Phase transitions in spherical models

Random Geometry in Heidelberg – May 16, 2022

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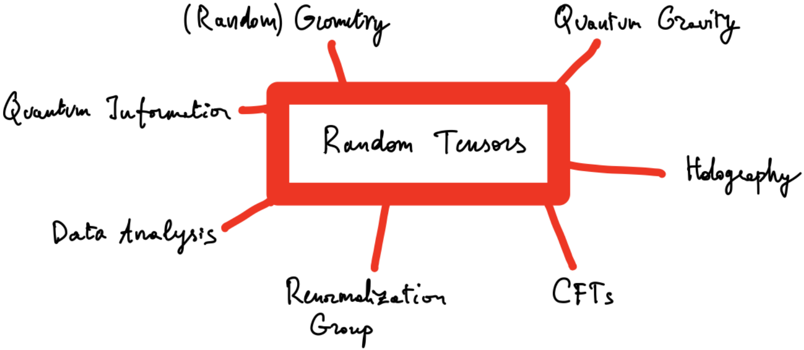
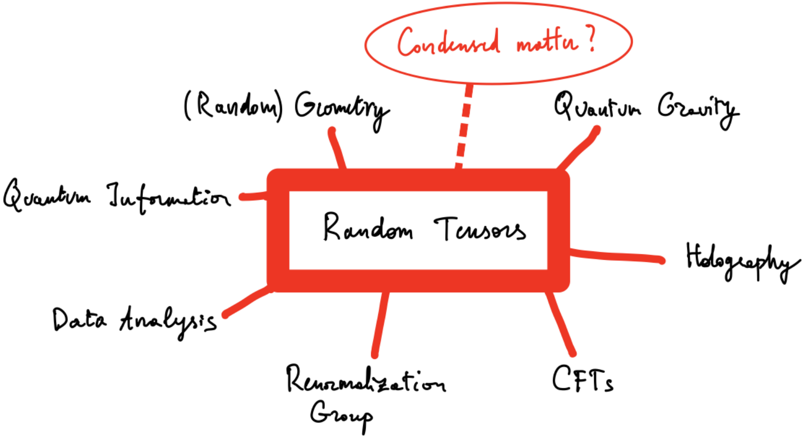
Motivations





Vectors and matrices dispose of a vast collection of interesting phases and transitions. What about tensors?

$$\boldsymbol{s} = (s_1, \ldots, s_N)\,; \quad Z(\beta) = \sum_{\boldsymbol{s}} e^{-\beta H[\boldsymbol{s}]}$$

Ising:

$$H[\boldsymbol{s}] = -\sum_{i \sim j} J s_i s_j - h \sum_i s_i\,; \quad s_i = \pm 1$$

$$d = 1 : \langle s_i s_j \rangle_\beta \leq C \exp(-c(\beta)|i-j|) \quad \text{[Ising 1924]}$$

$$d \geq 2 : \langle s_i s_j \rangle_\beta \geq c(\beta) > 0 \qquad (d = 2 \text{ [Onsager 1944]})$$

Spin glasses:

Edwards-Anderson [1975]:

$$H = \sum_{i \sim j} J_{ij} s_i s_j\,; \quad \overline{J_{ij}^2} \propto J^2$$

Sherrington-Kirkpatrick [SK 1972, Parisi 1979, Talagrand 2006]:

$$H = \sum_{1 \leq i < j \leq N} J_{ij} s_i s_j$$

Main properties: non-ergodicity, ultrametricity, non-selfaveraging.

Vector models: Spherical p-spin

$$H[\boldsymbol{s}] = -\sum_{1\le i_1<\cdots<i_p\le N} J_{i_1\ldots i_p} s_{i_1}\ldots s_{i_p} - h\sum_{i=1}^{N} s_i$$

$$\boldsymbol{s}^2 = s_1^2+\cdots+s_N^2 = N\,; \quad \overline{J^2_{i_1\ldots i_p}} = \frac{J^2 p!}{2N^{p-1}}$$

[Berlin Kac '52]: fixed coupling ($J_{i_1\ldots i_p} = J > 0$), nearest neighbour, large N:

$$d = \begin{cases} 1,2 & \text{disordered phase} \\ 3 & \text{spontaneous magnetization} \end{cases}$$

[Crisanti Sommers '92 (1)]: The annealed average introduces n replicas:

$$\beta F = -\overline{\ln Z} = -\lim_{n\to 0}\frac{\overline{Z^n}-1}{n}$$

$$\overline{Z^n} = \overline{\sum_{\boldsymbol{s}_1\cdots\boldsymbol{s}_n}\exp\left(-\beta\sum_{\alpha=1}^{n} H[\boldsymbol{s}_\alpha]\right)}$$

$$= \sum_{\boldsymbol{s}_1\cdots\boldsymbol{s}_n}\exp\left(\frac{(\beta J)^2 p!}{4N^{p-1}}\sum_{1\le i_1<\cdots<i_p\le N} s_{i_1\alpha}\ldots s_{i_p\alpha}s_{i_1\beta}\ldots s_{i_p\beta} + \beta h\sum_{i\alpha}^{N} s_{i\alpha}\right)$$

Introducing the overlap matrix:

$$q_{\alpha\beta} = \frac{1}{N} \sum_i s_{i\alpha} s_{i\beta}\,; \quad (Q)_{\alpha\beta} = q_{\alpha\beta} \quad \text{(order parameter)}$$

with Lagrange multipliers $(\boldsymbol{\lambda})_{\alpha\beta} = \lambda_{\alpha\beta}$, gives:

$$\overline{Z^n} = \int_{Q>0} \prod_{\alpha<\beta} \mathrm{d}q_{\alpha\beta} \int_{-i\infty}^{i\infty} \mathrm{d}\boldsymbol{\lambda} \exp\left(-NG[Q,\boldsymbol{\lambda}]\right) \,,$$

$$G[Q,\boldsymbol{\lambda}] = -\frac{(\beta J)^2}{4} \sum_{\alpha\beta} q^p_{\alpha\beta} + \frac{1}{2} \sum_{\alpha\beta} \lambda_{\alpha\beta} q_{\alpha\beta}$$
$$- \ln \int \prod_\alpha \mathrm{d}s_\alpha \exp\left(\frac{1}{2} \sum_{\alpha\beta} \lambda_{\alpha\beta} s_\alpha s_\beta + \beta h \sum_\alpha s_\alpha \right) + \mathcal{O}\left(\frac{1}{N}\right) \,.$$

After integration over s (Gaussian) and λ at first orders in $1/N$ and n:

$$\overline{Z^n} = \int_{Q>0} \prod_{\alpha\beta} \sqrt{\frac{N}{2\pi}} \mathrm{d}q_{\alpha\beta} \exp\left[-NG_0(Q) - G_1(Q) + \mathcal{O}\left(\frac{1}{N}\right)\right] ,$$

$$2G_0(Q) = -\frac{\mu}{p} \sum_{\alpha\beta} q_{\alpha\beta}^p - b^2 \sum_{\alpha\beta} q_{\alpha\beta} - \ln\det Q + \frac{b^4}{2} \left(\sum_{\alpha\beta} q_{\alpha\beta}\right)^2 ,$$

$$2G_1(Q) = \frac{\mu}{2}(p-1) \sum_{\alpha\beta} q_{\alpha\beta}^{p-2} \left(1 + 2q_{\alpha\beta}^2 - 2b^4 (\sum_{\gamma\delta} q_{\alpha\gamma} q_{\beta\delta})^2\right) + \ln\det Q ,$$

$$\mu = \frac{(\beta J)^2}{2} p , \quad b = \beta h .$$

Extremize (maximum!) and integrate over Gaussian fluctuations:

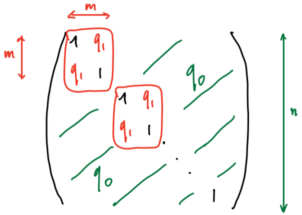
$$\frac{\beta F}{N} = -s(\infty) + \frac{1}{n} G_0(Q_*) + \frac{1}{Nn} \left(G_1(Q_*) + \frac{1}{2} \sum_{\nu} n_\nu \ln \Lambda_\nu\right) + \mathcal{O}\left(\frac{1}{N^2}\right) .$$

We will look for stable solutions ($\Lambda_\nu > 0$).

Replica symmetric ansatz:

$$q_{\alpha\beta} = \begin{cases} 1 & (\alpha = \beta) \\ q & (\alpha \neq \beta) \end{cases}$$

1 step replica-symmetry-broken ansatz ($0 \leq q_0 \leq q_1 \leq 1$):

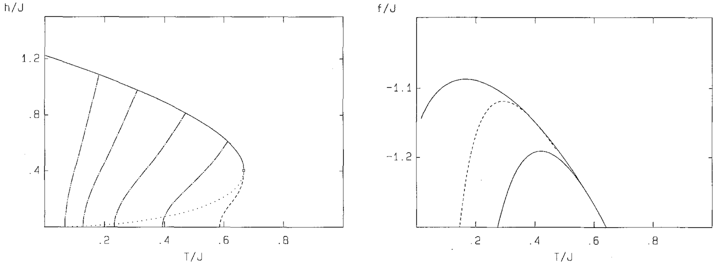


In the SK model, full RSB mechanism was needed [Parisi '79], with a monotone functional:

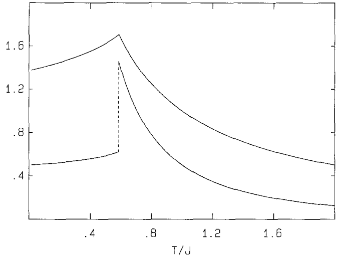
$$q(x) \quad (0 \leq x \leq 1).$$

Other methods: TAP equations, Langevin dynamics, supersymmetry...

Vector models: Spherical p-spin



Phase diagram and free energy (for $h = 0$) for $p = 3$. [Taken from (1)]



Susceptibility (upper), specific heat (lower) for $h = 0$ and $p = 3$. [Taken from (1)]

Matrix models: Spin softening

Can matrix models serve to describe glasses?

[Cugliandolo et al '94, Anninos et al '14, Hartnoll et al '19 (2)]

$$H[S] = \mathrm{tr}\left[V(SS^t)\right] \ , \quad S_{aB} = \pm 1 \quad (1 \leq a \leq N_1; 1 \leq B \leq N_2)$$

Spin softening:

$$Z(\beta) = \sum_{\boldsymbol{S}=\{\pm 1\}^{N_1 N_2}} e^{-\beta H[S]} \overset{N_i \to \infty}{\to} \int dM \delta(\mathrm{tr}\, MM^t - N_1 N_2) e^{-\beta H(M)} \ .$$

Remarks:

- Emergent continuous symmetry at high T.
- Bad approximation for non-singlet correlators at low T.

Indeed:

$$Z(\beta) = \int \mathrm{d}\boldsymbol{G}\mathrm{d}\boldsymbol{\lambda} e^{-\beta \operatorname{tr} V(G)} \int \mathrm{d}\boldsymbol{S}\delta(S_{aB}^2 - 1) e^{-i\lambda_{ab}\left(G_{ab}-(SS^t)_{ab}\right)} \; .$$

Inserting the trivial relation: $\sum_a \mu_a \left(N_2 - (SS^t)_{aa}\right) = 0$,

$$\int \mathrm{d}\boldsymbol{S}\delta(S_{aB}^2 - 1) e^{i\lambda_{ab}(SS^t)_{ab}} = e^{iN_2 \sum_a \mu_a} z(\mu, \boldsymbol{\lambda})^{N_2} \; ,$$

such that, with $\tilde{\lambda}_{ab} = 2i(\lambda_{ab} - \mu_a \delta_{ab})$:

$$\begin{aligned} z(\mu, \boldsymbol{\lambda}) &= \sharp \sum_{s \in \{\pm 1\}^{N_1}} \frac{1}{\sqrt{\det \tilde{\lambda}}} \int \mathrm{d}\boldsymbol{w} e^{-\frac{1}{2} w_a \tilde{\boldsymbol{\lambda}}^{-1} w_b + \sum_a w_a s_a} \\ &= \sharp \frac{1}{\sqrt{\det \tilde{\boldsymbol{\lambda}}}} \int \mathrm{d}\boldsymbol{w} e^{-\frac{1}{2} w_a \tilde{\boldsymbol{\lambda}}^{-1} w_b + \sum_a \ln(2\cosh(w_a))} \\ &\overset{!}{\approx} \sharp \frac{1}{\sqrt{\det\left(1 - \tilde{\boldsymbol{\lambda}}\right)}} \; . \end{aligned}$$

The last line holds when we have chosen the μ_a such that:

$$\left(\frac{1}{1-\tilde{\boldsymbol{\lambda}}}\right)_{aa} = 1 \quad \forall a.$$

In this way, we can obtain a matrix integral:

$$z(\mu, \boldsymbol{\lambda})^{N_2} = \int \mathrm{d}M e^{-\frac{1}{2}M_{aC}(1-\tilde{\boldsymbol{\lambda}}^*)_{ab}M_{bC}} \; .$$

The integral over $\boldsymbol{\lambda}$ gives $\delta(MM^t - G)$, hence the result[1]

$$Z(\beta) \approx \int dM \delta(\operatorname{tr} MM^t - N_1 N_2) e^{-\frac{1}{2}\operatorname{tr} MM^t - \beta \operatorname{tr} V(MM^t)} \; .$$

[1] Numerics helped set $\mu_a = \mu \; \forall a$.

Matrix models: Topological transition

For simplicity, we assume: $N_1 = N_2 = N$.
Diagonalizing M, with eigenvalues $(x_i)_{1 \leq i \leq N}$:

$$Z(\beta) = \int d\mu \prod dx_i \exp N^2 \Big[i\mu \left(1 - \frac{1}{N} \sum_i x_i^2 \right) - \frac{\beta}{N} \sum_i V(x_i) + \frac{1}{2N^2} \sum_{i \neq j} \log \left| x_i^2 - x_j^2 \right| \Big] .$$

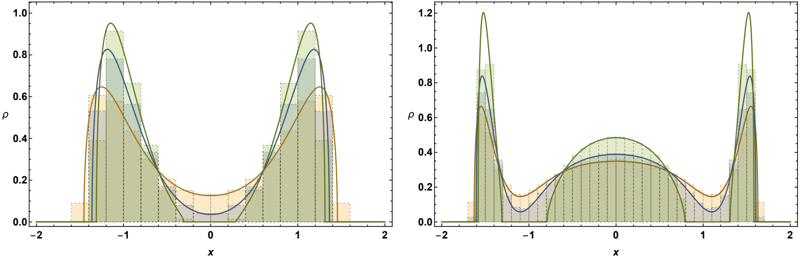
The saddle-point equations are:

$$\frac{1}{N} \sum_i x_i^2 = 1 \,, \quad i\mu x_i + \frac{\beta}{2} V'(x_i) = \frac{1}{N} \sum_{j \neq i} \frac{x_i}{x_i^2 - x_j^2}$$

$$\int dx \rho(x) x^2 = 1 \,, \quad i\mu x + \frac{\beta}{2} V'(x) = P \int dy \frac{\rho(y)}{x - y}$$

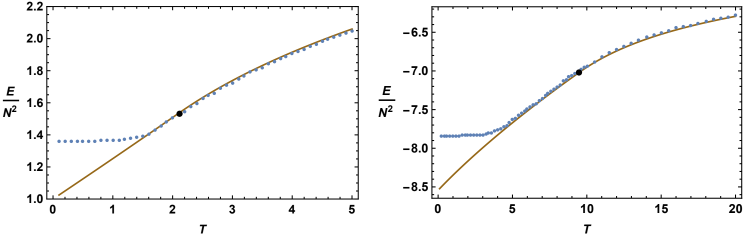
- $\beta \to 0$: Gaussian limit, Wigner semi-circular law ,
- $\beta \to \infty$: V dominates log, $\rho(x) = \sum s_* \delta(x - x_*)$.

Indeed, one- and multi-cut solutions can be obtained exactly for different potentials and match with finite N numerics.



Spectral density for $H = \mathrm{tr}(SS^t)^3$ and $H = -3\,\mathrm{tr}(SS^t)^4 + \mathrm{tr}(SS^t)^5$ [Taken from (2)].

Generically (except for $N = 2^k$), there is a glass transition (not described by the matrix!) and the topological transition happens for various potentials.



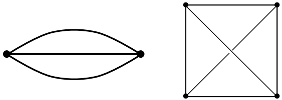
Energy density for $H = \text{tr}(SS^t)^3$ and $H = -3\,\text{tr}(SS^t)^4 + \text{tr}(SS^t)^5$ [Taken from (2)].

NB: If the potential is unbounded below, there is a ferromagnetic ground state (low energy), again not described by the matrix integral (high entropy).

$$T_{abc} \quad (1 \leq a, b, c \leq N) \; ; \quad \sum_{a,b,c=1}^{N} T_{abc}^2 = N^{3/2}$$

Imposing a spherical constraint removes the negative directions of the tetrahedron:

$$S(T, \mu, \lambda) = \frac{\mu}{2} T_{abc} T_{abc} + \frac{\lambda}{4} T_{abc} T_{ade} T_{fbe} T_{fdc}$$



$$Z_{sph}(\lambda) = \int_0^{\infty} \mathrm{d}\mu \int \mathrm{d}T \exp \left[N^3 \mu - N^{3/2} S(T, \mu, \lambda) \right]$$

$$\frac{G_{sph}(\lambda; abc)}{N^{3/2}} = \frac{\int_0^{\infty} \mathrm{d}\mu \int \mathrm{d}T T_{abc} T_{abc} \exp \left[N^3 \mu - N^{3/2} S(T, \mu, \lambda) \right]}{Z_{sph}(\lambda)}$$

Tensor model II

Rescaling $\tilde{T} = \sqrt{\mu} T$, one finds:

$$\frac{\int \mathrm{d}\mu \exp\left[N^3\left(\mu - \left(\frac{1}{2} + \frac{1}{N^3}\right)\ln\mu - F_{CT}(\lambda/\mu^2)\right)\right] G_{CT}(\lambda/\mu^2)}{\int \mathrm{d}\mu \exp\left[N^3\left(\mu - \left(\frac{1}{2} + \frac{1}{N^3}\right)\ln\mu - F_{CT}(\lambda/\mu^2)\right)\right]}$$

with [Carrozza Tanasa '15]:

$$Z_{CT}(\lambda) = \int dT e^{-N^{3/2} S(T,1,\lambda)} = \exp\Big[-N^3 F_{CT}(\lambda)\Big].$$

The large-N saddle-point equation

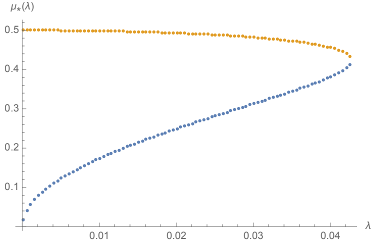
$$1 - 1/(2\mu_*) - \partial_\mu F_{CT}(\lambda/\mu^2)|_{\mu=\mu_*} = \mathcal{O}(1/N)$$

becomes

$$\mu_* - 3/2 + G_{CT}(\lambda/\mu_*^2) = 0$$
$$(\lambda < 0.0448)$$

using Schwinger-Dyson equation:

$$G_{CT}(\lambda) = 1 + \lambda\partial_\lambda F_{CT}(\lambda).$$



Tensor model III

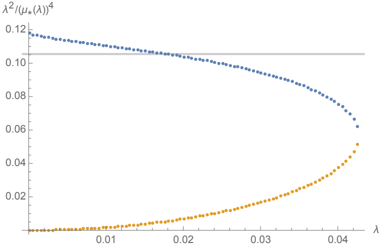
Additionally, at LO (melonic approximation):

$$G_{CT}(\lambda) = \sum_{p \in N} C_p \lambda^{2p}, \quad C_p = \frac{(4p)!}{p!(3p+1)!}, \quad |\lambda|^2 < \lambda_c^2 = \frac{3^3}{4^4} \ ,$$

$$G_{CT}(\lambda) \underset{\lambda \to \lambda_c^-}{\sim} \frac{4}{3} - K\sqrt{1 - \frac{\lambda^2}{\lambda_c^2}} \ .$$

Returning to the spherical model:

$$\frac{G_{sph}(\lambda)}{N^{3/2}} \sim \frac{e^{N^3(\mu_* - 1/2 \ln \mu_* - F_{CT}(\lambda/\mu_*^2))}}{Z_{sph}(\lambda)} G_{CT}(\lambda/\mu_*^2) \, ; \quad 0 \le \frac{\lambda^2}{\mu_*^4} < \frac{3^3}{4^4} \ .$$



Conclusions

What happens after $\lambda = 0.0448$?
Low temperature/high coupling ground state?
Phase transition?
Good order parameter?

Beyond the melonic limit?

- Mean field theory around non-trivial vacuum?
  2PI effective action [Benedetti, Gurau '18] with reduced symmetry [Benedetti, Costa '19]
- Numerics? Monte Carlo [Jha '21], bootstrap equations [Lin '20]

Thank you!