# Trisections in colored tensor models

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# Outline


1. Motivations and Background
2. Heegaard Splittings and Trisections in SMOOTH category
3. Heegaard Splittings and Trisections in PL category

# Random geometrical path integral formulations

Partition function of quantum gravity may be given by

$$Z_{\text{gravity}} \sim \sum_{\text{top}} \int \mathcal{D}\mathbf{g}\; e^{-S_{\text{gravity}}} \quad \to \quad \sum_{\text{random triangulations}} e^{-S_{\text{discretised gravity}}}$$

$d = 2$: Matrix models

- Large $N$ expansion of partition function is controlled by a topological invariant (genus $g$),

$$Z = \sum_{g} N^{2-2g} Z_g , \quad \text{where } Z_g \sim |\lambda - \lambda_c|^{(2-\gamma)\chi(g)/2} f_g \, .$$

- A clear relation between topology and the critical behavior in the continuum limit ($N \to \infty$, $\lambda \to \lambda_c$ while $\kappa$ being constant),

$$Z = \sum_{g} \kappa^{2g-2} f_g , \quad \text{where } \kappa^{-1} = N|\lambda - \lambda_c|^{(2-\gamma)/2} \, .$$

# Random geometrical path integral formulations

Partition function of quantum gravity may be given by

$$Z_{\text{gravity}} \sim \sum_{\text{top}} \int \mathcal{D}\mathbf{g}\; e^{-S_{\text{gravity}}} \quad \to \quad \sum_{\text{randomtriangulations}} e^{-S_{\text{discretised gravity}}}$$

$d > 2$: Tensor models

- Large $N$ limit is controlled by Gurau degree $\omega_{\mathcal{G}}$, having strong influence from combinatorics and encoding geometrical information via the number of bicolor cycles (faces), $F_{\mathcal{G}}$,

$$Z \sim \sum_{\mathcal{G}} N^{d-\frac{\mathbf{2}}{(d-\mathbf{1})!}\omega_{\mathcal{G}}}, \quad \text{where } \omega_{\mathcal{G}} = \frac{(d-1)!}{2}\left(\frac{d(d-1)}{2}p + d - F_{\mathcal{G}}\right) = \sum_{\text{jackets},\mathcal{J}} g_{\mathcal{J}} \geq 0\,.$$

- A systematic analysis of subleading contributions ($\omega_{\mathcal{G}} > 0$) turns out to be hard. (how to go beyond melons ?)

**Q.** Can we do something different if we learn new topological constructions?

# Colored tensor models

Consider combinatorially nonlocal 0-dimensional field theories of size $N$ random tensors with $d$-indices $T : \mathbb{Z}_N^d \to \mathbb{C}$, where the $d = 4$ action for $(d+1) = (4+1)$-colored tensors with $(d = 4)$-indices is given by

$$S[T, \bar{T}] = N^{4/2}\Big( \sum_{c=0}^{4} \sum_{a_i \in \mathbb{Z}_N} T^{c}_{a_1 a_2 a_3 a_4} \bar{T}^{c}_{a_1 a_2 a_3 a_4} + \lambda \sum_{a_{ij} \in \mathbb{Z}_N} T^{0}_{a_{01} a_{02} a_{03} a_{04}} T^{1}_{a_{10} a_{14} a_{13} a_{12}} T^{2}_{a_{21} a_{20} a_{24} a_{23}} T^{3}_{a_{32} a_{31} a_{30} a_{34}} T^{4}_{a_{43} a_{42} a_{41} a_{40}} + \bar{\lambda} \sum_{a_{ij} \in \mathbb{Z}_N} \bar{T}^{0}_{a_{01} a_{02} a_{03} a_{04}} \bar{T}^{1}_{a_{10} a_{14} a_{13} a_{12}} \bar{T}^{2}_{a_{21} a_{20} a_{24} a_{23}} \bar{T}^{3}_{a_{32} a_{31} a_{30} a_{34}} \bar{T}^{4}_{a_{43} a_{42} a_{41} a_{40}} \Big),$$

with $a_{ij} = a_{ji}$ and the probability measure is given by

$$d\mu = \int \mathcal{N} \prod_{c, a_i} dT^{i}{}_{a_1 a_2 a_3 a_4}\, d\bar{T}^{i}{}_{a_1 a_2 a_3 a_4}\; e^{-S[T, \bar{T}]} . \tag{1}$$

The tensors transform under the symmetry of $U(N)^4$:

$$T_{a_1 a_2 a_3 a_4} \to T'_{a_1 a_2 a_3 a_4} = T_{lmnp} U^{l}_{a_1} U^{m}_{a_2} U^{n}_{a_3} U^{p}_{a_4}$$
$$\bar{T}_{a_1 a_2 a_3 a_4} \to \bar{T}'_{a_1 a_2 a_3 a_4} = \bar{T}_{lmnp} (U^{\dagger})^{l}_{a_1} (U^{\dagger})^{m}_{a_2} (U^{\dagger})^{n}_{a_3} (U^{\dagger})^{p}_{a_4} .$$

$(d+1)$-colored graphs (also, called graph encoding manifolds (GEM) ) are dual to simplicial triangulations of piecewise linear (PL) $d$-dimensional pseudo-manifolds [Bandieri, Gagliardi 1982; Ferri, Gagliardi, Grasselli 1986].

Illustration below of $d$-simplices in $d = 2$, 3, 4 dimensions, where we embedded $d + 1$-colored graphs.

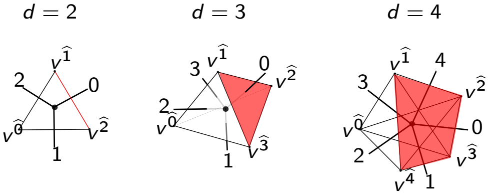


Figure: Colored representation.

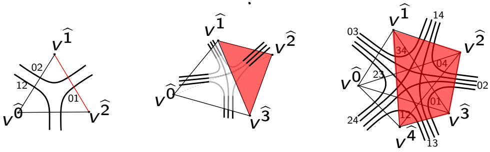


Figure: Stranded representation.

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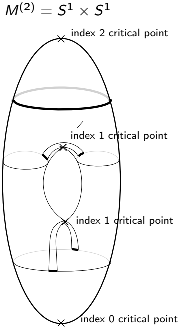
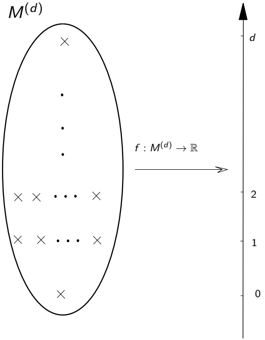
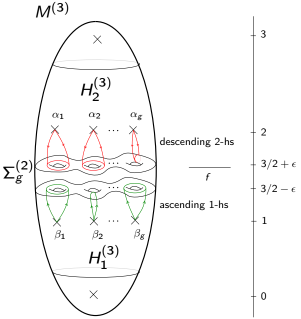
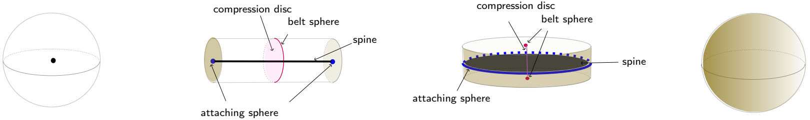
# Morse theory pictures



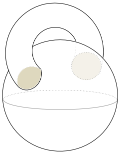
- An $i$-**handle** is a thickening of $\mathrm{D}^i$, and is therefore $\mathrm{D}^i \times \mathrm{D}^{d-i}$ glued along $\mathrm{S}^{i-1} \times \mathrm{D}^{d-i}$. e.g., in 3-dimensions,



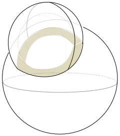

0-handle ($\mathrm{D}^3$) | 1-handle ($\mathrm{D}^1 \times \mathrm{D}^2$ glued along $\mathrm{S}^0 \times \mathrm{D}^2$) | 2-handle ($\mathrm{D}^2 \times \mathrm{D}^1$ glued along $\mathrm{S}^1 \times \mathrm{D}^1$) | 3-handle ($\mathrm{D}^3$ glued along $\mathrm{S}^2$)

Figure: Compression discs and spines are transversal. Belt sphere for 1-handle to become an attaching curve, and attaching sphere for 2-handle to become an attaching curve.

to obtain, e.g.,



0-h ∪ 1-h

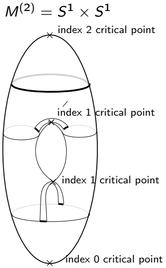


0-h ∪ 2-h

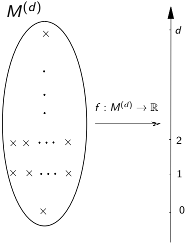
etc.

# Handle decomposition (in SMOOTH category)

- In Morse theory, passing through an index $i$ critical point means attaching an $i$-handle, i.e., topology changes when passing an $i$-handle.




- Morse theory picture naturally provides us with the **handle decomposition** of a closed $d$-manifold: $M^{(d)} =$ 0-handle $\cup$ $g$ 1-handles $\cup$ $h$ 2-handles $\cup \cdots \cup$ $d$-handle.



# Heegaard splittings in 3-dimensions (in SMOOTH category)

- A special case is when $H^{(d)} = $ 0-handle $\cup\, g$ 1-handles (therefore with a boundary), then $H^{(d)}$ is called 1-**handlebody** with its genus defined as $g$. In a 1-handlebody, there is a **1-skeletal spine** (i.e., a graph, or equivalently a set of nodes and lines) where the manifold collapses onto.

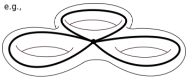
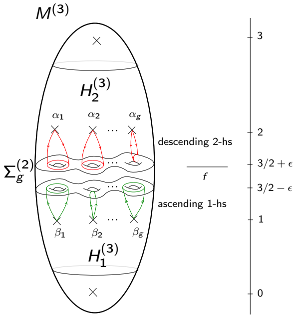


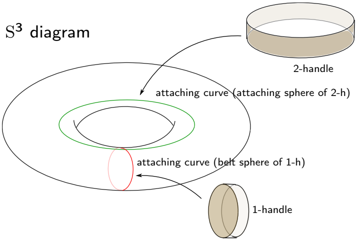

Figure: $g = 3$ 1-handlebody (0-handle $\cup$ three 1-handles).

- Definition. A **Heegaard splitting** of a compact connected oriented 3-manifold $M^{(3)}$ is the triple $(\Sigma^{(2)}, H_1^{(3)}, H_2^{(3)})$, where $\Sigma = \partial H_1 = \partial H_2 = H_1 \cap H_2$, called a Heegaard surface, is a compact connected closed oriented 2-dimensional surface, while $H_1^{(3)}$ and $H_2^{(3)}$ are 1-**handlebodies** whose union is $M^{(3)} = H_1 \cup H_2$. $g(\Sigma) = g(H_1) = g(H_2)$. **min($g$) is a topological invariant**.



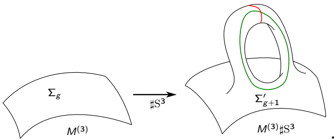
## Heegaard splittings in 3-dimensions (in SMOOTH category)

- One can represent $M^{(3)}$ with a **Heegaard diagram** which consists of a Heegaard surface and $\alpha$ and $\beta$ attaching curves, e.g.,

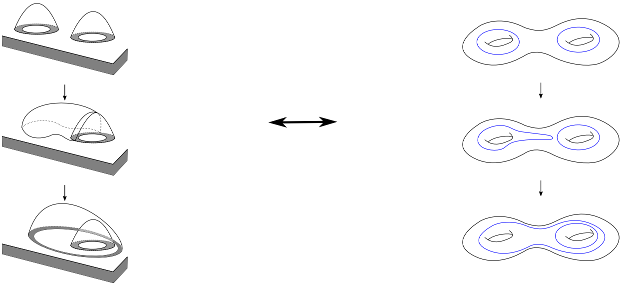



# Heegaard splittings in 3-dimensions (in SMOOTH category)

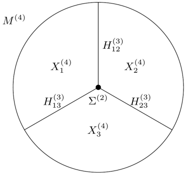
- Two diagrams for a given $M^{(3)}$ can be transformed into each other by a finite sequence of moves of stabilisations/handle-cancelations and handle slides. In particular, **stabilisations** can be visualised as




- handle slides. There is a freedom to where one glues handles on a given Heegaard surface. The corresponding Heegaard diagrams are equivalent.



# Trisections in 4-dimensions (in SMOOTH category)




A **trisection** is given by [Gay and Kirby 2012]

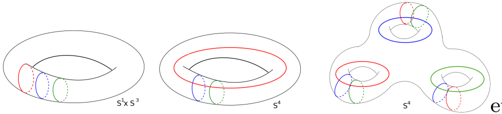
- $M^{(4)} = X_1^{(4)} \cup X_2^{(4)} \cup X_3^{(4)}$
- $H_{ij}^{(3)} = X_i^{(4)} \cap X_j^{(4)}$ and $\partial X_i^{(4)} = H_{ij}^{(3)} \cup H_{ik}^{(3)}$.
- $\Sigma^{(2)} = X_1^{(4)} \cap X_2^{(4)} \cap X_3^{(4)}$ is a closed surface.
- All $X_i^{(4)}$ and $H_{ij}^{(3)}$ are 1-handlebodies.

Some remarks follow:

- $(H_{ij}^{(3)}, H_{jk}^{(3)}, \Sigma^{(2)})$ forms a Heegaard splitting.
- $g(H_{12}^{(3)}) = g(H_{13}^{(3)}) = g(H_{23}^{(3)}) = g(\Sigma^{(2)})$.
- $\min g(\Sigma^{(2)})$, **trisection genus, is an invariant**.

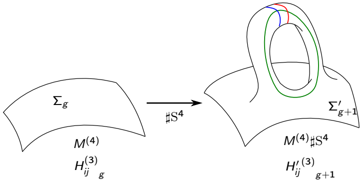
# Trisections in 4-dimensions (in SMOOTH category)

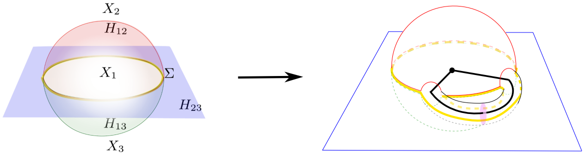
- One can represent $M^{(4)}$ in a **trisection diagram** with a 2-dim central surface and $\alpha$, $\beta$, $\gamma$-attaching curves, e.g.,



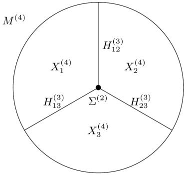

etc.

- Again, one can perform a **stabilisation**, which will not alter the overall 4-manifold nor spoil the trisection.



# Trisections in 4-dimensions (in SMOOTH category)




*Extending theorem (Montesinos).*
Given a 4-dimensional 1-handlebody $X$ of genus $g$ (so, $g$-number of 1-handles and a 0-handle) and a homeomorphism $\phi : \partial X \to \partial X$, there exists a unique homeomorphism $\Phi : X \to X$ extending $\phi$.

Trisections, then, allow us to fully determine $M^{(4)}$ by the three 3-dimensional 1-handlebodies $H_{ij}^{(3)}$ which in turn, can be represented by means of Heegaard diagrams.

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# Jackets as Heegaard surfaces (in PL category)

- Given a colored graph $\mathcal{G}$, *Gurau degree* $\omega(\mathcal{G})$ is given as the sum of genera of all *jackets* of $\mathcal{G}$, $\omega(\mathcal{G}) = \sum_{\mathcal{J}} g_{\mathcal{J}}$, where a jacket $\mathcal{J}_\eta$ is an embedded 2-subcomplex of a colored graph $\mathcal{G}$, labeled by a permutation $\eta$ of the set $\{0, \dots, d\}$ such that $\mathcal{J}_\eta$ and $\mathcal{G}$ have the same node and line sets and the bicolored cycle (face) set of $\mathcal{J}_\eta$ is a subset of that of $\mathcal{G}$.

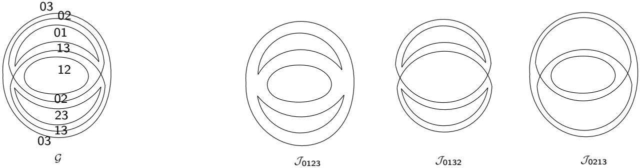
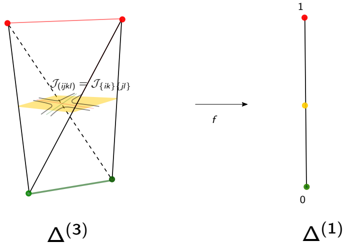


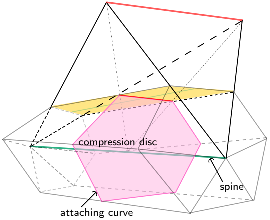

Figure: The elementary melon graph for tensor models with 3-indices, and its three jackets, in stranded representation

Regular embeddings (= jackets) of the graph define surfaces in the simplicial complex, and define matrix models within tensor models.

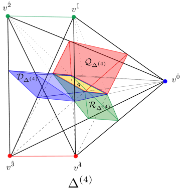
# Jackets as Heegaard surfaces (in PL category)

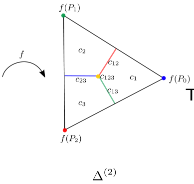
Splitting of a 3-simplex and a visualisation of how the 3-dim 1-handlebody can be formed [Ryan 2012]:



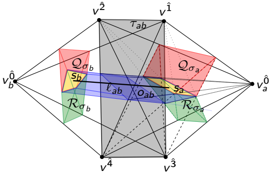



**1.** Start with mapping a 4-simplex $\Delta^{(4)}$ onto a triangle by partitioning the sets of vertices in three sets $P_0 = \{v^{\hat{0}}\}$, $P_1 = \{v^{\hat{1}}, v^{\hat{2}}\}$ and $P_2 = \{v^{\hat{3}}, v^{\hat{4}}\}$ [Bell, Hass, Rubinstein, Tillmann, 2017; Casali, Cristofori, 2019]




Take two 4-simplices $\sigma_a$ and $\sigma_b$.




Notice

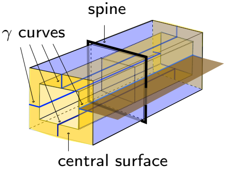
- the nested Heegaard splitting-like structures inside a trisection-like structure already present in a single 4-simplex; see the triples $(s, \mathcal{Q}_{\Delta^{(4)}}, \mathcal{R}_{\Delta^{(4)}})$, $(s, \mathcal{Q}_{\Delta^{(4)}}, \mathcal{D}_{\Delta^{(4)}})$, and $(s, \mathcal{R}_{\Delta^{(4)}}, \mathcal{D}_{\Delta^{(4)}})$.
- $s = \Delta^{(4)} \cap K(\mathcal{J}(\mathcal{B}^{\hat{0}}))$, where $K(\mathcal{J}(\mathcal{B}^{\hat{0}}))$ is the realisation of the **jacket of** $\hat{0}$**-bubbles**.

The problem is that each of the 3-dimensional pieces which are to be $H_{ij}^{(3)}$s in a trisection is not connected once we look at the induced structure on a whole colored tensor model graph.

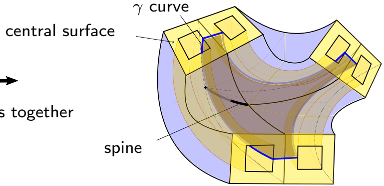
# Constructing trisections in colored tensor model graphs (in PL cat.)

**2.** So, carve out a neighborhood of embedded 0-color lines of the colored graph [Martini, Toriumi 2021].

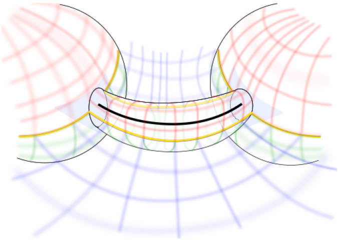
For example, let us take $\mathcal{D}_{\sigma_a}$ and $\mathcal{D}_{\sigma_b}$, from where we carved out such a neighborhood.




e.g., three such objects together




Via the carving operation above, we connected the isolated duals of 4-bubbles, also ensuring the spines of 3-dimensional and 4-dimensional pieces are 1-skeletons.



# Constructing trisections in colored tensor model graphs (in PL cat.)

Now let us analyse the genus of the central surface of the trisection constructed above.

Obtain a graph $\tilde{\mathcal{G}}$ derived from a colored graph $\mathcal{G}$, by contracting all the $\hat{0}-$bubbles to points which will then become the nodes of $\tilde{\mathcal{G}}$, whose number is $|\mathcal{V}^{\hat{0}}|$.

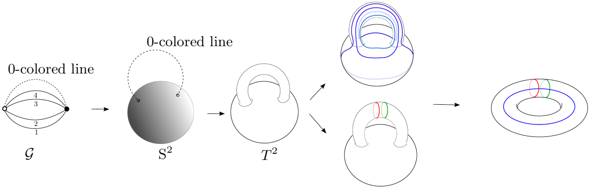
The **genus of the central surface** is given by

$$g_c = \sum_{a=1}^{|\mathcal{V}^{\hat{\mathbf{0}}}|} g_{\mathcal{J}(\mathcal{B}_a^{\hat{\mathbf{0}}})} + L_0 \qquad \Rightarrow \qquad \sum_{c=1}^{15} g_c = \omega(\mathcal{G}) + 3(4p+1)\,,$$

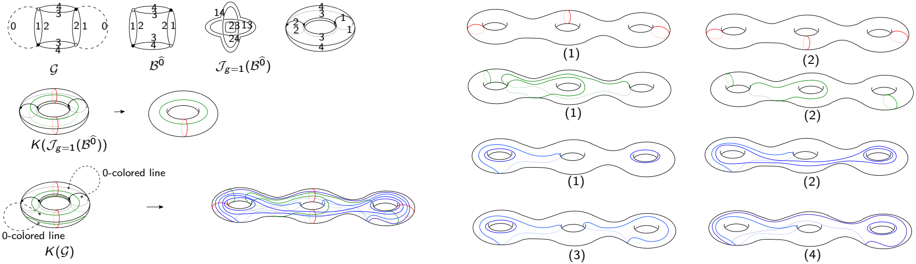
where $L_0$ the number of independent loops of $\tilde{\mathcal{G}}$, $2p$ is the number of nodes of the original colored graph $\mathcal{G}$, and $\omega(\mathcal{G})$ is the Gurau degree.

Drawing trisection diagrams based on colored tensor model graphs.

I.




II.



# Conclusion

- We formulated a construction of trisections on manifolds realised by colored tensor models with 4-indices.
- We found in this construction, a central surface of a trisection is realised via a jacket of a $4-$bubble.
- Some drawbacks in relation to initial hope is that in tensor models as a random geometric approach to quantum gravity, one is interested in taking a continuum limit, where we shall send the number of triangulations to infinity, i.e., the graph will become large, therefore, $g_c$ tends large as well, being far away from the topologically invariant trisection genus.
- It is also ambiguous whether classifying graphs according to $g_c$ is meaningful.