# A Galois group on meromorphic germs and locality evaluators

Sylvie Paycha
ongoing joint work with Li Guo and Bin Zhang

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# I. Starting point: analytic renormalisation

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$$f : \mathbb{C}^k \to \mathbb{C},\ f = \frac{h(z_1, \cdots, z_k)}{L_1^{s_1} \cdots L_m^{s_m}} \quad L_i = \sum_{j \in J_i} z_j, \quad J_i \subset \{1, \cdots, k\}, \quad h \text{ holom. at zero}$$

**Question:** How to evaluate them consistently at the poles $z_1 = \cdots = z_k = 0$ and what freedom of choice do we have in the choice of evaluator?

# Brain teaser: an example

How can we evaluate $f(z_1, z_2) := \frac{z_1 - z_2}{z_1 + z_2}$ at $z_1 = z_2 = 0$?

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## Iterated minimal subtraction schemes

- $\mathrm{ev}_{z_2=0}^{\mathrm{reg}} \circ \mathrm{ev}_{z_1=0}^{\mathrm{reg}} \left(f(z_1, z_2)\right) = -1;$

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## Iterated minimal substraction scheme

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define a family of linear forms $\mathcal{E}_k : \mathcal{M}^{\text{Feyn}}(\mathbb{C}^k) \to \mathbb{C}$, compatible with the filtration, which fulfill the following conditions

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3. $\mathcal{E}$ is invariant under permutations of the variables $\mathcal{E}_k \circ \sigma^* = \mathcal{E}_k$ for any $\sigma \in \Sigma_k$, with $\sigma^* f(z_1, \cdots, z_k) := f(z_{\sigma(1)}, \cdots, z_{\sigma(k)})$;

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4. (continuity) If $f_n(\vec{z}_k) \cdot L_1^{s_1} \cdots L_m^{s_m} \xrightarrow[n\to\infty]{\text{uniformly}} g(\vec{z}_k)$ as holomorphic germs, then $\mathcal{E}_k(f_n) \underset{n\to\infty}{\longrightarrow} \mathcal{E}_k(\lim_{n\to\infty} f_n)$.

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b) *a change of variable* $u = z_1 - z_2, v = z_1 + z_2$ *in* $f$ *and* $g$ *gives* $\tilde f(z_1, z_2) = \frac{z_1 - z_2}{z_1 + z_2}$, $\tilde g(z_1, z_2) = \left(\frac{z_1 - z_2}{z_1 + z_2}\right)^2$ *and we have* $\mathcal{E}_2^{\rm iter}(\tilde f) = 0$ *whereas* $\mathcal{E}_2^{\rm iter}(\tilde g) = 1$.

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Instead, we

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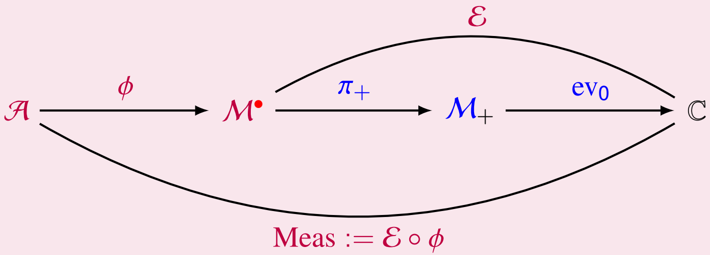
Instead, we

- build a coordinate free multivariable minimal subtraction scheme governed by an inner product $Q$;
- define a class of "generalised evaluators" called locality evaluators which contains multivariable minimal subtraction schemes;
- show that (on certain algebras of meromorphic germs) modulo the action of a Galois group, all locality evaluators are multivariable minimal subtraction schemes.

# II. Framework and protagonists

# The abstract setup



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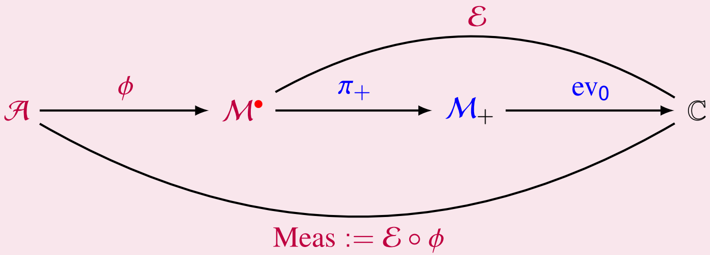


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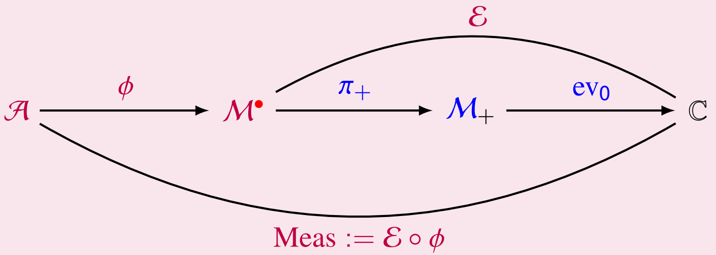


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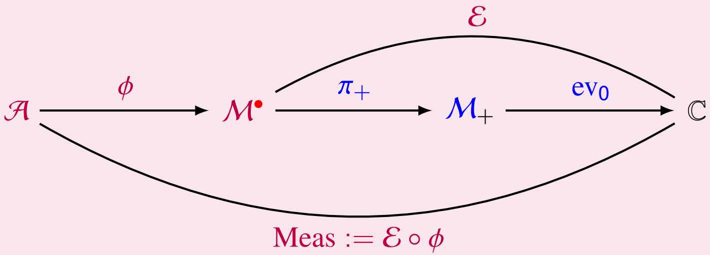


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- $(\mathcal{M}^\bullet, \cdot)$ is an algebra of meromorphic germs at zero with pointwise product;
- $\phi : (\mathcal{A}, \vee) \longrightarrow (\mathcal{M}^\bullet, \cdot)$ is a morphism given by Feynman integrals, branched zeta functions or conical zeta functions.

# Our main protagonists: Meromorphic germs

The target algebra $(\mathcal{M}^\bullet, \cdot)$ is an algebra of meromorphic germs at zero in several variables with linear poles:

$$f : \mathbb{C}^k \to \mathbb{C},\ f = \frac{h(z_1, \cdots, z_k)}{L_1^{s_1} \cdots L_m^{s_m}}, \quad L_i : \mathbb{C}^k \to \mathbb{C}, \text{linear}, \quad h \text{ holom. at zero} .$$

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- Laplace transforms on polyhedral cones [GPZ DJM 2017]:
  $L_i(\vec{z}) = \sum_{\ell=1}^{L_i} a_\ell\, z_\ell \rightsquigarrow \mathcal{M}$.

For $I \subset [[1, n]]$ we set $z_I := \sum_{i \in I} z_i$. Speer actually shows that Feynman amplitudes have poles of a nested form $J_i \subset J_{i+1}$:

Speer fractions: $$z_{I_1}^{s_1}(z_{I_1} + z_{I_2})^{s_2} \cdots (z_{I_1} + \cdots + z_{I_k})^{s_k}, \quad I_j \subset \{1, \cdots, k\}, \qquad (1)$$

# III. A guiding principle: locality

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**Claim: On certain algebras of meromorphic germs with a prescriped type of pole at zero, modulo a Galois transformation, any $\perp$-locality evaluator at the poles is determined by a multivariable minimal subtraction scheme.**

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## Speer's locality: separation of variables

$(z_1 - z_2) \perp^{\mathrm{Speer}} (z_3 + z_4) \Rightarrow (z_1 - z_2) \perp^Q (z_3 + z_4)$.

# IV. Statement and ingredients for its proof

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is the induced projection onto the holomorphic part.

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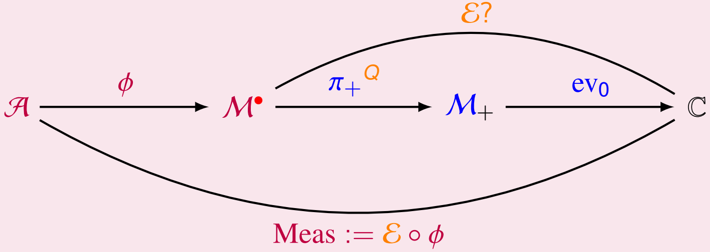

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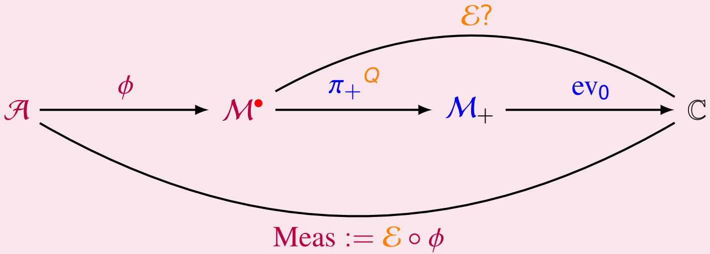


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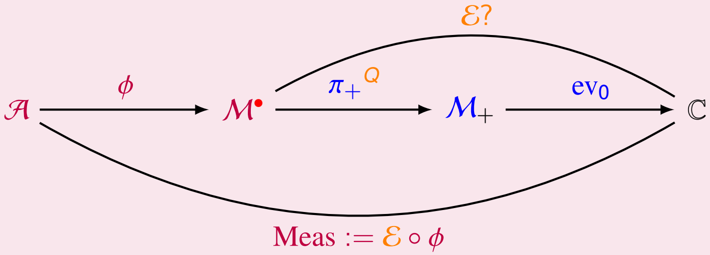


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$$\mathrm{Gal}^\perp(\mathcal{M}^\bullet) = \{T \in \mathrm{Aut}^{\perp^Q}(\mathcal{M}^\bullet) \mid T|_{\mathcal{M}_+} = \mathrm{Id}\}.$$

In practice, we consider a restricted Galois group singling out transformations which preserve the residue arising form the Laurent expansion.

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# Perspectives

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**Question**

What happens beyond $\perp^Q$-locality relations?

**THANK YOU FOR YOUR ATTENTION!**

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