv}(g) + \mathcal{S}_{local}(g) + \mathcal{S}_{\mathcal{A}}(g)$$

with arbitrary Weyl invariant term $S_{inv}(e^{2\sigma} g) = S_{inv}(g)$ and $\mathcal{S}_{local}(g) \sim \int R^2$ term.
The general form of the anomalous trace is

$$\left\langle T^{\mu}_{\mu}(x) \right\rangle_g = \mathcal{A} \equiv b\, C^2 + b' \left(E - \frac{2}{3} \Box R \right) + \sum_i \beta_i \mathcal{L},$$

and the Wess-Zumino consistency condition

$$\mathcal{S}_{\mathcal{A}}[e^{2\sigma} \bar{g}] = \mathcal{S}_{\mathcal{A}}[\bar{g}] + \Gamma_{WZ}[\bar{g}; \sigma],$$

$$\frac{\delta \Gamma_{WZ}[\bar{g}; \sigma]}{\delta \sigma(x)} = [\sqrt{-g} \mathcal{A}]_{g = e^{2\sigma} \bar{g}} \,.$$

Integrating the anomaly along the orbit of the local conformal group, one finds the form of the anomaly effective action in the local form (depending on σ). There are two cases in $d = 4$ in which one can invert $\sigma = \Sigma(g)$ to obtain

$$\mathcal{S}_{\mathcal{A}}[g, \Sigma(g)] = -\frac{b'}{4}\int d^4x\sqrt{-g}\,\Sigma\,\Delta_4\,\Sigma - \frac{1}{4}\int d^4x\sqrt{-g}\left[b\,C^2 + b'\left(E - \frac{2}{3}\Box R + \sum_i \beta_i \mathcal{L}\right)\right]\Sigma$$

with the two choices

$$\Sigma_{FV}(g) = 2\ln\left(1 + \frac{1}{6}G_{\Box - R/6}R\right), \qquad \Sigma_R(g) = -\frac{1}{2}G_{\Delta_4}\left(E - \frac{2}{3}\Box R\right),$$

where G_{Δ_4} and $G_{\Box - R/6}$ are the Green's functions of the operators Δ_4 and $\Box - R/6$ respectively. [E. Fradkin and G. A. Vilkovisky (1978); R. J. Riegert (1984)]

Momentum Space

No direct way to find the structure of the correlators.
One has to solve the CWI's

$$0=\left[\sum_{j=1}^{n}\Delta_j-(n-1)d-\sum_{j=1}^{n-1}p_j^{\alpha}\frac{\partial}{\partial p_j^{\alpha}}\right]\langle\mathcal{O}_1(p_1)\dots\mathcal{O}_n(\bar{p}_n)\rangle$$

$$0=\left[\sum_{j=1}^{n-1}\left(2(\Delta_j-d)\frac{\partial}{\partial p_{j\,\kappa}}-2p_j^{\kappa}\frac{\partial^2}{\partial p_j^{\alpha}\partial p_{j\,\kappa}}+p_j^{\kappa}\frac{\partial^2}{\partial p_{j\,\alpha}\partial p_j^{\alpha}}\right)\right]\langle\mathcal{O}_1(p_1)\dots\mathcal{O}_n(\bar{p}_n)\rangle$$

$$2\sum_{j=1}^{n}\sum_{h=1}^{r_j}\left[\delta^{\kappa\mu_{j_h}}\frac{\partial}{\partial p_j^{\alpha_{j_h}}}-\delta^{\kappa}_{\alpha_{j_h}}\frac{\partial}{\partial p_{j\,\mu_{j_h}}}\right]\langle\mathcal{O}^{\mu_{1_1}\mu_{1_2}\cdots\mu_{1_{r_1}}}(x_1)\dots\mathcal{O}^{\mu_{j_1}\mu_{j_2}\cdots\mu_{j_{r_j}}}(x_j)\dots\mathcal{O}^{\mu_{n_1}\mu_{n_2}\cdots\mu_{n_{r_n}}}(x_n)\rangle$$

Scalar 3-point function in momentum space

We have three invariants defined as $p_i = \sqrt{p_i^2}$

$$\langle \mathcal{O}_1(p_1)\mathcal{O}_2(p_2)\mathcal{O}_3(\bar{p}_3)\rangle = \Phi(p_1,p_2,p_3).$$

All the conformal WI's can be re-expressed in scalar form using the chain rules

$$\frac{\partial \Phi}{\partial p_i^\mu} = \frac{p_i^\mu}{p_i}\frac{\partial \Phi}{\partial p_i} - \frac{\bar{p}_3^\mu}{p_3}\frac{\partial \Phi}{\partial p_3}, \quad i = 1,2$$

where $\bar{p}_3^\mu = -p_1^\mu - p_2^\mu$ and $p_3 = \sqrt{(p_1+p_2)^2}$. By using this equation, the dilatation operator can be written as

$$\sum_{i=1}^{2} p_i^\mu \frac{\partial}{{p_i}^\mu}\Phi(p_1,p_2,p_3) = \left(p_1\frac{\partial}{\partial p_1} + p_2\frac{\partial}{\partial p_2} + p_3\frac{\partial}{\partial p_3}\right)\Phi(p_1,p_2,p_3).$$

and the special conformal Ward identities in d dimension

$$0 = K^\kappa \Phi(p_1,p_2,p_3) = \left(p_1^\kappa K_{13} + p_2^\kappa K_{23}\right)\Phi(p_1,p_2,p_3)$$

with

$$K_{ij} \equiv K_i - K_j, \qquad K_i \equiv \frac{\partial^2}{\partial p_i \partial p_i} + \frac{d+1-2\Delta_i}{p_i}\frac{\partial}{\partial p_i}, \quad i = 1,2,3.$$

The differential equations are explicitly solved and we have

$$\langle\mathcal{O}(p_1)\mathcal{O}(p_2)\mathcal{O}(p_3)\rangle = \left(p_3^2\right)^{-d+\frac{\Delta_t}{2}} C(\Delta_1,\Delta_2,\Delta_3,d)$$

$$\Big\{\Gamma\left(\Delta_1-\frac{d}{2}\right)\Gamma\left(\Delta_2-\frac{d}{2}\right)\Gamma\left(d-\frac{\Delta_1+\Delta_2+\Delta_3}{2}\right)\Gamma\left(d-\frac{\Delta_1+\Delta_2-\Delta_3}{2}\right)F_4\left(\frac{d}{2}-\frac{\Delta_1+\Delta_2-\Delta_3}{2},d-\frac{\Delta_t}{2},\frac{d}{2}-\Delta_1+1,\frac{d}{2}-\Delta_2+1;x,y\right)$$

$$+\Gamma\left(\frac{d}{2}-\Delta_1\right)\Gamma\left(\Delta_2-\frac{d}{2}\right)\Gamma\left(\frac{\Delta_1-\Delta_2+\Delta_3}{2}\right)\Gamma\left(\frac{d}{2}+\frac{\Delta_1-\Delta_2-\Delta_3}{2}\right)x^{\Delta_1-\frac{d}{2}}F_4\left(\frac{\Delta_1-\Delta_2+\Delta_3}{2},\frac{d}{2}-\frac{\Delta_2+\Delta_3-\Delta_1}{2},\Delta_1-\frac{d}{2}+1,\frac{d}{2}-\Delta_2+1;x,y\right)$$

$$+\Gamma\left(\Delta_1-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-\Delta_2\right)\Gamma\left(\frac{-\Delta_1+\Delta_2+\Delta_3}{2}\right)\Gamma\left(\frac{d}{2}+\frac{-\Delta_1+\Delta_2-\Delta_3}{2}\right)y^{\Delta_2-\frac{d}{2}}F_4\left(\frac{\Delta_2-\Delta_1+\Delta_3}{2},\frac{d}{2}-\frac{\Delta_1-\Delta_2+\Delta_3}{2},\frac{d}{2}-\Delta_1+1,\Delta_2-\frac{d}{2}+1;x,y\right)$$

$$+\Gamma\left(\frac{d}{2}-\Delta_1\right)\Gamma\left(\frac{d}{2}-\Delta_2\right)\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta_3}{2}\right)\Gamma\left(-\frac{d}{2}+\frac{\Delta_1+\Delta_2+\Delta_3}{2}\right)x^{\Delta_1-\frac{d}{2}}y^{\Delta_2-\frac{d}{2}}F_4\left(-\frac{d}{2}+\frac{\Delta_t}{2},\frac{\Delta_1+\Delta_2-\Delta_3}{2},\Delta_1-\frac{d}{2}+1,\Delta_2-\frac{d}{2}+1;x,y\right)\Big\}.$$

with $\quad x=\frac{p_1^2}{p_3^2}, \qquad y=\frac{p_2^2}{p_3^2}.$

or in a equivalent and compact form

$$\langle\mathcal{O}(p_1)\mathcal{O}(p_2)\mathcal{O}(p_3)\rangle = C_{123}\,p_1^{\Delta_1-\frac{d}{2}}p_2^{\Delta_2-\frac{d}{2}}p_3^{\Delta_3-\frac{d}{2}}\int_0^\infty dx\,x^{\frac{d}{2}-1}K_{\Delta_1-\frac{d}{2}}(p_1\,x)K_{\Delta_2-\frac{d}{2}}(p_2\,x)K_{\Delta_3-\frac{d}{2}}(p_3\,x)$$

In the limit $p_3 \gg p_1$ (expressible also as $p_3^2, p_2^2 \to \infty$ with $p_2^2/p_3^2 \to 1$ fixed), it behaves as

$$\langle\mathcal{O}(p_1)\mathcal{O}(p_2)\mathcal{O}(p_3)\rangle \propto f(d,\Delta_i)\,p_3^{\Delta_1+\Delta_2+\Delta_3-2d}\left(1+O\left(\frac{p_1}{p_3}\right)\right) \qquad \text{if } \Delta_1>\frac{d}{2}$$

$$\langle\mathcal{O}(p_1)\mathcal{O}(p_2)\mathcal{O}(p_3)\rangle \propto g(d,\Delta_i)\,p_3^{\Delta_2+\Delta_3-\Delta_1-d}\,p_1^{2\Delta_1-d}\left(1+O\left(\frac{p_1}{p_3}\right)\right) \qquad \text{if } \Delta_1<\frac{d}{2},$$

with $f(d,\Delta_i)$ and $g(d,\Delta_i)$ depending only on the scaling and spacetime dimensions.

4-point function in momentum space

The general solution in coordinate space

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \frac{h(u(x_i),v(x_i))}{\left(x_{12}^2\right)^{\frac{\Delta_1+\Delta_2}{2}}\left(x_{34}^2\right)^{\frac{\Delta_3+\Delta_4}{2}}}$$

where $h(u(x_i),v(x_i))$ remains unspecified in terms of cross ratios

$$u(x_i) = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v(x_i) = \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2}$$

In momentum space the four point function depends on six invariants

$$p_i = |\sqrt{p_i^2}|, \quad i = 1,\ldots,4, \quad s = |\sqrt{(p_1+p_2)^2}|, t = |\sqrt{(p_2+p_3)^2}|$$

$$\langle \mathcal{O}(p_1)\mathcal{O}(p_2)\mathcal{O}(p_3)\mathcal{O}(\bar{p}_4)\rangle = \Phi(p_1,p_2,p_3,p_4,s,t).$$

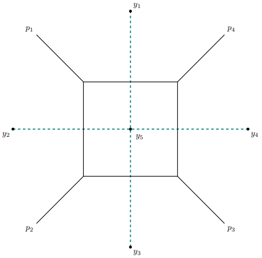
This correlation function, to be conformally invariant, has to verify the dilatation Ward Identity

$$\left[\sum_{i=1}^{4}\Delta_i - 3d - \sum_{i=1}^{3} p_i^\mu \frac{\partial}{\partial p_i^\mu}\right]\langle \mathcal{O}(p_1)\mathcal{O}(p_2)\mathcal{O}(p_3)\mathcal{O}(\bar{p}_4)\rangle = 0,$$

and the special conformal Ward Identities

$$\sum_{i=1}^{3}\left[2(\Delta_i - d)\frac{\partial}{\partial p_{i\,\kappa}} - 2p_i^\alpha \frac{\partial^2}{\partial p_i^\alpha \partial p_i^\kappa} + p_i^\kappa \frac{\partial^2}{\partial p_i^\alpha \partial p_{i\,\alpha}}\right]\langle \mathcal{O}(p_1)\mathcal{O}(p_2)\mathcal{O}(p_3)\mathcal{O}(\bar{p}_4)\rangle = 0.$$

We have found a particular solution of the 4 point function in momentum space using the dual conformal symmetry (DCC).



$$p_1^\mu = y_2^\mu - y_1^\mu, \quad p_2^\mu = y_3^\mu - y_2^\mu,$$
$$p_3^\mu = y_4^\mu - y_3^\mu, \quad p_4^\mu = y_1^\mu - y_4^\mu.$$

$$\langle \mathcal{O}(p_1)\mathcal{O}(p_2)\mathcal{O}(p_3)\mathcal{O}(\bar{p}_4)\rangle = \Phi(p_1,\dots,p_4,s,t)$$

$$0 = D(p_i)\Phi(p_1,\dots,p_4,s,t)$$
$$0 = K^\kappa(p_i)\Phi(p_1,\dots,p_4,s,t)$$

$$0 = D(y_i)\Phi(y_1,\dots,y_4)$$
$$0 = K^\kappa(y_i)\Phi(y_1,\dots,y_4)$$

$$D(y_i) = \left[\sum_{j=1}^{4}\Delta_j + \sum_{j=1}^{4} y_j^\alpha \frac{\partial}{\partial y_j^\alpha}\right]$$

$$K^\kappa(y_i) = \left[\sum_{j=1}^{4}\left(2\Delta_j y_j^\kappa + 2 y_j^\kappa y_j^\alpha \frac{\partial}{\partial y_j^\alpha} - y_j^2 \frac{\partial}{\partial y_{j\,\kappa}}\right)\right]$$

$$\langle \mathcal{O}(p_1)\mathcal{O}(p_2)\mathcal{O}(p_3)\mathcal{O}(p_4)\rangle = C\Big[I_{\frac{d}{2}-1\{\Delta-\frac{d}{2},\Delta-\frac{d}{2},0\}}(p_1\,p_3, p_2\,p_4, s\,t)$$
$$+ I_{\frac{d}{2}-1\{\Delta-\frac{d}{2},\Delta-\frac{d}{2},0\}}(p_2\,p_3, p_1\,p_4, s\,u) + I_{\frac{d}{2}-1\{\Delta-\frac{d}{2},\Delta-\frac{d}{2},0\}}(p_1\,p_2, p_3\,p_4, t\,u)\Big],$$

[C. Corianò, M. M. M., JHEP 1909 (2019) 107]

[Bzowski et al. , Implication of conformal invariance in momentum space, JHEP vol. 03, (2014), arXiv:1304.7760]

The separation of the stress energy tensor

$$T^{\mu\nu} = t^{\mu\nu} + t^{\mu\nu}_{loc} \tag{2}$$

where

$$\begin{aligned} t^{\mu\nu}(p) &= \Pi^{\mu\nu}_{\alpha\beta}(p)\, T^{\alpha\beta}(p) \equiv \left[\pi^{(\mu}_{\alpha}(p)\pi^{\nu)}_{\beta}(p) - \frac{1}{d-1}\pi^{\mu\nu}(p)\pi_{\alpha\beta}(p)\right] T^{\alpha\beta}(p) \\ t^{\mu\nu}_{loc}(p) &= \Sigma^{\mu\nu}_{\alpha\beta}(p) T^{\alpha\beta}(p) \equiv \left\{\frac{p_\beta}{p^2}\left[2p^{(\mu}\delta^{\nu)}_{\alpha} - \frac{p_\alpha}{d-1}\left(\delta^{\mu\nu} + (d-2)\frac{p^\mu p^\nu}{p^2}\right)\right] + \frac{\pi^{\mu\nu}(p)}{d-1}\delta_{\alpha\beta}\right\} T^{\alpha\beta}(p) \\ \pi^{\mu\nu}(p) &\equiv \delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \end{aligned} \tag{3}$$

allows to decompose the correlator in transverse-traceless, longitudinal-trace parts

$$\begin{aligned} \langle T^{\mu_1\nu_1}\, T^{\mu_2\nu_2}\, T^{\mu_3\nu_3}\rangle &= \langle t^{\mu_1\nu_1}\, t^{\mu_2\nu_2}\, t^{\mu_3\nu_3}\rangle + \langle T^{\mu_1\nu_1}\, T^{\mu_2\nu_2}\, t^{\mu_3\nu_3}_{loc}\rangle + \langle T^{\mu_1\nu_1}\, t^{\mu_2\nu_2}_{loc}\, T^{\mu_3\nu_3}\rangle \\ &+ \langle t^{\mu_1\nu_1}_{loc}\, T^{\mu_2\nu_2}\, T^{\mu_3\nu_3}\rangle - \langle T^{\mu_1\nu_1}\, t^{\mu_2\nu_2}_{loc}\, t^{\mu_3\nu_3}_{loc}\rangle - \langle t^{\mu_1\nu_1}_{loc}\, t^{\mu_2\nu_2}_{loc}\, T^{\mu_3\nu_3}\rangle - \langle t^{\mu_1\nu_1}_{loc}\, T^{\mu_2\nu_2}\, t^{\mu_3\nu_3}_{loc}\rangle + \langle t^{\mu_1\nu_1}_{loc}\, t^{\mu_2\nu_2}_{loc}\, t^{\mu_3\nu_3}_{loc}\rangle \end{aligned} \tag{4}$$

Invariance under Weyl transformation $\Longrightarrow$ Trace Ward Identities

$$g_{\mu\nu} \langle T^{\mu\nu}(x) \rangle_g = 0$$

$\downarrow$ Functional Differentiation, flat space limit and Fourier Transform

$$\delta_{\mu_1\nu_1} \langle T^{\mu_1\nu_1}(p_1) T^{\mu_2\nu_2}(p_2) T^{\mu_3\nu_3}(p_3) \rangle = -2 \langle T^{\mu_2\nu_2}(p_1+p_2) T^{\mu_3\nu_3}(p_3) \rangle - 2 \langle T^{\mu_2\nu_2}(p_2) T^{\mu_3\nu_3}(p_1+p_3) \rangle \tag{5}$$

Invariance under diffeomorphisms of the background metric $\Longrightarrow$ Transverse Ward Identities

$$\nabla^{(g)}_{\mu} \langle T^{\mu\nu}(x) \rangle_g = 0$$

$\downarrow$ Functional Differentiation, flat space limit and Fourier Transform

$$\begin{aligned} p_{1\nu_1} \langle T^{\mu_1\nu_1}(p_1)\, T^{\mu_2\nu_2}(p_2)\, T^{\mu_3\nu_3}(p_3) \rangle &= -p_2^{\mu_1} \langle T^{\mu_2\nu_2}(p_1+p_2) T^{\mu_3\nu_3}(p_3) \rangle - p_3^{\mu_1} \langle T^{\mu_2\nu_2}(p_2) T^{\mu_3\nu_3}(p_1+p_3) \rangle \\ &+ p_{2\alpha} \left[\delta^{\mu_1\nu_2} \langle T^{\mu_2\alpha}(p_1+p_2) T^{\mu_3\nu_3}(p_3) \rangle + \delta^{\mu_1\mu_2} \langle T^{\nu_2\alpha}(p_1+p_2) T^{\mu_3\nu_3}(p_3) \rangle \right] \\ &+ p_{3\alpha} \left[\delta^{\mu_1\nu_3} \langle T^{\mu_3\alpha}(p_1+p_3) T^{\mu_2\nu_2}(p_2) \rangle + \delta^{\mu_1\mu_3} \langle T^{\nu_3\alpha}(p_1+p_3) T^{\mu_2\nu_2}(p_2) \rangle \right]. \end{aligned} \tag{6}$$

In the decomposition

$$\langle T^{\mu_1\nu_1}\, T^{\mu_2\nu_2}\, T^{\mu_3\nu_3}\rangle = \boxed{\langle t^{\mu_1\nu_1}\, t^{\mu_2\nu_2}\, t^{\mu_3\nu_3}\rangle} + \underline{\langle t_{loc}^{\mu_1\nu_1}\, T^{\mu_2\nu_2}\, T^{\mu_3\nu_3}\rangle} + \underline{\langle T^{\mu_1\nu_1}\, t_{loc}^{\mu_2\nu_2}\, T^{\mu_3\nu_3}\rangle}$$
$$+ \underline{\langle T^{\mu_1\nu_1}\, T^{\mu_2\nu_2}\, t_{loc}^{\mu_3\nu_3}\rangle} - \underline{\langle T^{\mu_1\nu_1}\, t_{loc}^{\mu_2\nu_2}\, t_{loc}^{\mu_3\nu_3}\rangle} - \underline{\langle t_{loc}^{\mu_1\nu_1}\, t_{loc}^{\mu_2\nu_2}\, T^{\mu_3\nu_3}\rangle} - \underline{\langle t_{loc}^{\mu_1\nu_1}\, T^{\mu_2\nu_2}\, t_{loc}^{\mu_3\nu_3}\rangle} + \underline{\langle t_{loc}^{\mu_1\nu_1}\, t_{loc}^{\mu_2\nu_2}\, t_{loc}^{\mu_3\nu_3}\rangle} \tag{7}$$

the longitudinal and trace parts can be written in terms of two point functions

$$\langle t_{loc}^{\mu_1\nu_1}\, T^{\mu_2\nu_2}\, T^{\mu_3\nu_3}\rangle = \Sigma^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\langle T^{\alpha_1\beta_1}\, T^{\mu_2\nu_2}\, T^{\mu_3\nu_3}\rangle$$
$$= \left\{ \frac{p_{1\beta_1}}{p_1^2}\left[2p_1^{(\mu_1}\,\delta^{\nu_1)}_{\alpha_1} - \frac{p_{1\alpha_1}}{d-1}\left(\delta^{\mu_1\nu_1} + (d-2)\frac{p_1^{\mu_1}\, p_1^{\nu_1}}{p_1^2}\right)\right] + \frac{\pi^{\mu_1\nu_1}(p_1)}{d-1}\delta_{\alpha_1\beta_1}\right\}\langle T^{\alpha_1\beta_1}\, T^{\mu_2\nu_2}\, T^{\mu_3\nu_3}\rangle$$
$$= -\frac{2\,\pi^{\mu_1\nu_1}(p_1)}{(d-1)}\Big[\langle T^{\mu_2\nu_2}(p_1+p_2) T^{\mu_3\nu_3}(p_3)\rangle + \langle T^{\mu_2\nu_2}(p_2) T^{\mu_3\nu_3}(p_1+p_3)\rangle\Big]$$
$$+ \frac{1}{p_1^2}\left[2p_1^{(\mu_1}\,\delta^{\nu_1)}_{\alpha_1} - \frac{p_{1\alpha_1}}{d-1}\left(\delta^{\mu_1\nu_1} + (d-2)\frac{p_1^{\mu_1}\, p_1^{\nu_1}}{p_1^2}\right)\right]\Big\{ - p_2^{\alpha_1}\,\langle T^{\mu_2\nu_2}(p_1+p_2) T^{\mu_3\nu_3}(p_3)\rangle - p_3^{\alpha_1}\,\langle T^{\mu_2\nu_2}(p_2) T^{\mu_3\nu_3}(p_1+p_3)\rangle$$
$$+ p_{2\beta}\Big[\delta^{\alpha_1\mu_2}\,\langle T^{\beta\mu_2}(p_1+p_2) T^{\mu_3\nu_3}(p_3)\rangle + \delta^{\alpha_1\nu_2}\,\langle T^{\beta\mu_2}(p_1+p_2) T^{\mu_3\nu_3}(p_3)\rangle\Big]$$
$$+ p_{3\beta}\Big[\delta^{\alpha_1\mu_3}\,\langle T^{\nu_2\mu_2}(p_2) T^{\beta\nu_3}(p_1+p_3)\rangle + \delta^{\alpha_1\nu_3}\,\langle T^{\mu_2\nu_2}(p_2) T^{\mu_3\beta}(p_1+p_3)\rangle\Big]\Big\}$$

$\langle TTT \rangle$ decomposition

The transverse and traceless part can be decomposed in a minimal and manifestly symmetric form

$$\langle t^{\mu_1\nu_1}\, t^{\mu_2\nu_2}\, t^{\mu_3\nu_3} \rangle = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2)\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(p_3)\Big\{A_1\, p_2^{\alpha_1}\, p_2^{\beta_1}\, p_3^{\alpha_2}\, p_3^{\beta_2}\, p_1^{\alpha_3}\, p_1^{\beta_3}$$
$$+ A_2\,\delta^{\beta_1\beta_2} p_2^{\alpha_1}\, p_3^{\alpha_2}\, p_1^{\alpha_3}\, p_1^{\beta_3} + A_2\,(p_1 \leftrightarrow p_3)\,\delta^{\beta_2\beta_3} p_3^{\alpha_2}\, p_1^{\alpha_3}\, p_2^{\alpha_1}\, p_2^{\beta_1} + A_2\,(p_2 \leftrightarrow p_3)\,\delta^{\beta_3\beta_1} p_1^{\alpha_3}\, p_2^{\alpha_1}\, p_3^{\alpha_2}\, p_3^{\beta_2}$$
$$+ A_3\,\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2} p_1^{\alpha_3}\, p_1^{\beta_3} + A_3(p_1 \leftrightarrow p_3)\,\delta^{\alpha_2\alpha_3}\delta^{\beta_2\beta_3} p_2^{\alpha_1}\, p_2^{\beta_1} + A_3(p_2 \leftrightarrow p_3)\,\delta^{\alpha_3\alpha_1}\delta^{\beta_3\beta_1} p_3^{\alpha_2}\, p_3^{\beta_2}$$
$$+ A_4\,\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_3} p_2^{\beta_1}\, p_3^{\beta_2} + A_4(p_1 \leftrightarrow p_3)\,\delta^{\alpha_2\alpha_1}\delta^{\alpha_3\beta_1} p_3^{\beta_2}\, p_1^{\beta_3} + A_4(p_2 \leftrightarrow p_3)\,\delta^{\alpha_3\alpha_2}\delta^{\alpha_1\beta_2} p_1^{\beta_3}\, p_2^{\beta_1} + A_5\delta^{\alpha_1\beta_2}\delta^{\alpha_2\beta_3}\delta^{\alpha_3\beta_1}\Big\} \tag{8}$$

in terms of five independent form factors depending on three variables p_i, the magnitude of the momenta $p_i = \sqrt{p_i^2}$ with $i = 1, 2, 3$. These form factors have also some symmetry conditions, in particular A_1 and A_5 are S_3 invariant, and the others are symmetric under the permutation $p_1 \leftrightarrow p_2$.

What is the procedure to find these form factors in a general CFT?

[Bzowski et al. , Implication of conformal invariance in momentum space, JHEP vol. 03, (2014), arXiv:1304.7760]

Condition on the form factors

Imposing the dilatation and special conformal Ward identities to correlator one can find some contraints on the form factors in particular

Dilatation Ward Identities $\Longrightarrow$ A_i are homogeneous functions of the momenta of specific degree

$$\deg(A_i) = d - N_n$$

Special Conformal Ward Identities $\Longrightarrow$

- Second order differential equations on A_i
- First order differential equations on A_i

$$A_1(p_1, p_2, p_3) = \alpha_1 \, J_{6\{000\}} = \alpha_1 \int_0^\infty x^{\frac{d}{2}+5} \, p_1^{\frac{d}{2}} \, p_2^{\frac{d}{2}} \, p_3^{\frac{d}{2}} \, K_{\frac{d}{2}}(p_1 \, x) K_{\frac{d}{2}}(p_2 \, x) K_{\frac{d}{2}}(p_3 \, x)$$

...

The Special Conformal Ward Identities fix the 3-point function to depend on two undetermined constants α_1 and α_2 and 2-point function normalization c_T.

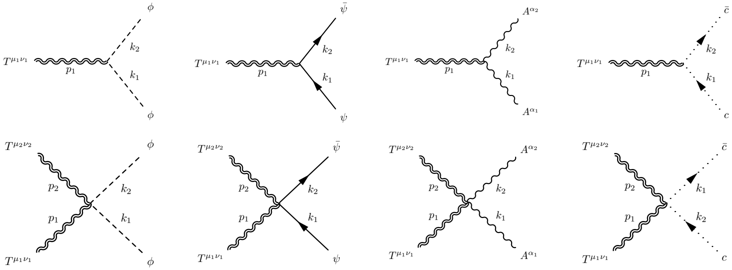
Perturbative Realization

The quantum actions for the scalar and fermion fields are respectively

$$S_{scalar} = \frac{1}{2}\int d^d x\,\sqrt{-g}\left[g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{(d-2)}{4(d-1)}\,R\,\phi^2\right],$$
$$S_{fermion} = \frac{i}{2}\int d^d x\,e\,e^\mu_a\left[\bar\psi\gamma^a(D_\mu\psi) - (D_\mu\bar\psi)\gamma^a\psi\right], \qquad D_\mu = \partial_\mu + \Gamma_\mu = \partial_\mu + \frac{1}{2}\Sigma^{ab}\,e^\sigma_a\nabla_\mu\,e_{b\,\sigma}. \tag{9}$$

The Σ^{ab} are the generators of the Lorentz group in the spin 1/2 representation. In $d = 4$ there is an additional conformal field theory

$$S_{abelian} = S_M + S_{gf} + S_{gh},$$
$$S_M = -\frac{1}{4}\int d^4x\,\sqrt{-g}\,F^{\mu\nu}F_{\mu\nu}, \quad S_{gf} = -\frac{1}{\xi}\int d^4x\,\sqrt{-g}\,(\nabla_\mu A^\mu)^2, \qquad S_{gh} = \int d^4x\,\sqrt{-g}\,\partial^\mu\bar c\,\partial_\mu c. \tag{10}$$

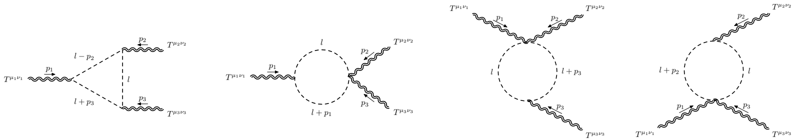


Feynman-'t Hooft gauge $\xi = 1$

The 3-point function we are interested in studying is

$$\langle T^{\mu_1\nu_1}(x_1)T^{\mu_2\nu_2}(x_2)T^{\mu_3\nu_3}(x_3)\rangle = 8\Bigg\{-\left\langle \frac{\delta S}{\delta g_{\mu_1\nu_1}(x_1)}\frac{\delta S}{\delta g_{\mu_2\nu_2}(x_2)}\frac{\delta S}{\delta g_{\mu_3\nu_3}(x_3)}\right\rangle + \left\langle \frac{\delta^2 S}{\delta g_{\mu_1\nu_1}(x_1)\delta g_{\mu_2\nu_2}(x_2)}\frac{\delta S}{\delta g_{\mu_3\nu_3}(x_3)}\right\rangle + \left\langle \frac{\delta^2 S}{\delta g_{\mu_1\nu_1}(x_1)\delta g_{\mu_3\nu_3}(x_3)}\frac{\delta S}{\delta g_{\mu_2\nu_2}(x_2)}\right\rangle + \left\langle \frac{\delta^2 S}{\delta g_{\mu_2\nu_2}(x_2)\delta g_{\mu_3\nu_3}(x_3)}\frac{\delta S}{\delta g_{\mu_1\nu_1}(x_1)}\right\rangle - \left\langle \frac{\delta^3 S}{\delta g_{\mu_1\nu_1}(x_1)\delta g_{\mu_2\nu_2}(x_2)\delta g_{\mu_3\nu_3}(x_3)}\right\rangle\Bigg\} \quad (11)$$

where the angle brackets denote the vacuum expectation value. In momentum space



$$V_S^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3) = \int \frac{d^d\ell}{(2\pi)^d}\frac{V_{T\phi\phi}^{\mu_1\nu_1}(\ell-p_2,\ell+p_3)V_{T\phi\phi}^{\mu_2\nu_2}(\ell,\ell-p_2)V_{T\phi\phi}^{\mu_3\nu_3}(\ell,\ell+p_3)}{\ell^2(\ell-p_2)^2(\ell+p_3)^2}$$

$$W_{S,1}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3) = \frac{1}{2}\int \frac{d^d\ell}{(2\pi)^d}\frac{V_{T\phi\phi}^{\mu_1\nu_1}(\ell,\ell+p_1)V_{TT\phi\phi}^{\mu_2\nu_2\mu_3\nu_3}(\ell,\ell+p_1)}{\ell^2(\ell+p_1)^2}$$

[Corianò and Maglio, The general 3-graviton vertex $\langle TTT\rangle$ of conformal field theorries in momentum space in $d=4$, Nucl.Phys. B937 (2018) 56-134]

One calculates the diagrams contributions in dimensional regularization and constructs the transverse and traceless part for each sectors as

$$\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)t^{\mu_3\nu_3}(p_3)\rangle_S = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2)\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(p_3)\Big[-V_S^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(p_1,p_2,p_3)+\sum_{j=1}^{3}W_{S,j}^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(p_1,p_2,p_3)\Big].$$

Using the properties

$$p^\alpha\Pi^{\mu\nu}_{\alpha\beta}(p)=0,\quad g^{\alpha\beta}\Pi^{\mu\nu}_{\alpha\beta}(p)=0,\quad p_3^{\alpha_1}\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)=-p_2^{\alpha_1}\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)$$
$$p_1^{\alpha_2}\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2)=-p_3^{\alpha_1}\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2),\quad p_2^{\alpha_3}\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(p_3)=-p_1^{\alpha_3}\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(p_3)$$

we reorganize the transverse and traceless part as

$$\begin{aligned}\langle t^{\mu_1\nu_1}\,t^{\mu_2\nu_2}\,t^{\mu_3\nu_3}\rangle_S &= \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2)\Pi^{\mu_3\nu_3}_{\alpha_3\beta_3}(p_3)\Big\{A_1^S\,p_2^{\alpha_1}p_2^{\beta_1}p_3^{\alpha_2}p_3^{\beta_2}p_1^{\alpha_3}p_1^{\beta_3}\\
&+A_2^S\,\delta^{\beta_1\beta_2}p_2^{\alpha_1}p_3^{\alpha_2}p_1^{\alpha_3}p_1^{\beta_3}+A_2^S\,(p_1\leftrightarrow p_3)\,\delta^{\beta_2\beta_3}p_3^{\alpha_2}p_1^{\alpha_3}p_2^{\alpha_1}p_2^{\beta_1}+A_2^S\,(p_2\leftrightarrow p_3)\,\delta^{\beta_3\beta_1}p_1^{\alpha_3}p_2^{\alpha_1}p_3^{\alpha_2}p_3^{\beta_2}\\
&+A_3^S\,\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2}p_1^{\alpha_3}p_1^{\beta_3}+A_3^S(p_1\leftrightarrow p_3)\,\delta^{\alpha_2\alpha_3}\delta^{\beta_2\beta_3}p_2^{\alpha_1}p_2^{\beta_1}+A_3^S(p_2\leftrightarrow p_3)\,\delta^{\alpha_3\alpha_1}\delta^{\beta_3\beta_1}p_3^{\alpha_2}p_3^{\beta_2}\\
&+A_4^S\,\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_3}p_2^{\beta_1}p_3^{\beta_2}+A_4^S(p_1\leftrightarrow p_3)\,\delta^{\alpha_2\alpha_1}\delta^{\alpha_3\beta_1}p_3^{\beta_2}p_1^{\beta_3}+A_4^S(p_2\leftrightarrow p_3)\,\delta^{\alpha_3\alpha_2}\delta^{\alpha_1\beta_2}p_1^{\beta_3}p_2^{\beta_1}+A_5^S\,\delta^{\alpha_1\beta_2}\delta^{\alpha_2\beta_3}\delta^{\alpha_3\beta_1}\Big\}\end{aligned} \tag{12}$$

The form factors A_i^S are functions of the kinematical invariants and the space-time dimensions d. Furthermore they are written in terms of master integrals B_0 and C_0.

Finally we construct the entire transverse and traceless part noticing that the number of fermion, scalar and gauge families is arbitrary and then the constants n_I, $I=F,S,G$ have to be considered.

$$\langle t^{\mu_1\nu_1}\,t^{\mu_2\nu_2}\,t^{\mu_3\nu_3}\rangle = \sum_{I=F,S,G} n_I\,\langle t^{\mu_1\nu_1}\,t^{\mu_2\nu_2}\,t^{\mu_3\nu_3}\rangle_I \tag{13}$$

Case $d = 4$

Divergences!

$$A_2^{Div} = \frac{\pi^2}{45\,\varepsilon}\left[26 n_G - 7 n_F - 2 n_S\right]$$

$$A_3^{Div} = \frac{\pi^2}{90\,\varepsilon}\left[3(p_1^2+p_2^2)\big(6n_F+n_S+12n_G\big)+p_3^2(11n_F+62n_G+n_S)\right]$$

$$A_4^{Div} = \frac{\pi^2}{90\,\varepsilon}\left[(p_1^2+p_2^2)\big(29n_F+98n_G+4n_S\big)+p_3^2(43n_F+46n_G+8n_S)\right]$$

$$A_5^{Div} = \frac{\pi^2}{180\,\varepsilon}\Big\{n_F\left(43p_1^4-14p_1^2(p_2^2+p_3^2)+43p_2^4-14p_2^2p_3^2+43p_3^4\right)$$
$$+2\Big[n_G\left(23p_3^2+26p_1^2(p_2^2+p_3^2)+23p_2^4+26p_2^2p_3^2+23p_3^4\right)+2n_S\left(2p_1^4-p_1^2(p_2^2+p_3^2)+2p_2^4-p_2^2p_3^2+2p_3^4\right)\Big]\Big\}$$

In perturbation theory the one loop counterterm Lagrangian is

$$S_{count} = -\frac{1}{\varepsilon}\sum_{I=F,S,G} n_I \int d^d x\,\sqrt{-g}\Big(\beta_a(I)\,C^2+\beta_b(I)\,E\Big) \tag{14}$$

where

$$V_{C^2}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3) = 8\int d^dx_1\,d^dx_2\,d^dx_3\,d^dx\left(\frac{\delta^3(\sqrt{-g}C^2)(x)}{\delta g_{\mu_1\nu_1}(x_1)\delta g_{\mu_2\nu_2}(x_2)\delta g_{\mu_3\nu_3}(x_3)}\right)_{flat} e^{-i(p_1x_1+p_2x_2+p_3x_3)}$$
$$\equiv 8\left[\sqrt{-g}\,C^2\right]^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3)$$

$$V_{E}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3) = 8\int d^dx_1\,d^dx_2\,d^dx_3\,d^dx\left(\frac{\delta^3(\sqrt{-g}E)(x)}{\delta g_{\mu_1\nu_1}(x_1)\delta g_{\mu_2\nu_2}(x_2)\delta g_{\mu_3\nu_3}(x_3)}\right)_{flat} e^{-i(p_1x_1+p_2x_2+p_3x_3)}$$
$$\equiv 8\left[\sqrt{-g}\,E\right]^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3).$$

Renormalized TTT

The renormalization of the transverse-traceless part of TTT ensures also the renormalization of the contributions coming from the two point function.

$$\begin{aligned}\langle t_{loc}^{\mu_1\nu_1}\, T^{\mu_2\nu_2}\, T^{\mu_3\nu_3}\rangle &= \left(\mathcal{I}^{\mu_1\nu_1}_{\alpha_1}(p_1)\, p_{1\beta_1} + \frac{\pi^{\mu_1\nu_1}(p_1)}{(d-1)}\delta_{\alpha_1\beta_1}\right)\langle T^{\alpha_1\beta_1}\, T^{\mu_2\nu_2}\, T^{\mu_3\nu_3}\rangle \\ &= -\frac{2\,\pi^{\mu_1\nu_1}(p_1)}{(d-1)}\Big[\langle T^{\mu_2\nu_2}(p_1+p_2)T^{\mu_3\nu_3}(p_3)\rangle + \langle T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_1+p_3)\rangle\Big] \\ &+ \mathcal{I}^{\mu_1\nu_1}_{\alpha_1}(p_1)\Big\{-p_2^{\alpha_1}\,\langle T^{\mu_2\nu_2}(p_1+p_2)T^{\mu_3\nu_3}(p_3)\rangle - p_3^{\alpha_1}\,\langle T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_1+p_3)\rangle \\ &+ p_{2\beta}\Big[\delta^{\alpha_1\mu_2}\,\langle T^{\beta\mu_2}(p_1+p_2)T^{\mu_3\nu_3}(p_3)\rangle + \delta^{\alpha_1\nu_2}\,\langle T^{\beta\mu_2}(p_1+p_2)T^{\mu_3\nu_3}(p_3)\rangle\Big] \\ &+ p_{3\beta}\Big[\delta^{\alpha_1\mu_3}\,\langle T^{\nu_2\mu_2}(p_2)T^{\beta_3\nu_3}(p_1+p_3)\rangle + \delta^{\alpha_1\nu_3}\,\langle T^{\mu_2\nu_2}(p_2)T^{\mu_3\beta}(p_1+p_3)\rangle\Big]\Big\}\end{aligned}$$

$$\begin{aligned}\langle t_{loc}^{\mu_1\nu_1}\, T^{\mu_2\nu_2}\, T^{\mu_3\nu_3}\rangle_{count} &= -\frac{1}{\varepsilon}\frac{(d-4)}{(d-1)}\pi^{\mu_1\nu_1}(p_1)\Big(4n_I\beta_b[E]^{\mu_2\nu_2\mu_3\nu_3}(p_2,p_3) + 4n_I\beta_a[C^2]^{\mu_2\nu_2\mu_3\nu_3}(p_2,p_3)\Big) \\ &+ \frac{1}{\varepsilon}\frac{2}{(d-1)}\pi^{\mu_1\nu_1}(p_1)\Big(4n_I\beta_a[C^2]^{\mu_2\nu_2\mu_3\nu_3}(p_1+p_2,p_3) + 4n_I\beta_a[C^2]^{\mu_2\nu_2\mu_3\nu_3}(p_2,p_1+p_3)\Big) \\ &- \frac{1}{\varepsilon}\mathcal{I}^{\mu_1\nu_1}_{\alpha_1}(p_1)\Big\{-4p_2^{\alpha_1}\,n_I\beta_a[C^2]^{\mu_2\nu_2\mu_3\nu_3}(p_1+p_2,p_3) - p_3^{\alpha_1}\,n_I\beta_a[C^2]^{\mu_2\nu_2\mu_3\nu_3}(p_2,p_1+p_3) \\ &+ 4p_{2\beta}\Big[\delta^{\alpha_1\mu_2}\,n_I\beta_a[C^2]^{\beta\nu_2\mu_3\nu_3}(p_1+p_2,p_3) + \delta^{\alpha_1\nu_2}\,n_I\beta_a[C^2]^{\mu_2\beta\mu_3\nu_3}(p_1+p_2,p_3)\Big] \\ &+ 4p_{3\beta}\Big[\delta^{\alpha_1\mu_3}\,n_I\beta_a[C^2]^{\mu_2\nu_2\beta\nu_3}(p_2,p_1+p_3) + \delta^{\alpha_1\nu_3}\,n_I\beta_a[C^2]^{\mu_2\nu_2\mu_3\beta}(p_2,p_1+p_3)\Big]\Big\}.\end{aligned}$$

The renormalized TTT is given by

$$\langle T^{\mu_1\nu_1}\, T^{\mu_2\nu_2}\, T^{\mu_3\nu_3}\rangle_{Ren} = \langle t^{\mu_1\nu_1}\, t^{\mu_2\nu_2}\, t^{\mu_3\nu_3}\rangle_{Ren} + \langle T^{\mu_1\nu_1}\, T^{\mu_2\nu_2}\, T^{\mu_3\nu_3}\rangle_{Ren\,l\,t} + \langle T^{\mu_1\nu_1}\, T^{\mu_2\nu_2}\, T^{\mu_3\nu_3}\rangle_{extra}$$

Anomaly poles

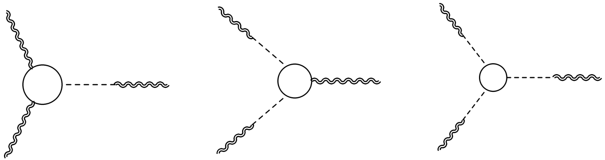
Anomalous Conformal Ward Identities $\Longrightarrow$ Explicit breaking of Conformal Invariance

[Corianò and Maglio, The general 3-graviton vertex $\langle TTT \rangle$ of conformal field theorries in momentum space in $d=4$, Nucl.Phys. B937 (2018) 56-134]

The extra part of the TTT after the renormalization is the anomaly part

$$\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}T^{\mu_3\nu_3}\rangle^{(4)}_{extra} = \left(\frac{\hat{\pi}^{\mu_1\nu_1}(p_1)}{3\,p_1^2}\left(4n_I\beta_b[E]^{\mu_2\nu_2\mu_3\nu_3}(p_2,\bar{p}_3)+4n_I\beta_a[C^2]^{\mu_2\nu_2\mu_3\nu_3}(p_2,\bar{p}_3)\right)+(\text{cycl. perm.})\right)$$
$$-\left(\frac{\hat{\pi}^{\mu_1\nu_1}(p_1)}{3\,p_1^2}\,\frac{\hat{\pi}^{\mu_2\nu_2}(p_2)}{3\,p_2^2}\delta_{\alpha_2\beta_2}\left(4n_I\beta_b[E]^{\alpha_2\beta_2\mu_3\nu_3}(p_2,\bar{p}_3)+4n_I\beta_a[C^2]^{\alpha_2\beta_2\mu_3\nu_3}(p_2,\bar{p}_3)\right)+(\text{cycl. perm.})\right)$$
$$+\frac{\hat{\pi}^{\mu_1\nu_1}(p_1)}{3\,p_1^2}\,\frac{\hat{\pi}^{\mu_2\nu_2}(p_2)}{3\,p_2^2}\,\frac{\hat{\pi}^{\mu_3\nu_3}(\bar{p}_3)}{3\,p_3^2}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}\left(4n_I\beta_b[E]^{\alpha_2\beta_2\alpha_3\beta_3}(p_2,\bar{p}_3)+4n_I\beta_a[C^2]^{\alpha_2\beta_2\alpha_3\beta_3}(p_2,\bar{p}_3)\right).$$

We have anomaly interactions!



Anomaly interactions mediated by the exchange of one, two or three poles.

Consider then the anomaly effective action in its non-local form for the two choices Σ_{FV} and Σ_R. We have calculated their variations to get the anomaly contribution to the $\langle TTT \rangle$

$$\mathcal{S}^{(3)}_{\rm anom}[\Sigma_{FV}] = \mathcal{S}^{(3)}_{\rm anom}[\Sigma_R] = \frac{b'}{9}\int dx \int dx' \int dx'' \Big\{ \left(\partial_\mu R^{(1)}\right)_x \left(\frac{1}{\Box}\right)_{xx'} \left(R^{(1)\mu\nu} - \frac{1}{3}\eta^{\mu\nu} R^{(1)}\right)_{x'} \left(\frac{1}{\Box}\right)_{x'x''} \left(\partial_\nu R^{(1)}\right)_{x''} \Big\}$$
$$- \frac{1}{6}\int dx \int dx' \left(b' E^{(2)} + b\,[C^2]^{(2)}\right)_x \left(\frac{1}{\Box}\right)_{xx'} R^{(1)}_{x'} + \frac{b'}{18}\int dx\, R^{(1)}\left(2R^{(2)} + (\sqrt{-g})^{(1)} R^{(1)}\right)$$

[Corianò, Maglio and Mottola, (2019)]

$$(\mathcal{A})\, \mathcal{S}_3^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1,p_2,p_3) = \frac{8}{3}\Big\{ \pi^{\mu_1\nu_1}(p_1)\left[b' E^{(2)} + b(C^2)^{(2)}\right]^{\mu_2\nu_2\mu_3\nu_3}(p_2,p_3) + (\text{cyclic})\Big\}$$
$$- \frac{16b'}{9}\Big\{ \pi^{\mu_1\nu_1}(p_1)\, Q^{\mu_2\nu_2}(p_1,p_2,p_3)\, \pi^{\mu_3\nu_3}(p_3) + (\text{cyclic})\Big\}$$
$$+ \frac{16b'}{27}\, \pi^{\mu_1\nu_1}(p_1)\, \pi^{\mu_2\nu_2}(p_2)\, \pi^{\mu_3\nu_3}(p_3) \Big\{ p_3^2\, p_1\cdot p_2 + (\text{cyclic})\Big\}$$

Conclusions

- Implications of conformal invariance in momentum space for 3-point function and a particular class of 4-point functions
- Exact Solutions in momentum space for the 4-point function by using the dual conformal invariance
- Matching of the exact solutions with the perturbative ones for the TTT
- Structure of the anomalous part for the TTT

Next:

- How to extend the analysis to higher point functions?
- What are the implications of the dual conformal symmetry?
- What is the anomalous structure of higher point functions?
- Do the two non-local anomaly effective actions in $d = 4$ give the same result at all orders?


That's all Folks!