Double scaling limit of multi-matrix models

joint work with V. Bonzom & A. Tanasa

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Aim of this talk

- Show how schemes are a general tool useful to enumerate specific families of graphs
- Stress the difficulties that can arise for certain model (e.g. reduced symmetry group).

This presentation is based on arXiv:2109.07238 and a companion paper that should appear on arXiv in the upcoming weeks.

Outline

The quartic $O(N)^3$ tensor model

The model

Introduced by S. Carrozza and A. Tanasa in [Carrozza-Tanasa (2015)]

- The tensor ϕ_{abc} is invariant under the action of $O(N)^3$:

$$\phi_{abc} \to \phi'_{a'b'c'} = \sum_{a,b,c=1}^{N} O^1_{a'a} O^2_{b'b} O^3_{c'c} \phi_{abc} \qquad O^i \in O(N)$$

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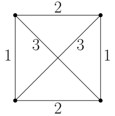
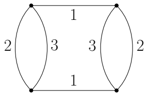
- Two different quartic invariants:

$$I_t(\phi) = \sum_{a,a',b,b',c,c'} \phi_{abc}\phi_{ab'c'}\phi_{a'bc'}\phi_{a'b'c} =$$



$$I_{p,1}(\phi) = \sum_{a,a',b,b',c,c'} \phi_{abc}\phi_{a'bc}\ \phi_{ab'c'}\phi_{a'b'c'} =$$



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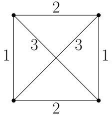

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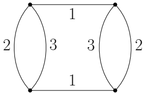


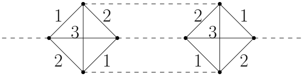
- The action reads

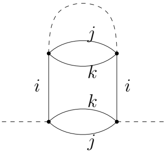
$$S_N(\phi) = -\frac{N^2}{2}\phi^2 + N^{5/2}\frac{\lambda_1}{4} I_t(\phi) + N^2\frac{\lambda_2}{4}\Big(I_{p,1}(\phi) + I_{p,2}(\phi) + I_{p,3}(\phi)\Big) \quad (1)$$

$$\omega(\bar{\mathcal{G}}) = 3 + \frac{3}{2} n_t(\bar{\mathcal{G}}) + 2 n_p(\bar{\mathcal{G}}) - F(\bar{\mathcal{G}}) \qquad (2)$$

$$\omega(\bar{\mathcal{G}}) = 3 + \frac{3}{2} n_t(\bar{\mathcal{G}}) + 2n_p(\bar{\mathcal{G}}) - F(\bar{\mathcal{G}}) \quad (2)$$

Melons





$$M(t, \mu) = 1 + tM^4 + t\mu M^2$$

with $(t, \mu) = (\lambda_1^2, \frac{3\lambda_2}{\lambda_1^2})$

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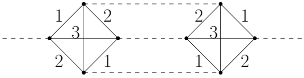

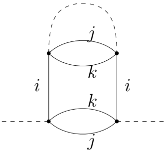

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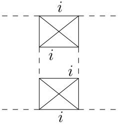
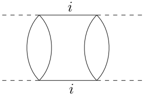
Dipoles





$$U(t,\mu) = tM^4 + \frac{1}{3} t\mu M^2$$

$$\omega(\bar{\mathcal{G}}) = 3 + \frac{3}{2} n_t(\bar{\mathcal{G}}) + 2n_p(\bar{\mathcal{G}}) - F(\bar{\mathcal{G}}) \tag{2}$$

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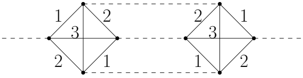

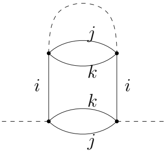

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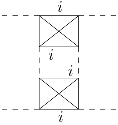

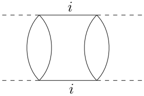

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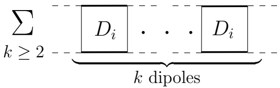
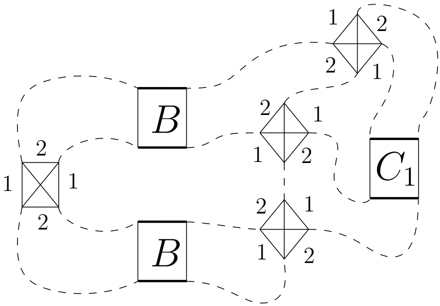
Chains



$$C_i(t,\mu) = \frac{U^2}{1-U}$$

$$B(t,\mu) = \frac{6U^2}{(1-3U)(1-U)}$$

Schemes



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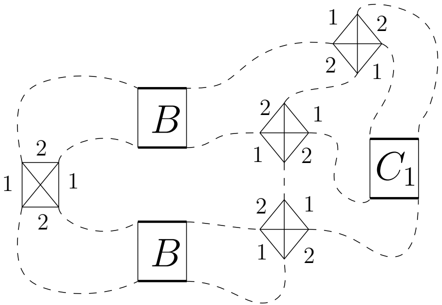


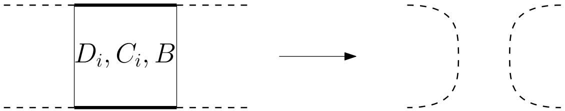
Theorem

The set of schemes of a given degree is finite in the quartic $O(N)^3$-invariant tensor model.

Overview of the proof

1. Study what happens when removing a dipole/chain in a scheme
2. Show that there can only be finitely many other faces

Chain removal



Two different cases:

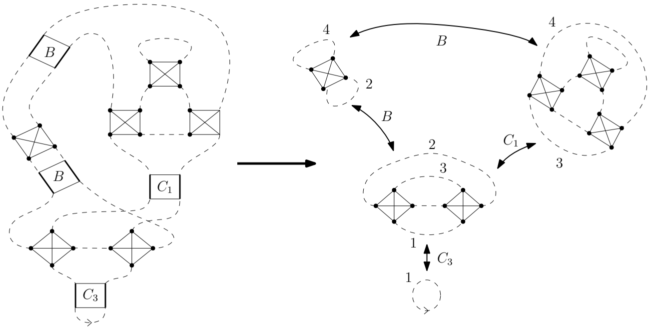
1. Removing the chain disconnects the scheme

$$\Delta\omega = 0$$

2. Removing the chain doesn't disconnect the scheme:

$$-3 \leq \Delta\omega \leq -1$$

One last definition..



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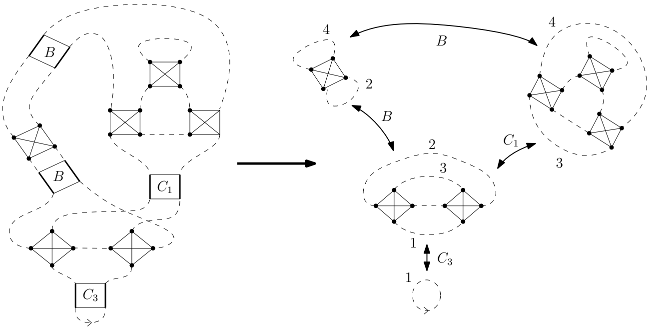

Definition

For a scheme $\mathcal{S}$, its *skeleton graph* $\mathcal{I}(\mathcal{S})$ is such that

- The vertex set of $\mathcal{I}(\mathcal{S})$ is the set of connected components obtained by removing all chain-vertices and dipole-vertices of $\mathcal{S}$.
- There is an edge between two vertices in $\mathcal{I}(\mathcal{S})$ if the two corresponding connected components are connected by a chain-vertex or a dipole-vertex. This edge is labeled by the type of chain- or dipole-vertex.

Properties of the skeleton graph

Lemma

For any scheme $\mathcal{S}$, define by convention $\omega(\mathcal{I}(\mathcal{S})) = \omega(\mathcal{S})$. We have the following properties.

1. *If $\bar{\mathcal{G}}^{(r)}$ for $r \in \{1, \ldots, p\}$ has vanishing degree, then the corresponding vertex in $\mathcal{I}(\mathcal{S})$ has valency at least equal to 3.*

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2. *Let $\mathcal{T} \subset \mathcal{I}(\mathcal{S})$ be a spanning tree. Let q be the number of edges of $\mathcal{I}(\mathcal{S})$ which are not in $\mathcal{T}$. Then, $\omega(\mathcal{T}) \leq \omega(\mathcal{S}) - q$.*

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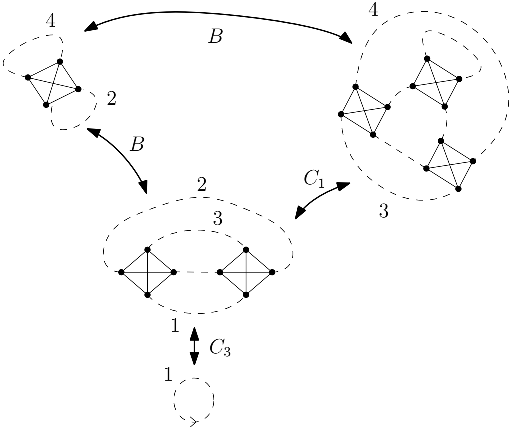
3. *$\omega(\mathcal{T}) = \omega(\mathcal{G}^{(0)}) + \sum_{r=1}^{p} \omega(\bar{\mathcal{G}}^{(r)})$, in other words, if $\mathcal{I}(\mathcal{S})$ is a tree, then the degree of $\mathcal{S}$ is the sum of the degrees of its components obtained by removing all chain- and dipole-vertices.*

There are finitely many chains in a scheme

$$N(\mathcal{S}) \leq 7\omega(\mathcal{S}) - 1 \quad (3)$$



What about scheme with no chains ?

Through each edge of color 0 passes exactly one face of each color.

$$\sum_{p\geq 1} pF_i^{(p)}(\mathcal{G}) = E_0(\mathcal{G}) = 2n(\mathcal{G}) \tag{4}$$

Using equation of the degree we get

$$\sum_{p\geq 5}(p-4)F^{(p)}(\mathcal{G}) = 4(\omega - 3) + 3F^{(1)}(\mathcal{G}) + 2F^{(2)}(\mathcal{G}) + F^{(3)}(\mathcal{G}) \tag{5}$$

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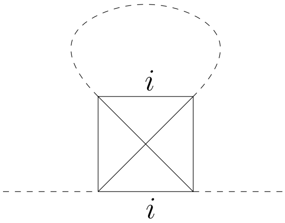
- We only have to show that there are finitely many "short faces"

Faces of length 1 and 2

1. Faces of length 1 are tadpoles

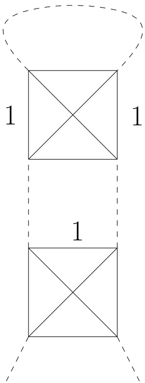


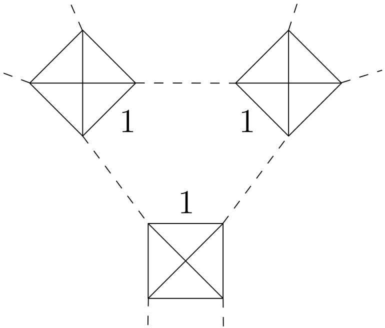
Removing the tadpole gives

$$\Delta\omega = -\frac{1}{2} \tag{6}$$

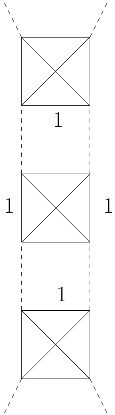
2. Faces of length 2 are dipoles, which have already been shown to be bounded.

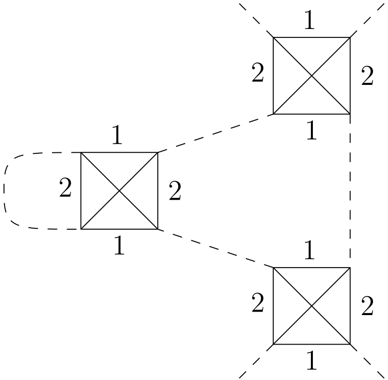
Faces of length 3





Self-intersecting faces of length 4





Non-self intersecting faces of length 4

- Too many cases to handle..

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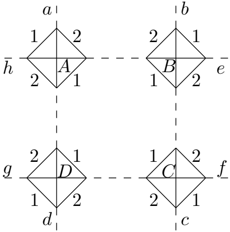
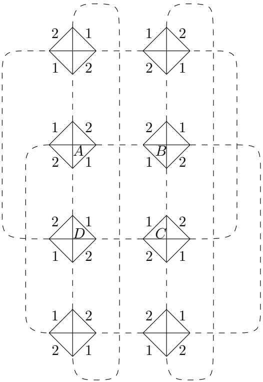
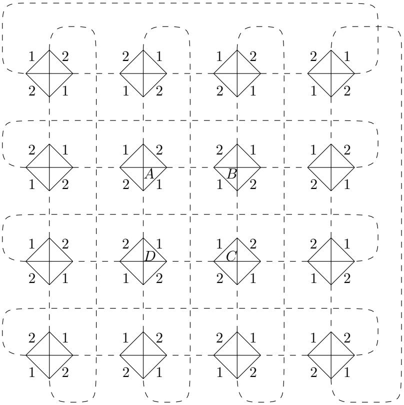
- Instead, we can show that all bubbles contributing to a face of length 4 are at bounded distance of a bubble which is in finite quantity in the scheme.



Two possible schemes





We have shown that

1. There are finitely many chains in a scheme of fixed degree
2. There are finitely many scheme with no chains.

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Theorem

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The quartic $U(N)^2 \times O(D)$ multi-matrix model

The model

This model model involves a vector of D complex matrices of size $N \times N$. It is required to be invariant under the following actions of $U(N)^2 \times O(D)$

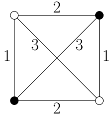
$$X_\mu \to U_1 X_\mu U_2^\dagger$$
$$X_\mu \to O_{\mu\mu'}\ X_{\mu'}$$

These types of models are sometimes called **matrix-tensor models** since each symmetry group acts on a distinct index.

Quartic invariant

The quartic invariants are

$$I_t(X, X^\dagger) = \sum_{\mu,\nu} \mathrm{Tr}\big(X_\mu X_\nu^\dagger X_\mu X_\nu^\dagger\big) = \tag{7}$$



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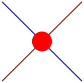
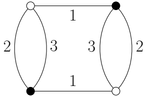


$$I_{p;1}(X, X^\dagger) = \sum_{\mu,\nu} \mathrm{Tr}\big(X_\mu X_\mu^\dagger X_\nu X_\nu^\dagger\big) = \tag{8}$$



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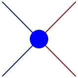
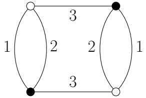


$$I_{p;3b}(X, X^\dagger) = \sum_{\mu,\nu} \mathrm{Tr}\big(X_\mu X_\nu^\dagger\big)\ \ \mathrm{Tr}\big(X_\nu X_\mu^\dagger\big) = \tag{9}$$



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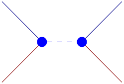

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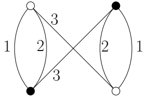

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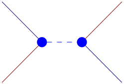

The action

The action of the model reads

$$S_{U(N)^2\times O(D)}(X_\mu, X_\mu^\dagger) = -ND\sum_{\mu=1}^{D}\mathrm{Tr}(X_\mu X_\mu^\dagger) + ND^{3/2}\frac{\lambda_1}{2}I_t(X_\mu, X_\mu^\dagger)$$
$$+ ND\frac{\lambda_2}{2}\left(I_{p;1}(X_\mu, X_\mu^\dagger) + I_{p;2}(X_\mu, X_\mu^\dagger)\right) + D^2\frac{\lambda_2}{2}\left(I_{p;3b}(X_\mu, X_\mu^\dagger) + I_{p;3nb}(X_\mu, X_\mu^\dagger)\right) \tag{11}$$

where each scaling is the unique choice such that there exists a large N, large D expansion and all interactions contribute at leading order of this expansion.

[Ferrari-Rivasseau-Villette]

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where each scaling is the unique choice such that there exists a large N, large D expansion and all interactions contribute at leading order of this expansion. [Ferrari-Rivasseau-Villette]

The double scaling limit of this model with tetrahedral interaction only (e.g. $\lambda_2 = 0$) has been studied in [Benedetti-Carrozza-Toriumi-Valette]

The large N, large D expansion

The free energy $F_{U(N)^2 \times O(D)}(\lambda_1, \lambda_2)$ admits a large N, large D expansion given by

$$F_{U(N)^2 \times O(D)}(\lambda_1, \lambda_2) = \sum_{g \in \mathbb{N}} N^{2-2h} \sum_{l \in \mathbb{N}} D^{-1+h-\frac{\ell}{2}} \sum_{\bar{\mathcal{G}} \in \bar{\mathbb{G}}_{g,l}} \mathcal{A}(\bar{\mathcal{G}}) \tag{12}$$

Sending both N and D to ∞ while holding $L = \frac{N}{\sqrt{D}}$ fixed we get

$$\tilde{F}_{U(N)^2 \times O(D)}(\lambda_1, \lambda_2) = \sum_{\bar{\mathcal{G}} \in \bar{\mathbb{G}}} L^{2-2h(\bar{\mathcal{G}})} D^{2-l(\bar{\mathcal{G}})/2} \mathcal{A}_{\bar{\mathcal{G}}}(\lambda_1, \lambda_2) \tag{13}$$

where

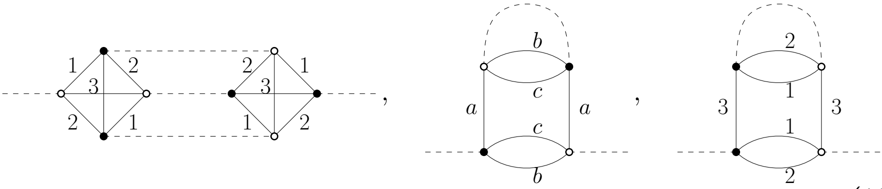
- $h(\bar{\mathcal{G}})$ is a non-negative integer called the **genus**,
- $l(\bar{\mathcal{G}})$ is a non-negative integer called the **grade**.

$$2 - 2h(\bar{\mathcal{G}}) = F_{01} + F_{02} - E_0 + n_t + n_1 + n_2 \tag{14}$$

$$1 + h(\bar{\mathcal{G}}) - \frac{l(\bar{\mathcal{G}})}{2} = F_{03} - E_0 + \frac{3}{2} n_t + n_1 + n_2 + 2(n_{3b} + n_{3nb}), \tag{15}$$

Leading order : the melonic limit

The elementary melons of our model are the following 2-point graphs:



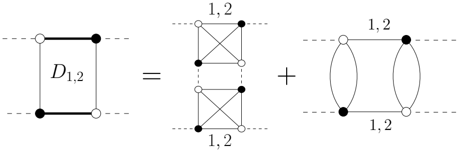
(16)

The generating function of melonic graph is given by

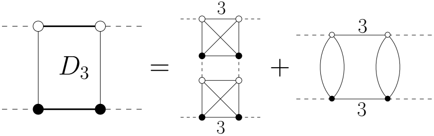
$$M(\lambda_1, \lambda_2) = 1 + \lambda_1^2 M(\lambda_1, \lambda_2)^4 + 4\lambda_2 M(\lambda_1, \lambda_2)^2$$

Dipoles

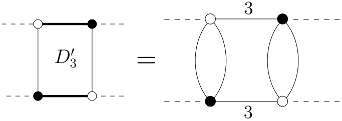
Dipoles are the 4-point graphs which can be obtained from an elementary melon by cutting one of its edges of color 0.



There are two different dipoles of color 3, depending if it is bipartite or not.

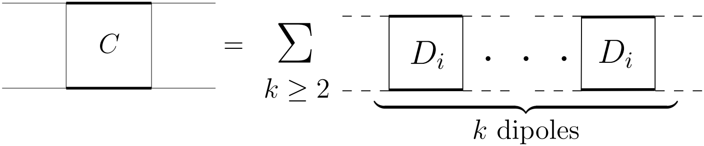


and



Chains

Chains are the 4-point functions obtained by connecting at least 2 dipoles side by side.



(17)

We distinguish different types of chains depending on the color of the dipoles involved:

- If a chain only has dipoles of color i then it is a **chain of color** i.
- Otherwise, the chain is said to be **broken**.

Generating function for dipoles and chains

Generating function for dipoles are

$$U(t,\mu) = tM(t,\mu)^4 + \frac{1}{4}t\mu M(t,\mu)^2 = M(t,\mu) - 1 - \frac{3}{4}t\mu M(t,\mu)^2 \tag{18}$$

$$V(t,\mu) = \frac{1}{4}t\mu M(t,\mu)^2 \tag{19}$$

And the generating function for chains is

$$C_1(t,\mu) = C_2(t,\mu) = U^2 \sum_{n\geq 0} U^n = \frac{U^2}{1-U} \tag{20}$$

$$C_3(t,\mu) = (U+V)^2 \sum_{n\geq 0} (U+V)^n = \frac{(U+V)^2}{1-(U+V)} \tag{21}$$

$$B(t,\mu) = \frac{-6U^3 - 8U^2V + 6U^2 - 2UV^2 + 4UV}{(1-3U-V)(1-U)(1-U-V)} \tag{22}$$

Schemes

For a graph $\mathcal{G} \in \bar{\mathbb{G}}$, we define the scheme of $\mathcal{G}$ is obtained by

1. Replacing every melonic 2-point subgraph with an edge of color 0,
2. Replacing every maximal chain with a chain-vertex of the same type,
3. Replacing every dipole with a dipole-vertex of the same type.

Theorem

For any genus and grade (h, l), the set of scheme $\mathbb{S}_{h,l}$ is a finite.

Proof via the $O(N)^3$ tensor model

From a scheme of our model of genus h and grade l, we can obtain a scheme of degree $\omega = h + \frac{l}{2}$ by forgetting the coloring of the vertices in the stranded representation.

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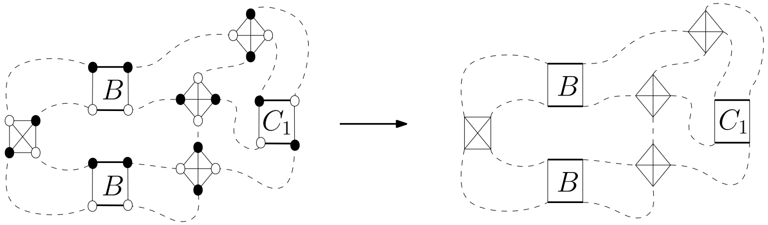

Double scaling limit and criticality

In the double scaling limit, coupling constants are sent to a critical point for the partition function. There are two possible types of singularity that could be encountered:

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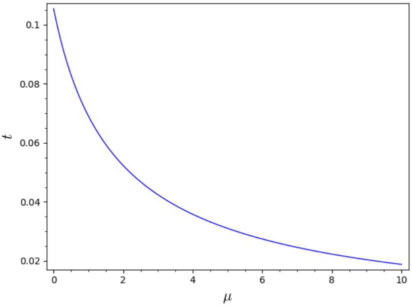
- Singular points for the melons generating function M.
- Singular points for the different types of chains.

Critical points for M



$$M(t,\mu) \underset{t\to t_c(\mu)}{\sim} M_c(\mu) - K(\mu)\sqrt{1-\frac{t}{t_c(\mu)}} \tag{23}$$

where $M_c(\mu)$ is the unique positive real root of the polynomial equation

$$-3x^3 + 4x^2 - \mu x + 2\mu = 0, \tag{24}$$

and

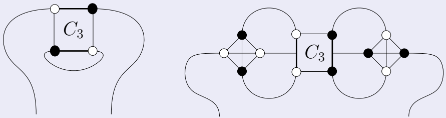
$$K(\mu) = \sqrt{\frac{M_c(\mu)^2\left(M_c(\mu)^2+\mu\right)}{6M_c(\mu)^2+\mu}}. \tag{25}$$

Dominant schemes

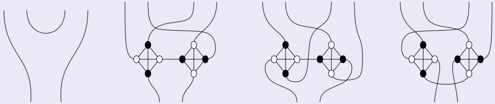
Theorem

A dominant scheme of genus $h > 0$ has $2h - 1$ broken chain-vertices, all separating. Such a scheme has the structure of a rooted binary plane tree where:

- *Edges correspond to broken chains.*
- *The root corresponds to the two external legs of the 2-point function.*
- *The h leaves are one of the two following graphs*



- *There are 7 types of internal nodes,*



Double scaling parameter

The contribution of a dominant scheme is given by

$$G_{\mathcal{T}}^{h}(t,\mu) = L^{2-2h}\big(C_{3,\substack{\circ\bullet\\ \bullet\circ}}(t,\mu) + tC_{3,\substack{\circ\circ\\ \bullet\bullet}}(t,\mu)\big)^{h}\,(1+6t)^{h-1}\,B(t,\mu)^{2h-1} \tag{26}$$

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Therefore we define the double scaling parameter as

$$\kappa(\mu)^{-1} = L^{2}\frac{1}{(1+6t)(C_{3,\substack{\circ\bullet\\\bullet\circ}}(t,\mu) + tC_{3,\substack{\circ\circ\\\bullet\bullet}}(t,\mu))}\left(\frac{1}{B(t,\mu)}\right)^{2} \tag{27}$$

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which gives

$$G^h_{dom}(t,\mu) = L\left(\frac{(C_{3,\substack{\circ\bullet\\\bullet\circ}} + tC_{3,\substack{\circ\circ\\\bullet\bullet}})}{1+6t}\bigg|_{(t_c(\mu),\mu)}\right)^{\frac{1}{2}} \kappa(\mu)^{h-\frac{1}{2}} \tag{28}$$

Final computation

The genus $h > 0$ contribution is obtained by summing over binary trees with h leaves

$$G^h_{dom}(t,\mu) = L\,\mathrm{Cat}_{h-1}\,M_c(t_c(\mu),\mu)\left(\frac{(C_{3,\substack{\circ\bullet\\\bullet\circ}} + tC_{3,\substack{\circ\circ\\\bullet\bullet}})}{1+6t}\bigg|_{(t_c(\mu),\mu)}\right)^{\frac{1}{2}}\kappa(\mu)^{h-\frac{1}{2}} \quad (29)$$

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Finally, the 2-point function at leading order in the double scaling limit is

$$G_2^{DS}(\mu) = M_c(t_c(\mu),\mu)\left(1 + L\left(\left.\frac{(C_{3,\substack{\circ\bullet\\\bullet\circ}} + tC_{3,\substack{\circ\circ\\\bullet\bullet}})}{1+6t}\right|_{(t_c(\mu),\mu)}\right)^{\frac{1}{2}} \frac{1-\sqrt{1-4\kappa(\mu)}}{2\kappa(\mu)^{\frac{1}{2}}}\right) \quad (30)$$

Conclusion for this model

- All the tools used for the $O(N)^3$ tensor model can be used for this matrix-tensor model.

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- Some details in the structure on the Feynman graphs still depend on the model.

The quartic bipartite $U(N) \times O(D)$ multi-matrix model

The model

The matrices are now required to be invariant under the action of $U(N) \times O(D)$

$$X_\mu \to U X_\mu U^\dagger$$
$$X_\mu \to O_{\mu\mu'}\ X_{\mu'}$$

We will restrict to tetrahedral interaction

$$I_t(X) = \mathrm{Tr}(X_\mu X_\nu X_\mu X_\nu) \tag{31}$$

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The action is

$$S_{U(N)\times O(D)}(X_\mu, X_\mu^\dagger) = -ND \sum_{\mu=1}^{D} \mathrm{Tr}(X_\mu^\dagger X_\mu) + \frac{\lambda}{4} N D^{\frac{3}{2}} \sum_{\mu,\nu=1}^{D} \mathrm{Tr}(X_\mu X_\nu X_\mu X_\nu) + \mathrm{Tr}(X_\mu^\dagger X_\nu^\dagger X_\mu^\dagger X_\nu^\dagger) \tag{32}$$

Free energy expansion

The free energy admits a large N, large D expansion given by

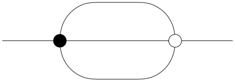
$$F_{U(N)\times O(D)}(\lambda) = \sum_{\bar{\mathcal{M}}\in\bar{\mathfrak{M}}} \left(\frac{N}{\sqrt{D}}\right)^{2-2h(\bar{\mathcal{M}})} D^{2-l(\bar{\mathcal{M}})/2}\ \mathcal{A}_{\bar{\mathcal{M}}}(\lambda) \tag{33}$$

where

$$2 - 2h(\bar{\mathcal{M}}) = F - E + V \tag{34}$$

$$\frac{l(\bar{\mathcal{M}})}{2} = 1 + h(\bar{\mathcal{M}}) + E - \frac{3V}{2} - \phi. \tag{35}$$

The melons



- Melons are defined by recursive insertion of the above elementary melons on its edges.
- Their generating function is given by

$$M(\lambda) = 1 + \lambda M(\lambda)^4 \tag{36}$$

Proof of melonic dominance

We have

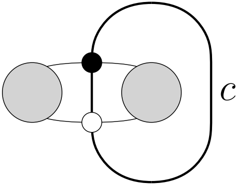
$$\frac{I(\bar{\mathcal{M}})}{2} = 1 + h + \sum_{n \geq 1} \Big(\frac{n}{2} - 1\Big) \phi_{2n}. \quad (37)$$

Proof of melonic dominance

We have

$$\frac{l(\bar{\mathcal{M}})}{2} = 1 + h + \sum_{n \geq 1} \Big(\frac{n}{2} - 1\Big) \phi_{2n}. \quad (37)$$

Therefore a map of vanising genus and grade has the following structure



Aparte: lower bound on grade for multi-matrix model

[Carrozza-Tanasa-Toriumi-Valette]

- Taking large N expansion gives

$$F = \sum_{h \geq 0} N^{2-2h} F_h \tag{38}$$

- The existence of the large D expansion is guaranteed if there exist $\eta(h) < \infty$ such that

$$F_h = \sum_{\ell \geq 0} D^{1+\eta(h)-\frac{\ell}{2}} \tag{39}$$

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  - All models where a large D expansion is defined have $\eta(h) = h$
  - In the traceless Hermitian matrix model, we cannot prove that $\eta(h)$ is bounded.

Dipoles and chains

There are two types of dipoles

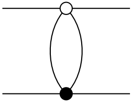

Figure: U-dipole

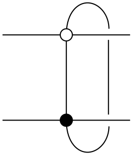

Figure: O-dipole

Their respective generating function are given by

$$D_O(t) = U(t) = tM(t)^4 = M(t) - 1 \quad (40)$$
$$D_U(t) = 2U(t) \quad (41)$$

Dipoles and chains

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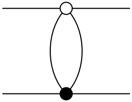

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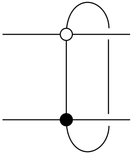


Figure: O-dipole

Their respective generating function are given by

$$D_O(t) = U(t) = tM(t)^4 = M(t) - 1 \quad (40)$$

$$D_U(t) = 2U(t) \quad (41)$$

$$B(t) = \frac{9U^2}{1-3U} - \frac{U^2}{1-U} - \frac{4U^2}{1-2U} = \frac{(4-6U)U^2}{(1-U)(1-2U)(1-3U)} \quad (42)$$

Schemes are maps obtained by

1. Replacing every melonic submap with an edge,
2. Replacing every maximal chain (resp. dipole) with a *chain-vertex* (resp. *dipole-vertex*), i.e. a 4-point vertex where we only keep track of the pairing of the half-edges by drawing them on the same side.

Theorem

The set $\mathbb{MS}_{h,l}$ of scheme of genus h and grade l is finite.

Sketch of proof

$$\frac{I(\bar{\mathcal{M}})}{2} = 1 + h + \sum_{n \geq 1} \left(\frac{n}{2} - 1 \right) \phi_{2n}. \tag{43}$$

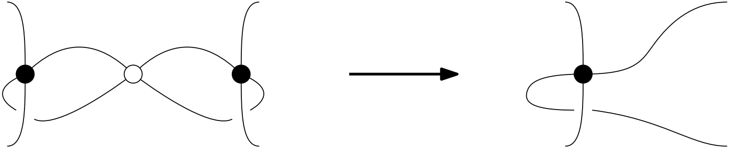
- Cycles of length 2 correspond to O-dipoles.

Sketch of proof

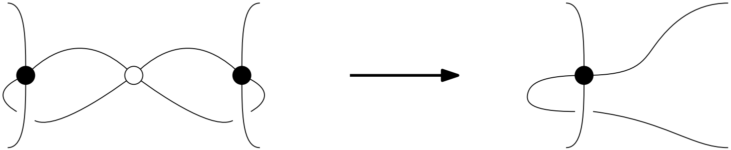
$$\frac{I(\bar{\mathcal{M}})}{2} = 1 + h + \sum_{n \geq 1} \left(\frac{n}{2} - 1 \right) \phi_{2n}. \tag{43}$$

- Cycles of length 2 correspond to O-dipoles.
- Cycles of length 4 are more involved..

Self-intersecting cycles of length 4



Self-intersecting cycles of length 4

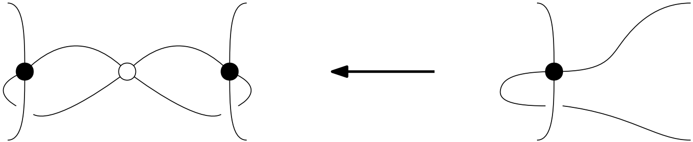


We first prove a weaker version of the theorem

Theorem

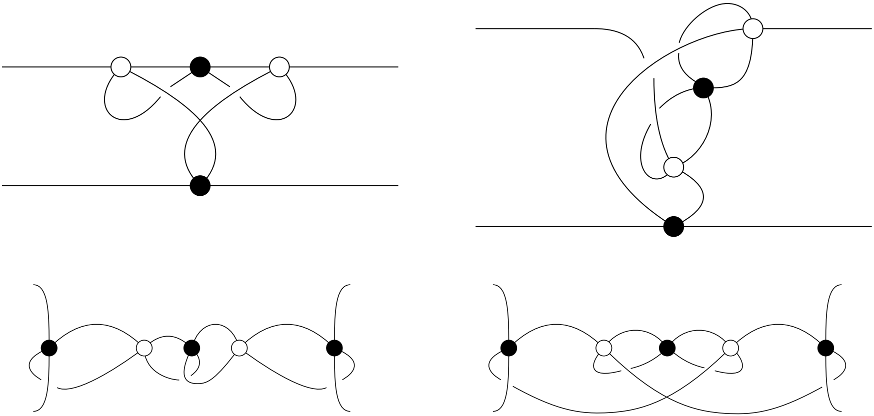
The set $\tilde{\mathbb{MS}}_{h,l}$ of scheme of genus h and grade l ***without self-intersecting 4-cycles*** *is finite.*

Reinserting self-intersecting cycles of length 4



$$\Delta g = -1 - \frac{1}{2}\Delta F$$

Double insertion of 4-si cycles



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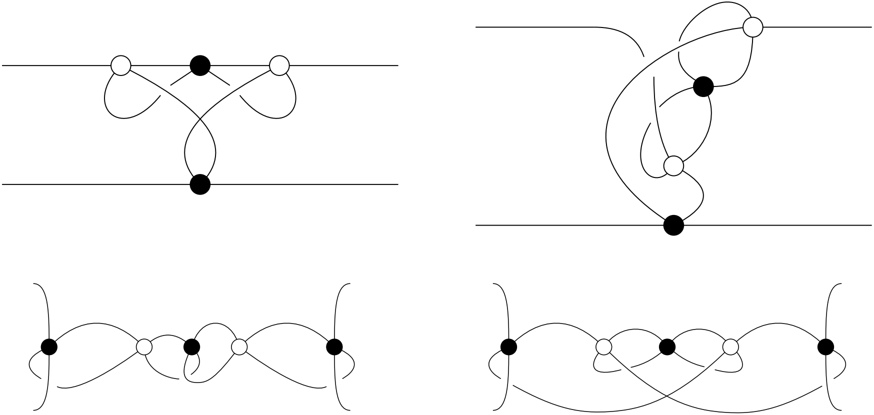


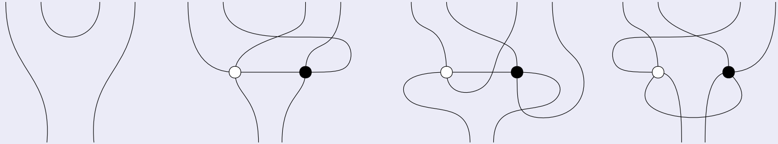
We are saved !

Dominant schemes

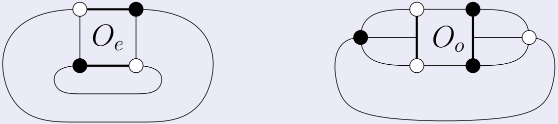
Theorem

A rooted dominant scheme of genus $h > 1$ has $2h - 1$ broken chains which are all separating. Such scheme has the structure of a binary plane tree where:

- *Each edge corresponds to a broken chain-vertex*
- *Each node corresponds to one of the seven 6-point subgraph:*



- *The root of the tree corresponds to the root of the map*
- *Each of the g leaves is one of the following two graph*



(44)

Double scaling limit

$$\kappa^{-1} = L^2 \frac{1}{\left(\frac{4}{5} + t_c \frac{6}{5}\right)(1 + 6t_c)} \frac{8}{27} \left(1 - \frac{t}{t_c}\right) \tag{45}$$

The contribution of a dominant scheme reads

$$G^g_{dom}(t) = \frac{4}{3} L \left(\frac{\frac{4}{5} + t_c \frac{6}{5}}{1 + 6t_c}\right)^{\frac{1}{2}} \kappa^{g - \frac{1}{2}} \tag{46}$$

Summing over the trees, then over the genus we finally get

$$G^{DS}_2(t) = 4/3 + \frac{2}{3} \frac{1}{L} \left(\frac{\frac{4}{5} + t_c \frac{6}{5}}{1 + 6t_c}\right)^{\frac{1}{2}} \frac{1 - \sqrt{1 - 4\kappa}}{\kappa^{\frac{1}{2}}} \tag{47}$$

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- The techniques used for tensor models can be applied to tensor-matrix models with few tweaks.
- These combinatorial methods are harder to apply to model with a reduced symmetry group.

Some open questions:

- Can we make these techniques less dependant on the model (e.g. symmetry group, interactions..) ?
- Can we obtain the same results without using diagrammatic methods ?

Thank you for your attention !