Solution of the $\lambda\Phi^4$-matrix model

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Triviality

- [Landau, Abrikosov, Khalatnikov 54] gave a heuristic argument that QED cannot exist as a renormalised QFT.

  The problem is called Landau ghost, triviality, positivity of β-function.
- This was considered as death of QFT, rescued only by the discovery of asymptotic freedom in non-Abelian Yang-Mills theory by Gross, Politzer, Wilczek in 1973.
- [Aizenman 81] and [Fröhlich 82] gave a rigorous proof of triviality for the $\lambda\phi^4$-model in $D = 4 + \epsilon$ dimensions.
- Triviality of $\lambda\Phi^4$ in dimension $D = 4$ remained an open conjecture until [Aizenman, Duminil-Copin 19].

This project started as attempt to circumvent triviality by constructing the $\lambda\phi^4_4$-model as limit from noncommutative geometry.

QFT on noncommutative geometries

Consider $\lambda\phi^4$ in an harmonic oscillator potential on Moyal space:

$$S(\phi) = \int_{\mathbb{R}^D} \frac{dx}{(8\pi)^{D/2}} \Big(\frac{1}{2}\phi\star(-\Delta+\Omega^2\|2\Theta^{-1}x\|^2+m^2)\phi+\frac{\lambda}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

where $(f \star g)(x) = \int_{\mathbb{R}^{2D}} \frac{dydk}{(2\pi)^D} f(x+\frac{1}{2}\Theta k)g(x+y)e^{\mathrm{i}\langle k,y\rangle}$

- $\exists$ matrix basis $(e_{kl}(x))_{k,l\in\mathbb{N}}$ with $(e_{kl}\star e_{mn})(x) = \delta_{lm}e_{kn}(x)$, $\int dx\; e_{kl}(x) = \sqrt{\det(2\pi\Theta)}\delta_{kl}$
- Expand $\phi(x) = \sum_{k,l=1}^{\infty} \Phi_{kl}e_{kl}(x)$ to get $\int_{\mathbb{R}^D} dx(\phi\star\phi\star\phi\star\phi)(x) = \sqrt{\det(2\pi\Theta)}\mathrm{Tr}(\Phi^4)$
- For $\Omega = 1$ also the kinetic term is matrix product:

$$S(\phi) = \sqrt{\det(\Theta/4)}\;\mathrm{Tr}\Big(E\Phi^2+\frac{\lambda}{4}\Phi^4\Big)$$

  $E = c_0\mathrm{id}+c_1\mathrm{diag}(0,1,1,2,2,2,3,3,3,3,\dots)$ in 4D
- need renormalisation $c_0 \mapsto m^2_{bare}$, $\Phi \mapsto \sqrt{Z}\Phi$

Reformulation as a matrix model

Setting $\sqrt{\det(\Theta/4)} \mapsto N$ and truncating to H_N (Hermitian $N \times N$-matrices) gives formally a measure

$$d\mu_{E,\lambda}(\Phi) = \frac{e^{-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)} d\mu_{E,0}(\Phi)}{\int_{H'_N} e^{-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)} d\mu_{E,0}(\Phi)}$$

on H'_N where $d\mu_{E,0}$ is Gaußian with

$$\int_{H'_N} d\mu_0(\Phi)\, e^{\mathrm{i}\Phi(M)} = \exp\Big(-\frac{1}{2N}\sum_{k,l=1}^{N} \frac{M_{kl}M'_{lk}}{E_k + E_l}\Big)$$

- Observe the analogy to the [Kontsevich 92] model which is a cubic deformation $\frac{1}{\mathcal{Z}} e^{-\frac{\mathrm{i}N}{6}\mathrm{Tr}(\Phi^3)} d\mu_{E,0}(\Phi)$ of the same $d\mu_{E,0}$.
- Correlation functions of the Kontsevich model generate intersection numbers of ψ- and κ-classes on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex curves.

Equations of motion

Fourier transform $\mathcal{Z}(M) := \int_{H'_N} d\mu_{E,\lambda}(\Phi)\, e^{\mathrm{i}\Phi(M)}$ satisfies

1. $$-N(E_p - E_q)\sum_{k=1}^{N} \frac{\partial^2 \mathcal{Z}(M)}{\partial M_{pk}\partial M_{kq}} = \sum_{k=1}^{N}\Big(M_{kp}\frac{\partial \mathcal{Z}(M)}{\partial M_{kq}} - M_{qk}\frac{\partial \mathcal{Z}(M)}{\partial M_{pk}}\Big)$$
2. $$\frac{1}{N}\frac{\partial \mathcal{Z}(M)}{\partial E_p} = \sum_{k=1}^{N}\frac{\partial^2 \mathcal{Z}(M)}{\partial M_{pk}\partial M_{kp}} + \mathcal{Z}(M)\int_{H'_N} d\mu_{E,\lambda}(\Phi)\frac{1}{N}\sum_{k=1}^{N}\Phi_{pk}\Phi_{kp}$$

- They allow to express $\sum_{k=1}^{N}\frac{\mathcal{Z}(M)}{\partial M_{pk}\partial M_{kq}}$ in Dyson-Schwinger equations by fewer derivatives, i.e. of same or lower order.
- (1) essentially due [Disertori, Gurau, Magnen, Rivasseau 06], can be used for $p \neq q$, whereas $p = q$ requires (2).
- This brings the two-point functions and their derivatives with respect to E to our interest.

Two-point functions

- Build for $a \neq b$ the two-point functions

$$G_{|ab|} = N \int_{H'_N} \Phi_{ab}\Phi_{ba} d\mu_{E,\lambda}(\Phi),$$
$$G_{|a|b|} = N^2 \int_{H'_N} \Phi_{aa}\Phi_{bb} d\mu_{E,\lambda}(\Phi)$$

- Fact: matrix models have an asymptotic $1/N$-expansion

$$G_{|ab|} = \sum_{g=0}^{\infty} N^{-2g} G^{(g)}_{|ab|} \quad \text{and} \quad G_{|a|b|} = \sum_{g=0}^{\infty} N^{-2g} G^{(g)}_{|a|b|}$$

where g is the genus. These $G^{(g)}_{|ab|}, G^{(g)}_{|a|b|}$ depend on $E = \mathrm{diag}(E_1, ..., E_N)$ and λ and give rise to

$$\Omega^{(g)}_{q_1,...,q_n} := \frac{(-N)^{n-1}\partial^{n-1}\left(\frac{1}{N}\sum_{k=1}^{N} G^{(g)}_{|kq_1|} + G^{(g-1)}_{|q_1|q_1|}\right)}{\partial E_{q_2}\cdots\partial E_{q_n}} + \frac{\delta_{g,0}\delta_{n,2}}{(E_{q_1}-E_{q_2})^2}$$

Dyson-Schwinger equation for planar 2-point function

Theorem [Grosse, W 09]

Let $e_1, ..., e_d$ be the pairwise different $\{E_k\}$ which arise with multiplicities $r_1, .., r_d$. There is a function $G^{(0)}$ of complex variables which interpolates $G^{(0)}(e_k, e_l) = G^{(0)}_{|kl|}$ and satisfies

$$\Big(\zeta+\eta+\frac{\lambda}{N}\sum_{l=1}^{d} r_l G^{(0)}(\zeta, e_l)\Big) G^{(0)}(\zeta,\eta) = 1+\frac{\lambda}{N}\sum_{k=1}^{d} r_k \frac{G^{(0)}(e_k,\eta)-G^{(0)}(\zeta,\eta)}{e_k-\zeta}$$

One can arrange an integral equation for $N \to \infty$. Depending on dimension, renormalisation constants are necessary:

$$\Big(\zeta+\eta+\mu^2_{bare}+\lambda\int_0^\infty dt\ \varrho_0(t)\, ZG^{(0)}(\zeta,t)\Big) ZG^{(0)}(\zeta,\eta)$$
$$= 1+\lambda\int_0^\infty dt\ \varrho_0(t)\,\frac{ZG^{(0)}(t,\eta)-ZG^{(0)}(\zeta,\eta)}{t-\zeta}$$

Solution

Theorem [Panzer-W 18 for $\varrho_0 = 1$, Grosse-Hock-W 19a]

1. Ansatz $G^{(0)}(\zeta,\eta) = \dfrac{e^{\mathcal{H}_\zeta[\tau_\eta(\bullet)]}\sin\tau_\eta(\zeta)}{Z\lambda\pi\varrho_0(\zeta)}$ Z=renormalisation, $\mathcal{H}_\zeta[f] = \frac{1}{\pi}\fint \frac{dp\, f(p)}{p-\zeta}$
2. $\tau_\eta(\zeta) = \mathrm{Im}\log\big(\eta + I(\zeta + \mathrm{i}\epsilon)\big)$ with $I(\zeta) = -R(-\mu^2 - R^{-1}(\zeta))$
3. $$R(z) = z - \lambda(-z)^{D/2}\int_0^\infty \frac{dt\ \varrho_\lambda(t)}{(\mu^2+t)^{D/2}(t+\mu^2+z)}$$
   $D = 2[\frac{\delta}{2}]$ at spectral dimension $\delta = \inf\Big(p :\ \int \frac{dt\,\varrho_0(t)}{(1+t)^{p/2}} < \infty\Big)$
4. ϱ_λ is implicit solution of $\varrho_0(R(\zeta)) = \varrho_\lambda(\zeta)$.

- Proof: [Cauchy 1831] residue theorem, [Lagrange 1770] inversion theorem, [Bürmann 1799] formula
- $\varrho_0(t) \equiv 1$ (2D Moyal) in terms of Lambert-W, $W(z)e^{W(z)} = z$:
  $I(\zeta) := \lambda W_0\big(\frac{1}{\lambda}e^{\frac{1+\zeta}{\lambda}}\big) - \lambda\log\big(1 - \lambda W_0\big(\frac{1}{\lambda}e^{\frac{1+\zeta}{\lambda}}\big)\big)$

$D=4$ Moyal space: $\varrho_0(t)=t$ [Grosse-Hock-W 19b]

- $\varrho_\lambda(x) \equiv \varrho_0(R(x)) = R(x) = x - \lambda x^2 \int_0^\infty \frac{dt\ \varrho_\lambda(t)}{(\mu^2+t)^2(t+x)}$
- If $\varrho_\lambda(t) \sim \varrho_0(t) = t$, then $R(x)$ bounded above. Consequently, R^{-1} would not be globally defined: triviality!
- Fredholm equation perturbatively solved by iterated integrals: Hyperlogarithms and $\zeta(2n)$ which can be summed to

$$\varrho_\lambda(x) = x \cdot {}_2F_1\left(\begin{matrix}\alpha_\lambda, 1-\alpha_\lambda \\ 2\end{matrix}\,\middle|\, -\frac{x}{\mu^2}\right)$$

$$\alpha_\lambda = \begin{cases} \frac{\arcsin(\lambda\pi)}{\pi} & \text{for } |\lambda| \leq \frac{1}{\pi} \\ \frac{1}{2} + \mathrm{i}\frac{\operatorname{arcosh}(\lambda\pi)}{\pi} & \text{for } \lambda \geq \frac{1}{\pi} \end{cases}$$

Corollary

The interaction alters the spectral dimension to $4 - 2\frac{\arcsin(\lambda\pi)}{\pi}$ and thus avoids the triviality problem.

Gives non-perturbative integral representation for $G^{(0)}(\xi,\eta)$.

All planar correlation functions

- A Catalan tuple is a tuple $\tilde{p} = (p_0, p_1, ..., p_k)$ with $p_i \geq 0$, $\sum_{j=0}^{l} p_j > l$ if $l < k$ and $\sum_{j=0}^{k} p_j = k$. We let $|\tilde{p}| = k$.
- We call a collection $\mathcal{T} = \langle \tilde{p}_0, \tilde{p}_1, ..., \tilde{p}_{n+1} \rangle$ of Catalan tuples a nested Catalan table (of length n) if its length tuple $(|\tilde{p}_0| + 1, |\tilde{p}_1|, ..., |\tilde{p}_{n+1}|)$ is itself a Catalan tuple.
- There are $\frac{1}{n+1}\binom{3n+1}{n}$ different Catalan tables of length n. Example $n = 1$: $\langle (1,0),(0),(0)\rangle$ and $\langle (0),(1,0),(0)\rangle$

Theorem [de Jong, Hock W 19]

The planar n-point function is a sum of terms of the form $\frac{\pm G^{(0)}_{b_p b_q} \cdots G^{(0)}_{b_r b_s}}{(E_{b_t} - E_{b_u}) \cdots (E_{b_v} - E_{b_w})}$ which are in bijection with nested Catalan tables of length $\frac{n-2}{2}$. The numerator is encoded in the length tuple, the denominator by two types of plane trees associated with the Catalan tuples it is made of.

The planar 2-point function for finite N

For finite N we make contact with complex algebraic geometry:

Theorem ([Schürmann-W 19], inspired by [Hock-Grosse-W 19])

Let $(\varepsilon_k, \varrho_k)$ be implicitly defined by $e_k = R(\varepsilon_k),\; r_k = R'(\varepsilon_k)\varrho_k$

for $R(z) = z - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{\varrho_k}{z+\varepsilon_k}$.

Then $G^{(0)}(\zeta,\eta) = \mathcal{G}^{(0)}(z,w)$ for $R(z)=\zeta$, $R(w)=\eta$ and

$$\boxed{\mathcal{G}^{(0)}(z,w) = \frac{S(R(z),R(w))}{(R(z)-R(-w))(R(w)-R(-z))}}$$

where

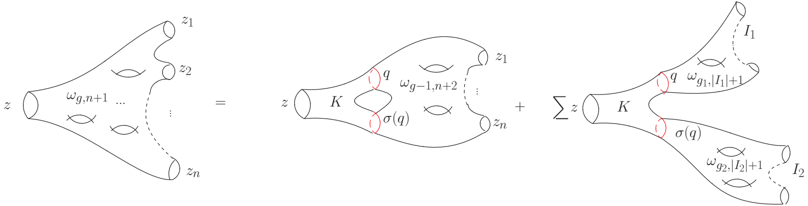
$$S(R(z),R(w)) = \frac{\prod_{j=0}^{d}(R(z)-R(-\hat{w}^j))}{\prod_{k=1}^{d}(R(z)-R(\varepsilon_k))}$$

is a symmetric rational function of $R(z), R(w)$. Here $u \in \{w=\hat{w}^0, \hat{w}^1, \dots, \hat{w}^d\}$ are all solutions of $R(u)=R(w)$.

Topological recursion [Eynard-Orantin 07]

- The exact solution by complex geometry suggests a link to topological recursion (TR).
- TR recursively constructs, starting from a spectral curve consisting of
  - a ramified covering $x : \Sigma \to \Sigma_0$ of Riemann surfaces,
  - meromorphic differentials $\omega_{0,1}=ydx$ on Σ and $\omega_{0,2}$ on $\Sigma\times\Sigma$,

  a family $\omega_{g,n}$ of meromorphic differentials on Σ^n, with poles at zeros of dx (ramification points), schematically by



Surprisingly many examples in mathematical physics, algebraic and enumerative geometry follow this TR-construction scheme.

The $\Omega_n^{(g)}$

In [Branahl, Hock, W 20] we derived DS-equations for complexifications $\Omega^{(g)}_{q_1,\dots,q_n} = \Omega^{(g)}_n(\varepsilon_{q_1},\dots,\varepsilon_{q_n})$ coupled to other functions. The Ω's are converted to meromorphic differentials $\omega_{g,n}(z_1,\dots,z_n) = \lambda^{2-2g-n}\Omega^{(g)}_n(z_1,\dots,z_m)\prod_{k=1}^n dR(z_k)$. We find

$$\omega_{0,2}(u,z) = \frac{du\,dz}{(u-z)^2} + \frac{du\,dz}{(u+z)^2}\,,$$

$$\omega_{0,3}(u_1,u_2,z) = -\sum_{i=1}^{2d} \frac{\Big(\frac{1}{(u_1-\beta_i)^2}+\frac{1}{(u_1+\beta_i)^2}\Big)\Big(\frac{1}{(u_2-\beta_i)^2}+\frac{1}{(u_2+\beta_i)^2}\Big)du_1\,du_2\,dz}{R'(-\beta_i)R''(\beta_i)(z-\beta_i)^2}$$
$$+\Big[d_{u_1}\Big(\frac{\omega_{0,2}(u_2,u_1)}{(dR)(u_1)}\frac{dz}{R'(-u_1)(z+u_1)^2}\Big) + u_1 \leftrightarrow u_2\Big]$$

with $\beta_{1,\dots,2d}$ the ramification points (solve $dR(\beta_i)=0$).

Observation

The blue terms correspond to topological recursion for $x(z)=R(z)$ and $y(z)=-R(-z)$, the red terms signal an extension to blobbed topological recursion [Borot-Shadrin 15].

Contact with topological recursion

The pattern continues:

$$\omega_{1,1}(z) = \sum_{i=1}^{2d} \frac{dz}{R'(-\beta_i)R''(\beta_i)} \Big\{ \frac{1}{8(z-\beta_i)^4} + \frac{\frac{1}{24}x_{1,i}}{(z-\beta_i)^3} + \frac{(x_{2,i}+y_{2,i}-x_{1,i}y_{1,i}-x_{1,i}^2-\frac{6}{\beta_i^2})}{48(z-\beta_i)^2} \Big\} - \frac{dz}{8(R'(0))^2 z^3} + \frac{R''(0)dz}{16(R'(0))^3 z^2}$$

where $x_{n,i} := \frac{R^{(n+2)}(\beta_i)}{R''(\beta_i)}$, $y_{n,i} := \frac{(-1)^n R^{(n+1)}(-\beta_i)}{R'(-\beta_i)}$.

Also found $\omega_{0,4}, \omega_{0,5}$ and then all $\omega_{0,n}$ described on next page. Recently extended to $\omega_{1,2}$.

Recursion of the genus-0 sector

Theorem [Hock-W 21]

$$\omega_{0,m+1}(u_1,\dots,u_m,z)$$
$$= \sum_{i=1}^{2d} \operatorname*{Res}_{q\to\beta_i} K_i(z,q) \sum_{I_1\uplus I_2=\{u_1,\dots,u_m\}} \omega_{0,|I_1|+1}(I_1,q)\omega_{0,|I_2|+1}(I_2,\sigma_i(q))$$
$$-\sum_{k=1}^{m} d_{u_k}\Big[\operatorname*{Res}_{q\to -u_k} \sum_{I_1\uplus I_2=\{u_1,\dots,u_m\}} \tilde{K}(z,q,u_k)d_{u_k}^{-1}\big(\omega_{0,|I_1|+1}(I_1,q)\omega_{0,|I_2|+1}(I_2,q)\big)\Big]$$

(for $m\geq 2$). Here σ_i is the local Galois involution near β_i, i.e. $R(z)=R(\sigma_i(z))$, $\sigma_i(\beta_i)=\beta_i$, $\sigma_i\neq \mathrm{id}$. The recursion kernels are

$$K_i(z,q) := \frac{\frac{1}{2}\Big(\frac{dz}{z-q}-\frac{dz}{z-\sigma_i(q)}\Big)}{dR(\sigma_i(q))(R(-\sigma_i(q))-R(-q))},$$
$$\tilde{K}(z,q,u) := \frac{\frac{1}{2}\Big(\frac{dz}{z-q}-\frac{dz}{z+u}\Big)}{dR(q)(R(u)-R(-q))}.$$

Work in progress: higher g (kernel K_0 for pole at $z=0$ is found)

Linear and quadratic loop equations

The best strategy seems to prove that the Dyson-Schwinger equations imply linear and quadratic loop equations

$$\sum_{j=0}^{d} \omega_{g,|I|+1}(I,\hat{z}^j) = f_1(R(z);I)$$

$$\sum_{\substack{j,k=0\\ j\neq k}}^{d} \Big(\omega_{g-1,|I|+2}(I,\hat{z}^j,\hat{z}^k) + \sum_{\substack{g_1+g_2=g\\ I_1\uplus I_2=I}} \omega_{g_1,|I_1|+1}(I_1,\hat{z}^j)\omega_{g_2,I_2|+1}(I_2,\hat{z}^k)\Big)$$

$$= f_2(R(z);I)$$

where f_1, f_2 are *holomorphic* at ramification points of R.

- Blobbed TR is the general solution of such loop equations [Borot, Shadrin 15]. To reduce to TR one needs more.
- In our case f_1, f_2 are of particular structure which completely fixes the poles at $z=-u_k$ and $z=0$.

Outlook I: Intersection numbers

Eynard proved in 2011 a general formula which expresses $\omega_{g,n}$ for *any* spectral curve $(x : \Sigma \to \Sigma_0, \omega_{0,1}, \omega_{0,2})$ with simple ramification points in terms of intersection numbers of ψ- and κ-classes on several copies of $\overline{\mathcal{M}}_{g,n}$.

- Extensions to higher-order ramifications are known.
- According to [Borot, Shadrin 15], Eynard's formula survives with some modifications to blobbed TR.

Thus, expressing our $\omega_{g,n}$ in terms of intersection numbers is an achievable goal in very near future.

- This expression can be interesting (like the ELSV formula) or not.
- We hope it captures aspects of the involution $z \mapsto -z$ which plays a decisive rôle in the residue formula.

Outlook II: Integrability

Consider $\tau(\{t_i\}) = N^2\mathcal{F}^{(0)} + \mathcal{F}^{(1)} + \sum_{g=2}^{\infty} N^{2-2g}\omega_{g,0}$

- $\mathcal{F}^{(0)}$ and $\mathcal{F}^{(1)}$ have been found in [Branahl, Hock 21].
- In the [Kontsevich 92] model, for $t_i = -\frac{(2i-1)!!}{N}\mathrm{Tr}(E^{-2i-1})$, τ satisfies the Hirota bilinear PDE of the KdV-hierarchy.
- [Eynard, Orantin 07] describe how to obtain from any spectral curve of TR a formal Hirota equation (order by order in $1/N^2$).
- Whether or not integrability extends to blobbed TR is not known.

We remain optimistic for our case because the recursion formula with residue kernel is very close to TR.