STRUCTURES
CLUSTER OF
EXCELLENCE

UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386

A Yang-Mills matrix theory

String Math 2022
University of Warsaw, July 2022

Carlos Pérez-Sánchez
ITP, University of Heidelberg
perez@thphys.uni-heidelberg.de

Based on:

1912.13288 [pl] ; 2007.10914 [pl] ; 2105.01025 [pl,de] ; 2111.02858 [pl,de]

pl TEAM Fundacja na Rzecz Nauki Polskiej
de ERC, indirectly & DFG-Structures Excellence Cluster

European Research Council
Established by the European Commission

FNP

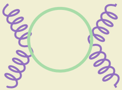
Introduction

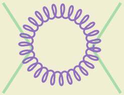
- Classical SU(n)-Yang-Mills theory is geometrically modelled on connections on a SU(n)-principal bundle on a smooth space M. We want to replace M by a ‘quantum space~~time~~’ model based on *noncommutative geometry* (NCG).

Introduction

- Classical SU(n)-Yang-Mills theory is geometrically modelled on connections on a SU(n)-principal bundle on a smooth space M. We want to replace M by a 'quantum space~~time~~' model based on *noncommutative geometry* (NCG).

- Why do we want SU(n)-Yang-Mills theory on that space? The natural quantum theory to consider could be 'gravity + matter' [Donà, Eichhorn, Percacci, *PRD* 2014]:





- From physics to NCG: The Standard Model from the Spectral Action

$$
\begin{gathered}
-\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4}g^2_s f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu + \\
\tfrac{1}{2}ig^2_s(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu + \bar{G}^a\partial^2 G^a + g_s f^{abc}\partial_\mu \bar{G}^a G^b g^c_\mu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - \\
M^2 W^+_\mu W^-_\mu - \tfrac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c^2_w}M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \\
\tfrac{1}{2}\partial_\mu H \partial_\mu H - \tfrac{1}{2}m^2_h H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2\phi^+\phi^- - \tfrac{1}{2}\partial_\mu\phi^0\partial_\mu\phi^0 - \\
\tfrac{1}{2c^2_w}M\phi^0\phi^0 - \beta_h[\tfrac{2M^2}{g^2} + \tfrac{2M}{g}H + \tfrac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)] + \tfrac{2M^4}{g^2}\alpha_h - \\
igc_w[\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + \\
Z^0_\mu(W^+_\nu\partial_\nu W^-_\mu - W^-_\nu\partial_\nu W^+_\mu)] - igs_w[\partial_\nu A_\mu(W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - A_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + A_\mu(W^+_\nu\partial_\nu W^-_\mu - \\
W^-_\nu\partial_\nu W^+_\mu)] - \tfrac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \tfrac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + \\
g^2 c^2_w(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s^2_w(A_\mu W^+_\mu A_\nu W^-_\nu - \\
A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w[A_\mu Z^0_\nu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - \\
2A_\mu Z^0_\mu W^+_\nu W^-_\nu] - g\alpha[H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-] - \tfrac{1}{8}g^2\alpha_h[H^4 + \\
(\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2] - \\
gMW^+_\mu W^-_\mu H - \tfrac{1}{2}g\tfrac{M}{c^2_w}Z^0_\mu Z^0_\mu H - \tfrac{1}{2}ig[W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - \\
W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)] + \tfrac{1}{2}g[W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) - \\
W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)] + \tfrac{1}{2}g\tfrac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) - \\
ig\tfrac{s^2_w}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_w MA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) - \\
ig\tfrac{1-2c^2_w}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + igs_w A_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \\
\tfrac{1}{4}g^2 W^+_\mu W^-_\mu[H^2 + (\phi^0)^2 + 2\phi^+\phi^-] - \tfrac{1}{4}g^2\tfrac{1}{c^2_w}Z^0_\mu Z^0_\mu[H^2 + (\phi^0)^2 + \\
2(2s^2_w - 1)^2\phi^+\phi^-] - \tfrac{1}{2}g^2\tfrac{s^2_w}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) -
\end{gathered}
$$

$$
\begin{gathered}
\tfrac{1}{2}ig^2\tfrac{s^2_w}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \tfrac{1}{2}g^2 s_w A_\mu\phi^0(W^+_\mu\phi^- + \\
W^-_\mu\phi^+) + \tfrac{1}{2}ig^2 s_w A_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\tfrac{s_w}{c_w}(2c^2_w - \\
1)Z^0_\mu A_\mu\phi^+\phi^- - g^1 s^2_w A_\mu A_\mu\phi^+\phi^- - \bar{e}^\lambda(\gamma\partial + m^\lambda_e)e^\lambda - \bar{\nu}^\lambda\gamma\partial\nu^\lambda - \\
\bar{u}^\lambda_j(\gamma\partial + m^\lambda_u)u^\lambda_j - \bar{d}^\lambda_j(\gamma\partial + m^\lambda_d)d^\lambda_j + igs_w A_\mu[-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \\
\tfrac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \tfrac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)] + \tfrac{ig}{4c_w}Z^0_\mu[(\bar{\nu}^\lambda\gamma^\mu(1 + \gamma^5)\nu^\lambda) + \\
(\bar{e}^\lambda\gamma^\mu(4s^2_w - 1 - \gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(\tfrac{4}{3}s^2_w - 1 - \gamma^5)u^\lambda_j) + (\bar{d}^\lambda_j\gamma^\mu(1 - \\
\tfrac{8}{3}s^2_w - \gamma^5)d^\lambda_j)] + \tfrac{ig}{2\sqrt{2}}W^+_\mu[(\bar{\nu}^\lambda\gamma^\mu(1 + \gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(1 + \\
\gamma^5)C_{\lambda\kappa}d^\kappa_j)] + \tfrac{ig}{2\sqrt{2}}W^-_\mu[(\bar{e}^\lambda\gamma^\mu(1 + \gamma^5)\nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\lambda\kappa}\gamma^\mu(1 + \\
\gamma^5)u^\lambda_j)] + \tfrac{ig}{2\sqrt{2}}\tfrac{m^\lambda_e}{M}[-\phi^+(\bar{\nu}^\lambda(1 - \gamma^5)e^\lambda) + \phi^-(\bar{e}^\lambda(1 + \gamma^5)\nu^\lambda)] - \\
\tfrac{g}{2}\tfrac{m^\lambda_e}{M}[H(\bar{e}^\lambda e^\lambda) + i\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda)] + \tfrac{ig}{2M\sqrt{2}}\phi^+[-m^\kappa_d(\bar{u}^\lambda_j C_{\lambda\kappa}(1 - \\
\gamma^5)d^\kappa_j) + m^\lambda_u(\bar{u}^\lambda_j C_{\lambda\kappa}(1 + \gamma^5)d^\kappa_j] + \tfrac{ig}{2M\sqrt{2}}\phi^-[m^\lambda_d(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1 + \\
\gamma^5)u^\kappa_j) - m^\kappa_u(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1 - \gamma^5)u^\kappa_j] - \tfrac{g}{2}\tfrac{m^\lambda_u}{M}H(\bar{u}^\lambda_j u^\lambda_j) - \\
\tfrac{g}{2}\tfrac{m^\lambda_d}{M}H(\bar{d}^\lambda_j d^\lambda_j) + \tfrac{ig}{2}\tfrac{m^\lambda_u}{M}\phi^0(\bar{u}^\lambda_j\gamma^5 u^\lambda_j) - \tfrac{ig}{2}\tfrac{m^\lambda_d}{M}\phi^0(\bar{d}^\lambda_j\gamma^5 d^\lambda_j)
\end{gathered}
$$

…this ‘fits’ in

$$\mathrm{Tr}(f(D/\Lambda)) + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle$$

Num. of generations and $\mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C}) \longrightarrow$ NCG $\longrightarrow$ *Classical* Lagrangian of the Standard Model

[Connes-Lott, *Nucl. Phys. B* ’91; . . . Chamseddine-Connes-Marcolli *ATMP* ’07 (Euclidean); J. Barrett *J. Math. Phys.* ’07 (Lorenzian), Connes-Chamseddine *JHEP* ’12]

Motivation

- From physics to NCG: The Standard Model from the Spectral Action

$$
D_F = \left(\begin{array}{cccc|ccccc|cccc|ccccc}
0 & 0 & \Upsilon_\nu^* & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_R^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Upsilon_e^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Upsilon_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \Upsilon_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^* & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^* & \otimes 1_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Upsilon_u & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Upsilon_d & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\Upsilon_R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_\nu^T & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_e^T & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^T & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^T & \otimes 1_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_u & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_d & 0 & 0 &
\end{array}\right) \in M_{96}(\mathbb{C})_{\text{s.a.}}
$$

Num. of generations and $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \rightarrowtail$ NCG $\rightarrowtail$ *Classical* Lagrangian of the Standard Model

[Connes-Lott, *Nucl. Phys. B* '91; . . . Chamseddine-Connes-Marcolli *ATMP* '07 (Euclidean); J. Barrett *J. Math. Phys.* '07 (Lorenzian), Connes-Chamseddine *JHEP* '12]

Motivation

- From physics to NCG: The Standard Model from the Spectral Action

$M_3(\mathbb{C}) \ni \Upsilon_e, \Upsilon_\nu, \dots, \Upsilon_d$

$$D_F = \begin{pmatrix}
0 & 0 & \Upsilon_\nu^* & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_R^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Upsilon_e^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Upsilon_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \Upsilon_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^* & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^* & \otimes 1_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Upsilon_u & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Upsilon_d & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Upsilon_R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_\nu^T & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_e^T & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^T & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^T & \otimes 1_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_u & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_d & 0 & 0 &
\end{pmatrix} \in M_{96}(\mathbb{C})_{\text{s.a.}}$$

$\begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}$

* One more non-zero entry in D_F
  $\langle J\psi, D_F \psi \rangle$
  $\Rightarrow$ not observed interaction

* all zeros from geometry

Num. of generations and $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \rightarrowtail$ NCG $\rightarrowtail$ *Classical* Lagrangian of the Standard Model

[Connes-Lott, *Nucl. Phys. B* '91; . . . Chamseddine-Connes-Marcolli *ATMP* '07 (Euclidean); J. Barrett *J. Math. Phys.* '07 (Lorenzian), Connes-Chamseddine *JHEP* '12]

Towards a quantum theory of noncommutative spaces

« *The far distant goal is to set up a functional integral evaluating spectral observables* $\mathscr{S}$ $\langle \mathscr{S} \rangle = \int \mathscr{S}\, \mathrm{e}^{-\operatorname{Tr} f(D/\Lambda) - \frac{1}{2}\langle J\psi, D\psi\rangle + \rho(e,D)} \mathrm{d}e\, \mathrm{d}\psi\, \mathrm{d}D$ »

[Eq. 1.892, Connes Marcolli, *NCG, QFT and motives*, 2007]

Towards a quantum theory of noncommutative spaces

« *The far distant goal is to set up a functional integral evaluating spectral observables* $\mathscr{S}$ $\quad \langle \mathscr{S} \rangle = \int \mathscr{S}\, \mathrm{e}^{-\operatorname{Tr} f(D/\Lambda) - \frac{1}{2}\langle J\psi, D\psi\rangle + \rho(e,D)} \mathrm{d}e\, \mathrm{d}\psi\, \mathrm{d}D$ »

[Eq. 1.892, Connes Marcolli, *NCG, QFT and motives*, 2007]

functional integral $\xrightarrow[\text{paradigm shift}]{}$ operator integral

$$\int_{\text{METRIC}} \mathrm{e}^{-\frac{1}{\hbar} S_{\text{EH}}[g]} \mathrm{d}g \xrightarrow{\text{Einstein-Hilbert} \to \text{spectral}} \int_{\text{DIRAC}} \mathrm{e}^{-\frac{1}{\hbar} \operatorname{Tr} f(D)} \mathrm{d}D$$

(hard to define for manifolds)

$f : \mathbb{R} \to \mathbb{R}$ with $f(D) \to \infty$ at large argument

Towards a quantum theory of noncommutative spaces

« *The far distant goal is to set up a functional integral evaluating spectral observables* $\mathscr{S}$ $\quad \langle \mathscr{S} \rangle = \int \mathscr{S}\, \mathrm{e}^{-\operatorname{Tr} f(D/\Lambda) - \frac{1}{2}\langle J\psi, D\psi\rangle + \rho(e,D)}\, \mathrm{d}e\, \mathrm{d}\psi\, \mathrm{d}D$ »

[Eq. 1.892, Connes Marcolli, *NCG, QFT and motives*, 2007]

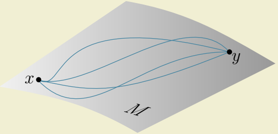
$$\text{functional integral} \xrightarrow[\text{paradigm shift}]{} \text{operator integral}$$

$$\int_{\text{METRIC}} \mathrm{e}^{-\frac{1}{\hbar} S_{\mathrm{EH}}[g]}\, \mathrm{d}g \xrightarrow{\text{Einstein-Hilbert} \to \text{spectral}} \int_{\text{DIRAC}} \mathrm{e}^{-\frac{1}{\hbar} \operatorname{Tr} f(D)}\, \mathrm{d}D$$

(hard to define for manifolds)

$f : \mathbb{R} \to \mathbb{R}$ with $f(D) \to \infty$ at large argument

- Possible application to (Euclidean) quantum gravity: *random noncommutative geometry*



Towards a quantum theory of noncommutative spaces

« *The far distant goal is to set up a functional integral evaluating spectral observables* $\mathscr{S}$ $\quad \langle \mathscr{S} \rangle = \int \mathscr{S}\, e^{-\operatorname{Tr} f(D/\Lambda) - \frac{1}{2}\langle J\psi, D\psi\rangle + \rho(e,D)}\, \mathrm{d}e\, \mathrm{d}\psi\, \mathrm{d}D$ »

[Eq. 1.892, Connes Marcolli, *NCG, QFT and motives*, 2007]

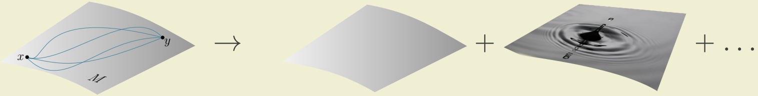
functional integral $\xrightarrow[\text{paradigm shift}]{}$ operator integral

$$\int_{\text{METRIC}} e^{-\frac{1}{\hbar} S_{\text{EH}}[g]}\, \mathrm{d}g \xrightarrow{\text{Einstein-Hilbert} \,\longrightarrow\, \text{spectral}} \int_{\text{DIRAC}} e^{-\frac{1}{\hbar}\operatorname{Tr} f(D)}\, \mathrm{d}D$$

(hard to define for manifolds)

$f : \mathbb{R} \longrightarrow \mathbb{R}$ with $f(D) \longrightarrow \infty$ at large argument

- Possible application to (Euclidean) quantum gravity: *random noncommutative geometry*



Quantum superposition of geometries

Towards a quantum theory of noncommutative spaces

« *The far distant goal is to set up a functional integral evaluating spectral observables* $\mathscr{S}$ $\quad \langle \mathscr{S} \rangle = \int \mathscr{S}\, \mathrm{e}^{-\operatorname{Tr} f(D/\Lambda) - \frac{1}{2}\langle J\psi, D\psi\rangle + \rho(e,D)}\, \mathrm{d}e\, \mathrm{d}\psi\, \mathrm{d}D$ »

[Eq. 1.892, Connes Marcolli, *NCG, QFT and motives*, 2007]

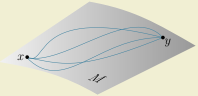
functional integral $\xrightarrow[\text{paradigm shift}]{}$ operator integral

$$\int_{\text{METRIC}} \mathrm{e}^{-\frac{1}{\hbar} S_{\text{EH}}[g]}\, \mathrm{d}g \xrightarrow{\text{Einstein-Hilbert} \,\longrightarrow\, \text{spectral}} \int_{\text{DIRAC}} \mathrm{e}^{-\frac{1}{\hbar}\operatorname{Tr} f(D)}\, \mathrm{d}D$$

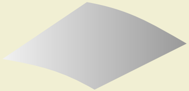
(hard to define for manifolds)

$f : \mathbb{R} \longrightarrow \mathbb{R}$ with $f(D) \longrightarrow \infty$ at large argument

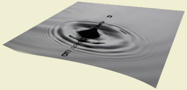
- Possible application to (Euclidean) quantum gravity: *random noncommutative geometry*



$\rightarrow \Big\{$



$+$



$+ \ldots \Big\} \times$

[Hokusai]

Quantum superposition of geometries (small perturbations + instantons)

- Origin of noncommutative topology Connes' differential *noncommutative (nc) geometry* = nc topology [Gelfand, Najmark *Mat. Sbornik* '43] + metric [A. Connes, *NCG* '94]

  {compact Hausdorff topological spaces} $\simeq$ {unital *commutative* C*-algebras}

- Origin of noncommutative topology Connes' differential *noncommutative (nc) geometry* = nc topology [Gelfand, Najmark *Mat. Sbornik* '43] + metric [A. Connes, *NCG* '94]

$$\begin{array}{ccc} \{\text{compact Hausdorff topological spaces}\} & \simeq & \{\text{unital } \textit{commutative } C^*\text{-algebras}\} \\ \downarrow & & \downarrow \\ \{\text{`noncommutative topological spaces'}\} & \simeq & \{\text{unital } \sout{\textit{commutative}}\ C^*\text{-algebras}\} \end{array}$$

- arguably, the 1st predecessor theorem of the spectral formalism is *Weyl's law* (1911) on the rate of growth of the Laplace spectrum (ordered $\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \ldots$) of $\Omega \subset \mathbb{R}^d$

$$\#\{i : \lambda_i \leqslant \Lambda\} = \frac{\operatorname{vol}(\text{unit ball})}{(2\pi)^d} \operatorname{vol}\Omega \cdot \Lambda^{d/2} + o(\Lambda^{d/2})$$

From this, you cannot answer positively Marek Kac' 1966-question. But you can 'hear the shape of Ω' knowing a *spectral triple*. [A, Connes, *JNCG* 2013] (and from it [Connes-van Suijlekom, *CMP* 2021] can hear an MP3; our story today is not entirely unrelated).

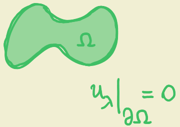
- Origin of noncommutative topology. Connes' differential *noncommutative (nc) geometry* = nc topology [Gelfand, Najmark *Mat. Sbornik* '43] + metric [A. Connes, *NCG* '94]

$$\{\text{compact Hausdorff topological spaces}\} \simeq \{\text{unital } \textit{commutative } C^*\text{-algebras}\}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\{\text{`noncommutative topological spaces'}\} \simeq \{\text{unital } \sout{\textit{commutative}}\ C^*\text{-algebras}\}$$

- arguably, the 1st predecessor theorem of the spectral formalism is *Weyl's law* (1911) on the rate of growth of the Laplace spectrum (ordered $\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \ldots$) of $\Omega \subset \mathbb{R}^d$



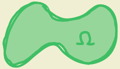
$$\#\{i : \lambda_i \leqslant \Lambda\} = \frac{\mathrm{vol(unit\ ball)}}{(2\pi)^d} \mathrm{vol}\,\Omega \cdot \Lambda^{d/2} + o(\Lambda^{d/2})$$

From this, you cannot answer positively Marek Kac' 1966-question. But you can 'hear the shape of Ω' knowing a *spectral triple.* [A, Connes, *JNCG* 2013] (and from it [Connes-van Suijlekom, *CMP* 2021] can hear an MP3; our story today is not entirely unrelated).

- **Origin of noncommutative topology** Connes' differential *noncommutative (nc) geometry* = nc topology [Gelfand, Najmark *Mat. Sbornik* '43] + metric [A. Connes, *NCG* '94]

$$\begin{array}{ccc} \{\text{compact Hausdorff topological spaces}\} & \simeq & \{\text{unital } \textit{commutative } C^*\text{-algebras}\} \\ \downarrow & & \downarrow \\ \{\text{`noncommutative topological spaces'}\} & \simeq & \{\text{unital } \sout{\textit{commutative}}\ C^*\text{-algebras}\} \end{array}$$

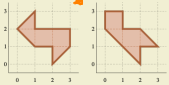
- arguably, the 1st predecessor theorem of the spectral formalism is *Weyl's law* (1911) on the rate of growth of the Laplace spectrum (ordered $\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \ldots$) of $\Omega \subset \mathbb{R}^d$



$u_\lambda|_{\partial\Omega} = 0$

$$\#\{i : \lambda_i \leqslant \Lambda\} = \frac{\text{vol(unit ball)}}{(2\pi)^d} \operatorname{vol}\Omega \cdot \Lambda^{d/2} + o(\Lambda^{d/2})$$

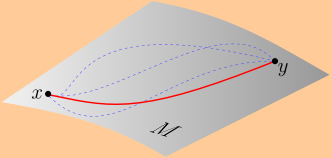
From this, you cannot answer positively Marek Kac' 1966-question. But you can 'hear the shape of Ω' knowing a *spectral triple*. [A, Connes, *JNCG* 2013] (and from it [Connes-van Suijlekom, *CMP* 2021] can hear an MP3; our story today is not entirely unrelated).



[Gordon, Webb, Wolpert, *Invent. Math.* '92]

Replace spin manifold (M, g) by $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

Connes' geodesic distance

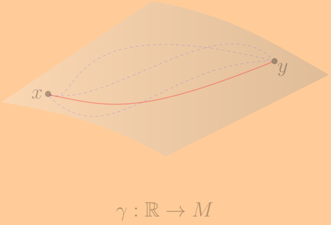


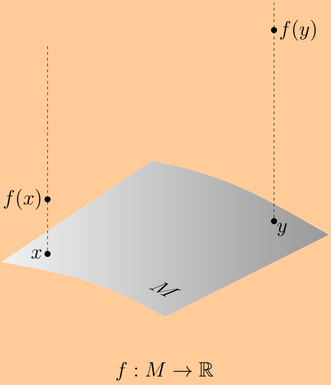
$\gamma : \mathbb{R} \to M$

$\inf_{\gamma \text{ as above}} \{ \int_\gamma \mathrm{d}s \} = d(x, y)$

Replace spin manifold (M, g) by $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

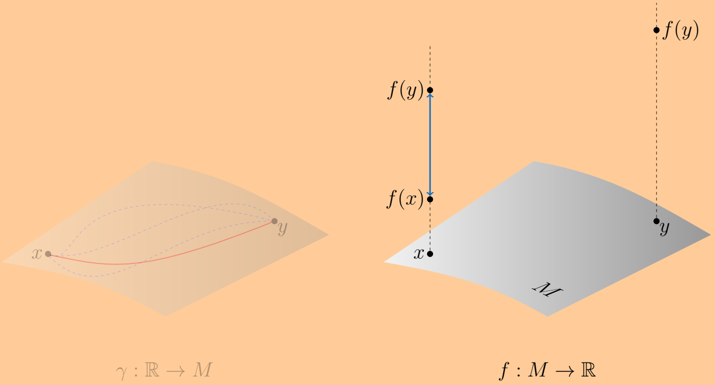
Connes' geodesic distance





Replace spin manifold (M, g) by $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

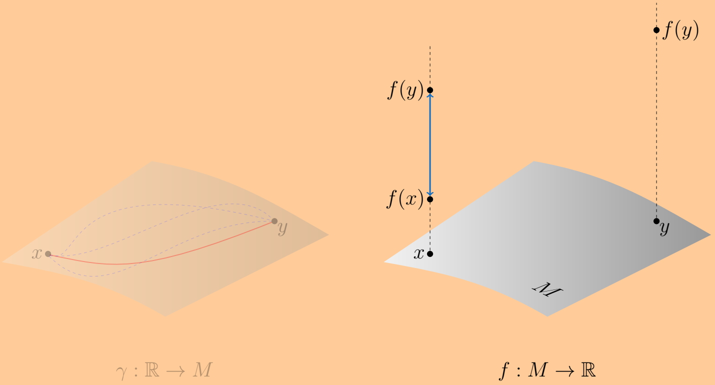
Connes' geodesic distance



$$|f(x) - f(y)|$$

Replace spin manifold (M, g) by $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

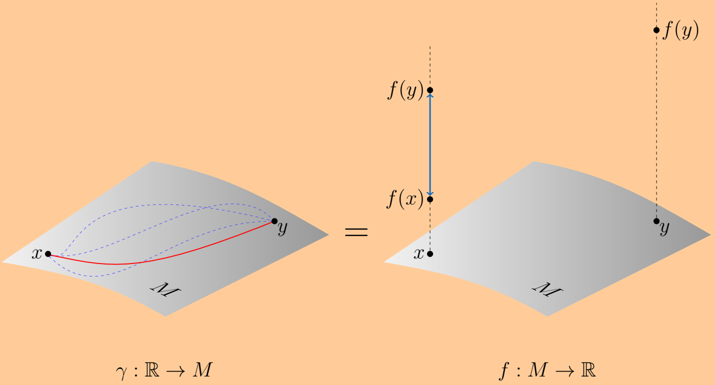
Connes' geodesic distance



$$\sup_{f \in C^\infty(M)} \left\{ |f(x) - f(y)| \; : \; ||D_M \circ f - f \circ D_M|| \leqslant 1 \right\}$$

Replace spin manifold (M, g) by $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

Connes' geodesic distance



$$\inf_{\gamma \text{ as above}} \{\textstyle\int_\gamma \mathrm{d}s\} = d(x, y) = \sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| \ : \ ||D_M \circ f - f \circ D_M|| \leqslant 1 \}$$

Commutative spectral triples

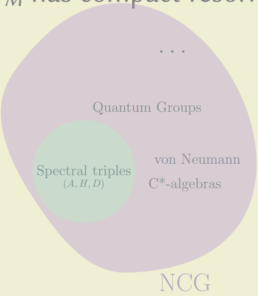
A spin manifold M yields (A_M, H_M, D_M)

- $A_M = C^\infty(M)$ is a comm. $*$-algebra
- $H_M := L^2(M, \mathbb{S})$ a repr. of A_M
- $D_M = -\mathrm{i}\gamma^\mu(\partial_\mu + \omega_\mu)$ is self-adjoint
- for each $a \in A_M$, $[D_M, a]$ is bounded, and in fact $[D_M, x^\mu] = -\mathrm{i}\gamma^\mu$

Commutative spectral triples

A spin manifold M yields (A_M, H_M, D_M)

- $A_M = C^\infty(M)$ is a comm. $*$-algebra
- $H_M := L^2(M, \mathbb{S})$ a repr. of A_M
- $D_M = -\mathrm{i}\gamma^\mu(\partial_\mu + \omega_\mu)$ is self-adjoint
- for each $a \in A_M$, $[D_M, a]$ is bounded, and in fact $[D_M, x^\mu] = -\mathrm{i}\gamma^\mu$
- D_M has compact resolvent . . .

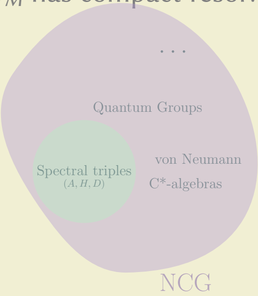


Commutative spectral triples

~~commutative~~

A spin manifold M yields (A_M, H_M, D_M)

- $A_M = C^\infty(M)$ is a comm. $*$-algebra
- $H_M := L^2(M, \mathbb{S})$ a repr. of A_M
- $D_M = -\mathrm{i}\gamma^\mu(\partial_\mu + \omega_\mu)$ is self-adjoint
- for each $a \in A_M$, $[D_M, a]$ is bounded, and in fact $[D_M, x^\mu] = -\mathrm{i}\gamma^\mu$
- D_M has compact resolvent . . .



A *spectral triple* (A, H, D) consists of

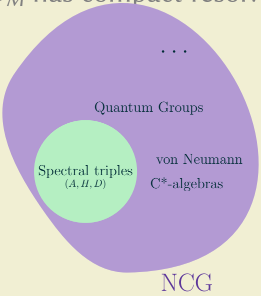
- a $*$-algebra A
- a representation H of A
- a self-adjoint operator D on H with compact resolvent and such that $[D, a]$ is bounded for each $a \in A$
- The 'commutative case' motivates

$$\Omega^1_D(A) := \Big\{ \sum_{\text{(finite)}} b[D, a] \mid a, b \in A \Big\}$$

Commutative spectral triples

A spin manifold M yields (A_M, H_M, D_M)

- $A_M = C^\infty(M)$ is a comm. $*$-algebra
- $H_M := L^2(M, \mathbb{S})$ a repr. of A_M
- $D_M = -\mathrm{i}\gamma^\mu(\partial_\mu + \omega_\mu)$ is self-adjoint
- for each $a \in A_M$, $[D_M, a]$ is bounded, and in fact $[D_M, x^\mu] = -\mathrm{i}\gamma^\mu$
- D_M has compact resolvent . . .



A *spectral triple* (A, H, D) consists of

- a $*$-algebra A
- a representation H of A
- a self-adjoint operator D on H with compact resolvent and such that $[D, a]$ is bounded for each $a \in A$
- The 'commutative case' motivates

$$\Omega^1_D(A) := \Big\{ \sum_{\text{(finite)}} b[D, a] \mid a, b \in A \Big\}$$

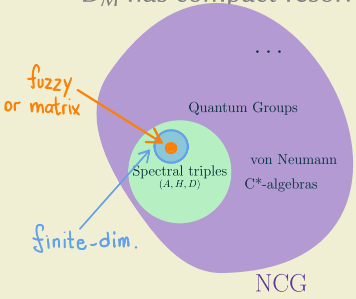
Connes' one-forms

- Reconstruction Theorem: Roughly formulated, states that commutative spectral triples$^{+axioms}$ are always Riemannian manifolds [A, Connes, *JNCG* '13] after efforts by [H. Figueroa, J. Gracia-Bondía-J. Varilly, A. Rennie-J. Várilly, '06]

Commutative spectral triples

There exists a ~~A spin~~ manifold M ~~yields~~ $(A, H, D) = (A_M, H_M, D_M)$ $\Leftarrow$

- $A_M = C^\infty(M)$ is a comm. $*$-algebra
- $H_M := L^2(M, \mathbb{S})$ a repr. of A_M
- $D_M = -\mathrm{i}\gamma^\mu(\partial_\mu + \omega_\mu)$ is self-adjoint
- for each $a \in A_M$, $[D_M, a]$ is bounded, and in fact $[D_M, x^\mu] = -\mathrm{i}\gamma^\mu$
- D_M has compact resolvent . . .



A abstract *spectral triple* (A, H, D) consists of

- a $*$-algebra A commutative
- a representation H of A
- a self-adjoint operator D on H with compact resolvent and such that $[D, a]$ is bounded for each $a \in A$
- The 'commutative case' motivates

$$\Omega^1_D(A) := \Big\{ \sum_{\text{(finite)}} b[D, a] \mid a, b \in A \Big\}$$

- RECONSTRUCTION THEOREM: Roughly formulated, states that commutative spectral triples$^{+axioms}$ are always Riemannian manifolds [A, Connes, *JNCG* '13] after efforts by [H. Figueroa, J. Gracia-Bondía-J. Varilly, A. Rennie-J. Várilly, '06]

NCG toolkit in high energy physics

- On a spectral triple (A, H, D) the (bosonic) classical action is given by

  $$S(D) = \mathrm{Tr}_H f(D/\Lambda)$$ [Chamseddine-Connes *CMP* '97]

  for a bump function f, Λ a scale. It's computed with heat kernel expansion

  [P. Gilkey, *J. Diff. Geom.* '75]

NCG toolkit in high energy physics

- On a spectral triple (A, H, D) the (bosonic) classical action is given by

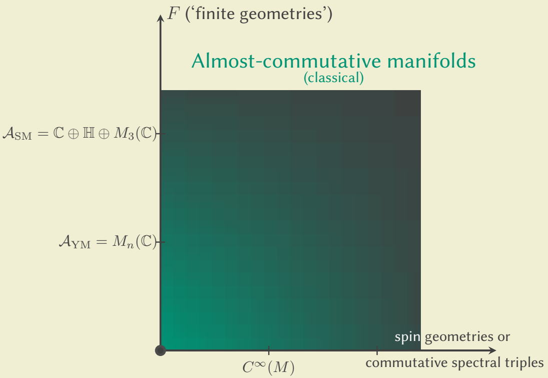
$$S(D) = \mathrm{Tr}_H f(D/\Lambda)$$ [Chamseddine-Connes *CMP* '97]

for a bump function f, Λ a scale. It's computed with heat kernel expansion

[P. Gilkey, *J. Diff. Geom.* '75]

- Realistic, classical models come from *almost-commutative manifolds* $M \times F$, where F is a finite-dim. spectral triple

$$(C^\infty(A_F), H_M \otimes H_F, D_M \otimes 1_F + \gamma_5 \otimes D_F)$$



NCG toolkit in high energy physics

- On a spectral triple (A, H, D) the (bosonic) classical action is given by

  $$S(D) = \mathrm{Tr}_H f(D/\Lambda)$$ [Chamseddine-Connes *CMP* '97]

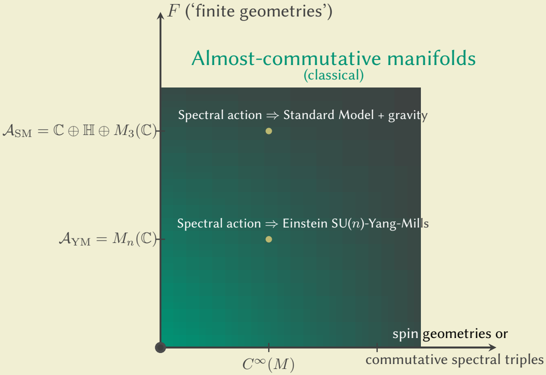
  for a bump function f, Λ a scale. It's computed with heat kernel expansion

  [P. Gilkey, *J. Diff. Geom.* '75]

- Realistic, classical models come from *almost-commutative manifolds* $M \times F$, where F is a finite-dim. spectral triple

  $$(C^\infty(A_F), H_M \otimes H_F, D_M \otimes 1_F + \gamma_5 \otimes D_F)$$

- applications require (A, H, D) to have a *reality* $J : H \to H$ antiunitary$^{+\text{ axioms}}$, implementing a right A-action on H



NCG toolkit in high energy physics

- On a spectral triple (A, H, D) the (bosonic) classical action is given by

$$S(D) = \mathrm{Tr}_H f(D/\Lambda)$$ [Chamseddine-Connes *CMP* '97]

  for a bump function f, Λ a scale. It's computed with heat kernel expansion

  [P. Gilkey, *J. Diff. Geom.* '75]

- Realistic, classical models come from *almost-commutative manifolds* $M \times F$, where F is a finite-dim. spectral triple

$$(C^\infty(A_F), H_M \otimes H_F, D_M \otimes 1_F + \gamma_5 \otimes D_F)$$

- applications require (A, H, D) to have a *reality* $J : H \to H$ antiunitary$^{\text{axioms}}$, implementing a right A-action on H

- *connections*: if S^G is a G-invariant functional on M

$$S^G \rightsquigarrow S^{\mathrm{Maps}(M,G)}$$

$$\mathrm{d} \rightsquigarrow \mathrm{d} + \mathbb{A} \qquad \mathbb{A} \in \Omega^1(M) \otimes \mathfrak{g}$$

$$\mathbb{A}' = u\mathbb{A}u^{-1} + u\mathrm{d}u^{-1} \qquad u \in \mathrm{Maps}(M, G)$$

NCG toolkit in high energy physics

- On a spectral triple (A, H, D) the (bosonic) classical action is given by

$$S(D) = \mathrm{Tr}_H f(D/\Lambda) \quad \text{[Chamseddine-Connes } \textit{CMP} \text{ '97]}$$

for a bump function f, Λ a scale. It's computed with heat kernel expansion

[P. Gilkey, *J. Diff. Geom.* '75]

- Realistic, classical models come from *almost-commutative manifolds* $M \times F$, where F is a finite-dim. spectral triple

$$(C^\infty(A_F), H_M \otimes H_F, D_M \otimes 1_F + \gamma_5 \otimes D_F)$$

- applications require (A, H, D) to have a *reality* $J : H \to H$ antiunitary$^{\text{axioms}}$, implementing a right A-action on H

- *connections*: if S^G is a G-invariant functional on M

$$S^G \rightsquigarrow S^{\mathrm{Maps}(M,G)}$$

$$\mathrm{d} \rightsquigarrow \mathrm{d} + \mathbb{A} \quad \mathbb{A} \in \Omega^1(M) \otimes \mathfrak{g}$$

$$\mathbb{A}' = u\mathbb{A}u^{-1} + u\mathrm{d}u^{-1} \quad u \in \mathrm{Maps}(M, G)$$

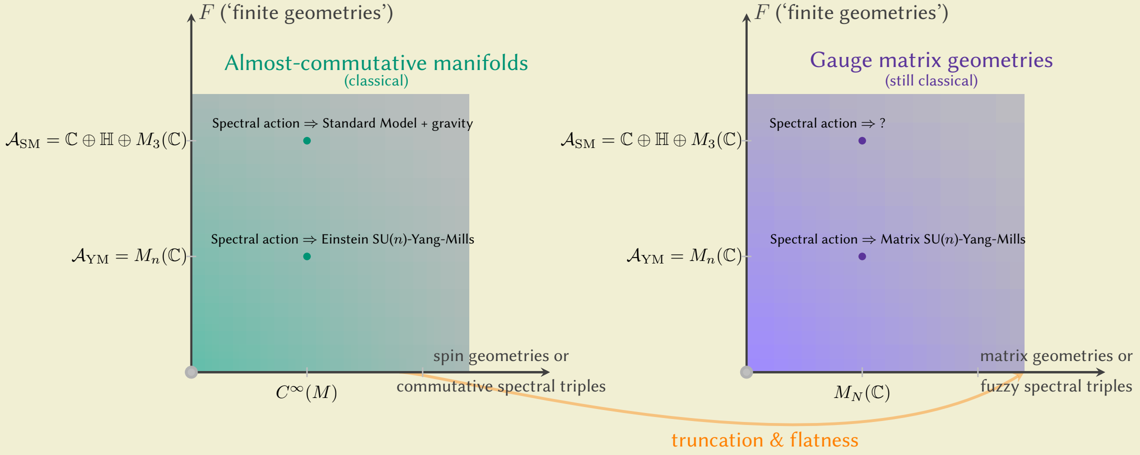
- given (A, H, D) and a Morita equivalent algebra B (i.e. $\mathrm{End}_A(E) \cong B$) yields new $(B, E \otimes_A H, \text{new } D\text{'s})$. For $A = B$, in fact a tower

$$\{(A, H, D + \omega \pm J\omega J^{-1})\}_{\omega \in \Omega^1_D(A)}$$

$$D_\omega \mapsto \mathrm{Ad}(u) D_\omega \mathrm{Ad}(u)^* = D_{\omega_u}$$

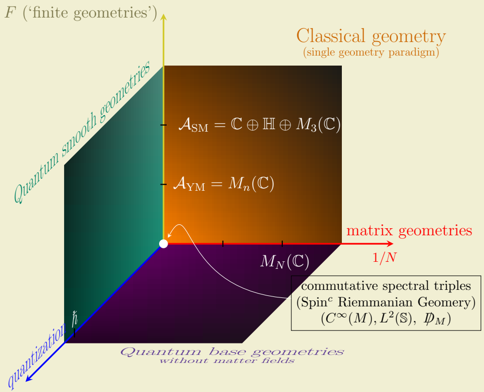
$$\omega \mapsto \omega_u = u\omega u^* + u[D, u^*] \quad u \in \mathcal{U}(A)$$

Main Result



Matrix Yang-Mills functional only using spectral triples [CP 2105.01025 *Ann. Henri Poincaré* **23** '22]. This obeys spectral triple axioms (unlike e.g. [Alekseev, Recknagel, Schomerus, *JHEP*, 00]) and quantisation leads to a pure matrix model.

Organisation



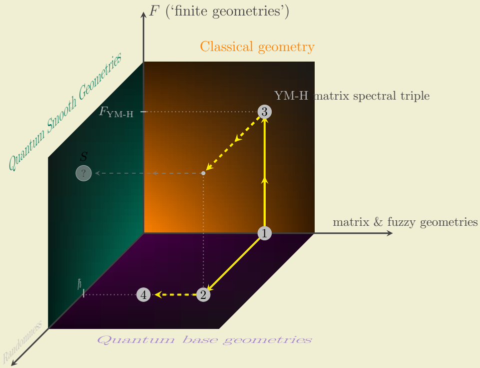
Aim: Make sense of

$$\mathcal{Z} = \int_{\text{Dirac}} e^{-\operatorname{Tr}_H f(D)} \mathrm{d}D$$

- *Plane* $(\hbar, 1/N, 0)$ *of 'base geometries'*
- *Plane* $(\hbar, 0, F) = \lim_{N\to\infty}(\hbar, 1/N, F)$
- *Plane* $(0, 1/N, F) = \lim_{\hbar\to 0}(\hbar, 1/N, F)$ *of classical geometries*

[CP 2105.01025]

Organisation



1 Matrix Geometries
[J. Barrett, *J. Math. Phys.* 2015]

2 Dirac ensembles [J. Barrett, L. Glaser, *J. Phys. A* 2016] and how to compute the spectral action [CP 1912.13288]

3 Gauge matrix spectral triples *(this talk)*
[CP 2105.01025]

4 Functional Renormalisation [CP 2007.10914] and [CP 2111.02858] *(not this talk)*

II. Fuzzy Geometries and Multimatrix Models

A *fuzzy geometry* of signature (p, q), so $\eta = \mathrm{diag}(+_p, -_q)$, consists of

- $A = M_N(\mathbb{C})$
- $H = \mathbb{S} \otimes M_N(\mathbb{C})$, with $\mathbb{S}$ a $\mathbb{C}\ell(p, q)$-module

. . . +axioms (omitted) that can be solved for D . . .

II. Fuzzy Geometries and Multimatrix Models

A *fuzzy geometry* of signature (p, q), so $\eta = \mathrm{diag}(+_p, -_q)$, consists of

- $A = M_N(\mathbb{C})$
- $H = \mathbb{S} \otimes M_N(\mathbb{C})$, with $\mathbb{S}$ a $\mathbb{C}\ell(p, q)$-module

... +axioms (omitted) that can be solved for D ...

- Fixing conventions for γ's, D in even dimensions: [J. Barrett, *J. Math. Phys.* '15]

$$D = \sum_J \Gamma^J_{\text{s.a.}} \otimes \{H_J, \ \cdot\ \} + \sum_J \Gamma^J_{\text{anti.}} \otimes [L_J, \ \cdot\ ]$$

multi-index J monot. increasing, $|J|$ odd, $H_J^* = H_J$, $L_J^* = -L_J$

II. Fuzzy Geometries and Multimatrix Models

A *fuzzy geometry* of signature (p, q), so $\eta = \mathrm{diag}(+_p, -_q)$, consists of

- $A = M_N(\mathbb{C})$
- $H = \mathbb{S} \otimes M_N(\mathbb{C})$, with $\mathbb{S}$ a $\mathbb{C}\ell(p, q)$-module

  . . . +axioms (omitted) that can be solved for D . . .

- Fixing conventions for γ's, D in even dimensions: [J. Barrett, *J. Math. Phys.* '15]

$$D = \sum_J \Gamma^J_{\text{s.a.}} \otimes \{H_J, \ \cdot \ \} + \sum_J \Gamma^J_{\text{anti.}} \otimes [L_J, \ \cdot \ ]$$

  multi-index J monot. increasing, $|J|$ odd, $H_J^* = H_J$, $L_J^* = -L_J$

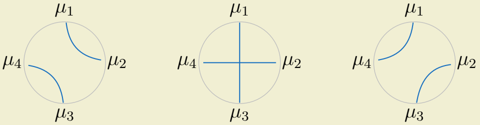
- Examples: [J. Barrett, L. Glaser, *J. Phys. A* 2016]
  - $D_{(1,1)} = \gamma^1 \otimes [L, \ \cdot \ ] + \gamma^2 \otimes \{H, \ \cdot \ \}$
  - $D_{(0,4)} = \sum_\mu \gamma^\mu \otimes [L_\mu, \ \cdot \ ] + \gamma^{\hat{\mu}} \otimes \{H_{\hat{\mu}}, \ \cdot \ \}$ $\qquad$ ($\hat{\mu}$ = omit μ from (0123))

  so we will get double traces from $\mathrm{Tr}_H = \mathrm{Tr}_{\mathbb{S}} \otimes \mathrm{Tr}_{M_N(\mathbb{C})} = \mathrm{Tr}_{\mathbb{S}} \otimes \mathrm{Tr}_N^{\otimes 2}$

Notation: $\mathrm{Tr}_V\, X$ is the trace on operators $X : V \to V$, $\mathrm{Tr}_V\, 1 = \dim V$. So $\mathrm{Tr}_N\, 1 = N$ but $\mathrm{Tr}_{M_N(\mathbb{C})}(1) = N^2$.

- $\mathrm{Tr}_H = \mathrm{Tr}_{\mathbb{S}} \otimes \mathrm{Tr}_{M_N^{\mathbb{C}}}$, and a tool to organize the first trace is chord diagrams:

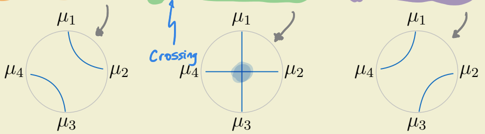
$$\mathrm{Tr}_{\mathbb{S}}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}) = \dim \mathbb{S}(\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + (-)\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \eta^{\mu_2\mu_3}\eta^{\mu_1\mu_4})$$



([CP 1912.13288] appeared two weeks after [Sati-Schreiber, 1912.10425] who relate fuzzy spaces to chord diagrams too)

- $\mathrm{Tr}_H = \mathrm{Tr}_{\mathbb{S}} \otimes \mathrm{Tr}_{M_N^{\mathbb{C}}}$, and a tool to organize the first trace is chord diagrams:

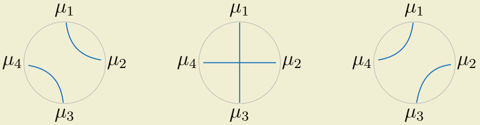
$$\mathrm{Tr}_{\mathbb{S}}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}) = \dim \mathbb{S}\left(\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + (-)\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \eta^{\mu_2\mu_3}\eta^{\mu_1\mu_4}\right)$$



([CP 1912.13288] appeared two weeks after [Sati-Schreiber, 1912.10425] who relate fuzzy spaces to chord diagrams too)

- $\mathrm{Tr}_H = \mathrm{Tr}_{\mathbb{S}} \otimes \mathrm{Tr}_{M_N^{\mathbb{C}}}$, and a tool to organize the first trace is chord diagrams:

$$\mathrm{Tr}_{\mathbb{S}}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}) = \dim \mathbb{S}(\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + (-)\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \eta^{\mu_2\mu_3}\eta^{\mu_1\mu_4})$$



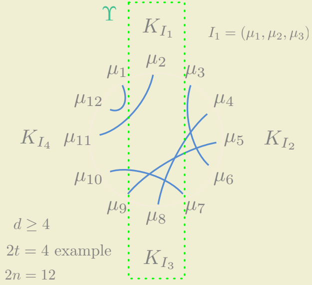
([CP 1912.13288] appeared two weeks after [Sati-Schreiber, 1912.10425] who relate fuzzy spaces to chord diagrams too)

- for dimension-d geometries, the combinatorial formula [CP' 19] reads

$$\frac{1}{\dim \mathbb{S}} \mathrm{Tr}(D^{2t}) = \overbrace{\sum_{I_1,\ldots,I_{2t}\in\Lambda_d^-}}^{\text{if } J\in\Lambda_d,\, \mathrm{d}x^J \neq 0 \text{ on } \mathbb{R}^d} \Bigg\{ \overbrace{\sum_{\substack{\chi\in \mathrm{CD}_{2n} \\ 2n=\sum_i |I_i|}} \chi^{I_1\ldots I_{2t}}}^{\text{decorated chord diags}}$$

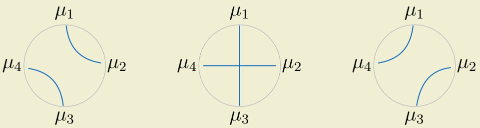
$$\times \Bigg(\sum_{\Upsilon\in\mathscr{P}_{2t}} \mathrm{sgn}(I_\Upsilon) \times \mathrm{Tr}_N(K_{I_{\Upsilon^c}}) \times \mathrm{Tr}_N\big[(K^T)_{I_\Upsilon}\big]\Bigg)\Bigg\}$$

$\mathscr{P}_{2t} = 2^{\{1,\ldots,2t\}}$, $K_I^* = \pm K_I$, $\mathrm{sgn}(I_\Upsilon) \in \mathbb{Z}_2$



- $\mathrm{Tr}_H = \mathrm{Tr}_{\mathbb{S}} \otimes \mathrm{Tr}_{M_N^{\mathbb{C}}}$, and a tool to organize the first trace is chord diagrams:

$$\mathrm{Tr}_{\mathbb{S}}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}) = \dim \mathbb{S}(\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + (-)\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \eta^{\mu_2\mu_3}\eta^{\mu_1\mu_4})$$



([CP 1912.13288] appeared two weeks after [Sati-Schreiber, 1912.10425] who relate fuzzy spaces to chord diagrams too)

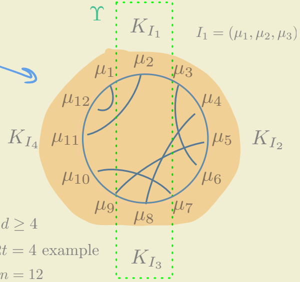
- for dimension-d geometries, the combinatorial formula [CP' 19] reads

$$\frac{1}{\dim \mathbb{S}} \mathrm{Tr}(D^{2t}) = \overbrace{\sum_{I_1,\ldots,I_{2t}\in\Lambda_d^-}}^{\text{if } J\in\Lambda_d,\ \mathrm{d}x^J \neq 0 \text{ on } \mathbb{R}^d} \Bigg\{ \overbrace{\sum_{\substack{\chi\in \mathrm{CD}_{2n} \\ 2n=\sum_i |I_i|}} \chi^{I_1\ldots I_{2t}}}^{\text{decorated chord diags}} \in \mathbb{Z}$$

$$\times \Bigg(\sum_{\Upsilon\in\mathscr{P}_{2t}} \mathrm{sgn}(I_\Upsilon) \times \mathrm{Tr}_N(K_{I_{\Upsilon^c}}) \times \mathrm{Tr}_N\left[(K^T)_{I_\Upsilon}\right]\Bigg)\Bigg\}$$

$\mathscr{P}_{2t} = 2^{\{1,\ldots,2t\}}$, $K_I^* = \pm K_I$, $\mathrm{sgn}(I_\Upsilon) \in \mathbb{Z}_2$

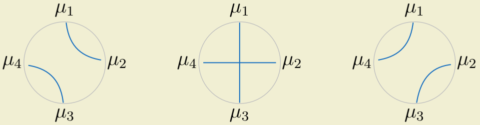
from chords



$d \geq 4$
$2t = 4$ example
$2n = 12$

- $\mathrm{Tr}_H = \mathrm{Tr}_{\mathbb{S}} \otimes \mathrm{Tr}_{M_N^{\mathbb{C}}}$, and a tool to organize the first trace is chord diagrams:

$$\mathrm{Tr}_{\mathbb{S}}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}) = \dim \mathbb{S}(\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + (-)\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \eta^{\mu_2\mu_3}\eta^{\mu_1\mu_4})$$

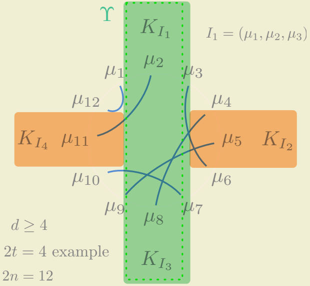


([CP 1912.13288] appeared two weeks after [Sati-Schreiber, 1912.10425] who relate fuzzy spaces to chord diagrams too)

- for dimension-d geometries, the combinatorial formula [CP' 19] reads

$$\frac{1}{\dim \mathbb{S}} \mathrm{Tr}(D^{2t}) = \overbrace{\sum_{I_1,\ldots,I_{2t}\in\Lambda_d^-}}^{\text{if } J\in\Lambda_d,\ \mathrm{d}x^J \neq 0 \text{ on } \mathbb{R}^d} \Bigg\{ \overbrace{\sum_{\substack{\chi\in \mathrm{CD}_{2n}\\ 2n=\sum_i |I_i|}} \chi^{I_1\ldots I_{2t}}}^{\text{decorated chord diags}} \times \Bigg(\sum_{\Upsilon\in\mathscr{P}_{2t}} \mathrm{sgn}(I_\Upsilon) \times \mathrm{Tr}_N(K_{I_{\Upsilon^c}}) \times \mathrm{Tr}_N\big[(K^T)_{I_\Upsilon}\big] \Bigg) \Bigg\}$$

$\mathscr{P}_{2t} = 2^{\{1,\ldots,2t\}}$, $K_I^* = \pm K_I$, $\mathrm{sgn}(I_\Upsilon) \in \mathbb{Z}_2$



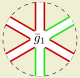
Multimatrix models with multi-traces

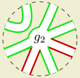
- The chord-diagram description holds in general dimension and signature [CP '19]

$$\mathcal{Z} = \int_{\text{DIRAC}} \mathrm{e}^{-\operatorname{Tr}_H f(D)} \mathrm{d}D \quad (\hbar = 1)$$
$$= \int_{M_{p,q}} \mathrm{e}^{-N \operatorname{Tr}_N P - \operatorname{Tr}_N^{\otimes 2}(Q_{(1)} \otimes Q_{(2)})} \mathrm{d}\mathbb{X}_{\text{LEB}}$$

- $\mathbb{X} \in M_{p,q}$ = products of $\mathfrak{su}(N)$ and $\mathcal{H}_N$
- $\mathrm{d}\mathbb{X}_{\text{LEB}}$ is the Lebesgue measure on $M_{p,q}$
- P, $Q_{(i)}$ in $\mathbb{C}_{\langle k \rangle} = \mathbb{C}\langle \mathbb{X} \rangle$ nc-polynomials

- $\mathcal{Z}_{\text{FORMAL}}$ leads to colored ribbon graphs

$g_1 \operatorname{Tr}_N(ABBBAB) \quad \leftrightarrow$ 

$g_2 \operatorname{Tr}_N^{\otimes 2}(AABABA \otimes AA) \quad \leftrightarrow$ 

(labelled cylinder)

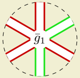
Multimatrix models with multi-traces

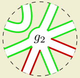
- The chord-diagram description holds in general dimension and signature [CP '19]

$$\mathcal{Z} = \int_{\text{Dirac}} e^{-\operatorname{Tr}_H f(D)} \mathrm{d}D \quad (\hbar = 1)$$
$$= \int_{M_{p,q}} e^{-N \operatorname{Tr}_N P - \operatorname{Tr}_N^{\otimes 2}(Q_{(1)} \otimes Q_{(2)})} \mathrm{d}\mathbb{X}_{\text{Leb}}$$

- $\mathbb{X} \in M_{p,q}$ = products of $\mathfrak{su}(N)$ and $\mathcal{H}_N$
- $\mathrm{d}\mathbb{X}_{\text{Leb}}$ is the Lebesgue measure on $M_{p,q}$
- P, $Q_{(i)}$ in $\mathbb{C}_{\langle k \rangle} = \mathbb{C}\langle \mathbb{X} \rangle$ nc-polynomials

- $\mathcal{Z}_{\text{formal}}$ leads to colored ribbon graphs

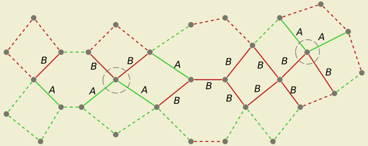
$g_1 \operatorname{Tr}_N (ABBBAB) \quad \leftrightarrow$ 

$g_2 \operatorname{Tr}_N^{\otimes 2} (AABABA \otimes AA) \quad \leftrightarrow$ 

(labelled cylinder)

- Multitrace: 'touching interactions' [Klebanov, *PRD* '95], wormholes [Ambjørn-Jurkiewicz-Loll-Vernizzi, *JHEP* '01], 'stuffed maps' [G. Borot *Ann. Inst. Henri Poincaré Comb. Phys. Interact.* '14], AdS/CFT [Witten, `hep-th/0112258`]

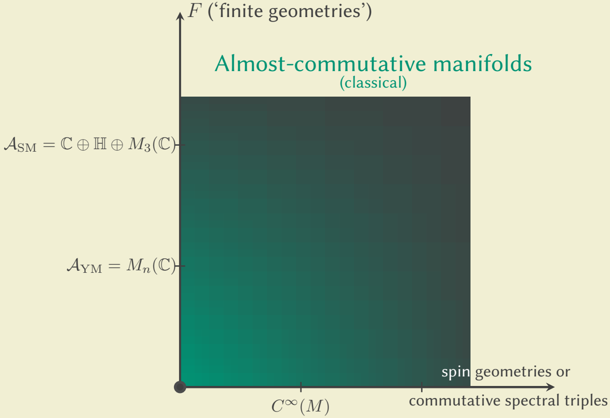
- Ribbon graphs: Enumeration of maps [Brezin, Itzykson, Parisi, Zuber, *CMP* '78], here 'face-worded'



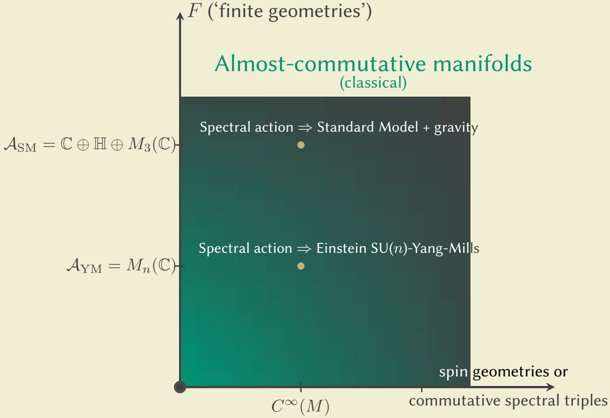
& intersection num. of ψ-classes [Kontsevich, *CMP*, '92]

$$\sum_{a_1+\ldots+a_n=\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdot \psi_2^{a_2} \cdots \psi_n^{a_n} \prod_{j=1}^{n} \frac{(2a_j-1)!!}{s_j^{2a_j+1}}$$
$$= \sum_{G \text{ trivalent of type } (g,n)} \frac{2^{2g-2+n}}{\#\mathrm{Aut}(G)} \prod_{e \in G} \frac{1}{s_{L(e)} + s_{R(e)}}$$

III. Yang-Mills-Higgs matrix theory



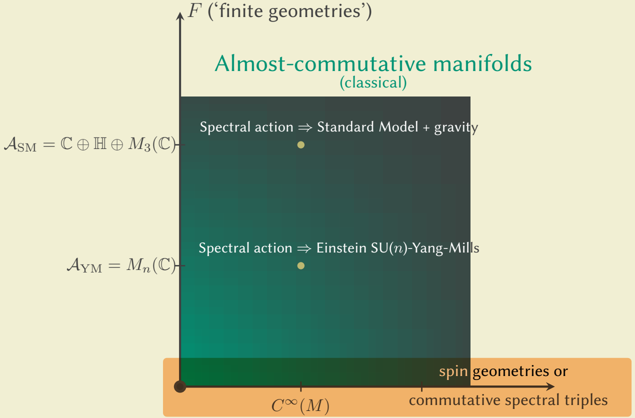
III. Yang-Mills-Higgs matrix theory



$$\mathcal{Z}_{\mathrm{AC}} \overset{?}{=} \int_{\mathrm{DIRAC}} e^{-\frac{1}{\hbar}\operatorname{Tr} f(D)} \mathrm{d}D$$

(hard for almost-commutative manifolds)

III. Yang-Mills-Higgs matrix theory



$$\mathcal{Z}_{\mathrm{AC}} \stackrel{?}{=} \int_{\mathrm{Dirac}} e^{-\frac{1}{\hbar}\operatorname{Tr} f(D)} \mathrm{d}D$$

(hard for almost-commutative manifolds)

replace fin-dim approx

Definition [CP 2105.01025]. We define a *gauge matrix spectral triple* $G_\ell \times F$ as the spectral triple product of a fuzzy geometry G_ℓ with a finite geometry $F = (A_F, H_F, D_F)$, $\dim A_F < \infty$.

LEMMA-DEFINITION [CP 2105.01025]. Consider a gauge matrix spectral triple $G_\ell \times F$ with

$$F = (M_n(\mathbb{C}), M_n(\mathbb{C}), D_F)$$

and G_ℓ Riemannian ($d = 4$) fuzzy geometry on $M_N(\mathbb{C})$, whose fluctuated Dirac op. is

$$D_\omega = \sum_{\mu=0}^{3} \overbrace{\gamma^\mu \otimes (\mathscr{l}_\mu + \mathscr{a}_\mu) + \gamma^{\hat\mu} \otimes (\mathscr{x}_\mu + \mathscr{s}_\mu)}^{D_{\text{gauge}}} + \overbrace{\gamma \otimes \Phi}^{D_{\text{Higgs}}}, \qquad \mathscr{a}_\mu = \text{‘gauge potential’}, \mathscr{x}_\mu = \text{spin connection?}$$

The *field strength* is given by

$$\mathscr{F}_{\mu\nu} := [\overbrace{\mathscr{l}_\mu + \mathscr{a}_\mu}^{\mathscr{d}_\mu}, \mathscr{l}_\nu + \mathscr{a}_\nu] =: [\mathsf{F}_{\mu\nu}, \ \cdot \ ]$$

Lemma-Definition [CP 2105.01025]. Consider a gauge matrix spectral triple $G_\ell \times F$ with

$$F = (M_n(\mathbb{C}), M_n(\mathbb{C}), D_F)$$

and G_ℓ Riemannian ($d = 4$) fuzzy geometry on $M_N(\mathbb{C})$, whose fluctuated Dirac op. is

$$D_\omega = \sum_{\mu=0}^{3} \overbrace{\gamma^\mu \otimes (\mathscr{l}_\mu + \mathscr{a}_\mu) + \gamma^{\hat\mu} \otimes (\mathscr{x}_\mu + \mathscr{s}_\mu)}^{D_{\text{gauge}}} + \overbrace{\gamma \otimes \Phi}^{D_{\text{Higgs}}}, \qquad \mathscr{a}_\mu = \text{'gauge potential'}, \mathscr{x}_\mu = \text{spin connection?}$$

The *field strength* is given by

$$\mathscr{F}_{\mu\nu} := [\overbrace{\mathscr{l}_\mu + \mathscr{a}_\mu}^{\mathscr{d}_\mu}, \mathscr{l}_\nu + \mathscr{a}_\nu] =: [\mathsf{F}_{\mu\nu}, \ \cdot\ ]$$

Lemma. The gauge group $\mathrm{G}(A) \cong \mathrm{PU}(N) \times \mathrm{PU}(n)$ acts as follows

$$\mathsf{F}_{\mu\nu} \mapsto \mathsf{F}^u_{\mu\nu} = u\mathsf{F}_{\mu\nu}u^* \text{ for all } u \in \mathrm{G}(A)$$

Lemma-Definition [CP 2105.01025]. Consider a gauge matrix spectral triple $G_\ell \times F$ with

$$F = (M_n(\mathbb{C}), M_n(\mathbb{C}), D_F)$$

and G_ℓ Riemannian ($d = 4$) fuzzy geometry on $M_N(\mathbb{C})$, whose fluctuated Dirac op. is

$$D_\omega = \sum_{\mu=0}^{3} \overbrace{\gamma^\mu \otimes (\ell_\mu + a_\mu) + \gamma^{\hat{\mu}} \otimes (x_\mu + s_\mu)}^{D_{\text{gauge}}} + \overbrace{\gamma \otimes \Phi}^{D_{\text{Higgs}}}, \qquad a_\mu = \text{'gauge potential'}, x_\mu = \text{spin connection?}$$

The *field strength* is given by

$$\mathscr{F}_{\mu\nu} := [\overbrace{\ell_\mu + a_\mu}^{d_\mu}, \ell_\nu + a_\nu] =: [\mathsf{F}_{\mu\nu}, \ \cdot\ ]$$

Lemma. The gauge group $\mathrm{G}(A) \cong \mathrm{PU}(N) \times \mathrm{PU}(n)$ acts as follows

$$\mathsf{F}_{\mu\nu} \mapsto \mathsf{F}^u_{\mu\nu} = u\mathsf{F}_{\mu\nu}u^* \text{ for all } u \in \mathrm{G}(A)$$

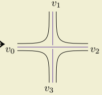
The content of the quadratic-quartic Spectral Action ...

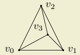
Meaning	Random matrix case, flat $d = 4$ Riem.	Smooth operator
	Tr = trace of ops. $M_N \otimes M_n \to M_N \otimes M_n$	
Derivation	$\ell_\mu = [L_\mu \otimes 1_n, \ \cdot \ ]$	∂_i
Gauge potential	$a_\mu = [A_\mu, \ \cdot \ ]$	$\mathbb{A}_i$
Covariant derivative	$d_\mu = \ell_\mu + a_\mu$	$\mathbb{D}_i = \partial_i + \mathbb{A}_i$

Meaning	Random matrix case, flat $d = 4$ Riem. Tr = trace of ops. $M_N \otimes M_n \to M_N \otimes M_n$	Smooth operator
Derivation	$\ell_\mu = [L_\mu \otimes 1_n, \ \cdot\ ]$	∂_i
Gauge potential	$a_\mu = [A_\mu, \ \cdot\ ]$	$\mathbb{A}_i$
Covariant derivative	$d_\mu = \ell_\mu + a_\mu$	$\mathbb{D}_i = \partial_i + \mathbb{A}_i$
Field strength	$[d_\mu, d_\nu] = \overbrace{[\ell_\mu, \ell_\nu]}^{\not\equiv 0} + [\ell_\mu, a_\nu] - [\ell_\nu, a_\mu] + [a_\mu, a_\nu]$	$[\mathbb{D}_i, \mathbb{D}_j] = \overbrace{[\partial_i, \partial_j]}^{\equiv 0} + \partial_i \mathbb{A}_j - \partial_j \mathbb{A}_i + [\mathbb{A}_i, \mathbb{A}_j]$
Yang-Mills action	$-\frac{1}{4}\mathrm{Tr}(\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu})$	$-\frac{1}{4}\int_M \mathrm{Tr}_{\mathfrak{su}(n)}(\mathbb{F}_{ij}\mathbb{F}^{ij})\mathrm{vol}$
Higgs field	Φ	h
Higgs potential	$\mathrm{Tr}(f_2\Phi^2 + f_4\Phi^4)$	$\int_M (f_2\lvert h\rvert^2 + f_4\lvert h\rvert^4)\mathrm{vol}$
Gauge-Higgs coupling	$-\mathrm{Tr}(d_\mu\Phi d^\mu\Phi)$	$-\int_M \lvert\mathbb{D}_i h\rvert^2\mathrm{vol}$

Meaning	Random matrix case, flat $d=4$ Riem.	Smooth operator
	Tr = trace of ops. $M_N \otimes M_n \to M_N \otimes M_n$	
Derivation	$\ell_\mu = [L_\mu \otimes 1_n, \ \cdot \ ]$	∂_i
Gauge potential	$a_\mu = [A_\mu, \ \cdot \ ]$	$\mathbb{A}_i$
Covariant derivative	$d_\mu = \ell_\mu + a_\mu$	$\mathbb{D}_i = \partial_i + \mathbb{A}_i$
Field strength	$[d_\mu, d_\nu] = \overbrace{[\ell_\mu, \ell_\nu]}^{\not\equiv 0} + [\ell_\mu, a_\nu] - [\ell_\nu, a_\mu] + [a_\mu, a_\nu]$	$[\mathbb{D}_i, \mathbb{D}_j] = \overbrace{[\partial_i, \partial_j]}^{\equiv 0} + \partial_i \mathbb{A}_j - \partial_j \mathbb{A}_i + [\mathbb{A}_i, \mathbb{A}_j]$
Yang-Mills action	$-\frac{1}{4}\mathrm{Tr}(\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu})$	$-\frac{1}{4}\int_M \mathrm{Tr}_{\mathfrak{su}(n)}(\mathbb{F}_{ij}\mathbb{F}^{ij})\mathrm{vol}$
Higgs field	Φ $\Downarrow$ forces $f_4 = 1$	h
Higgs potential	$\mathrm{Tr}(f_2\Phi^2 + f_4\Phi^4)$	$\int_M (f_2\lvert h\rvert^2 + f_4\lvert h\rvert^4)\mathrm{vol}$
Gauge-Higgs coupling	$-\mathrm{Tr}(d_\mu \Phi d^\mu \Phi)$	$-\int_M \lvert\mathbb{D}_i h\rvert^2 \mathrm{vol}$

$f(D) = \sum_{p \geq 1} f_p D^p$

and propagators and $\sim (\ell_\mu)_{ij}(\ell_\nu)_{jm}(\ell^\mu)_{ml}(\ell^\nu)_{li} \leftrightarrow$ 

$\sim$ 

Conclusion

- spectral triple $\equiv$ spin manifiold, after relaxing commutativity
- spin $M \times \{\text{finite spectral triple}\} \equiv$ almost-commutative (reproduces classical Standard Model, but hard to quantize)
- *fuzzy* or *matrix* geometry $\approx$ finite spectral triple + $\mathbb{C}\ell$-action; [CP 19] computes spectral action

Conclusion

- spectral triple $\equiv$ spin manifiold, after relaxing commutativity
- spin $M \times \{\text{finite spectral triple}\} \equiv$ almost-commutative (reproduces classical Standard Model, but hard to quantize)
- *fuzzy* or *matrix* geometry $\approx$ finite spectral triple + $\mathbb{C}\ell$-action; [CP 19] computes spectral action
- fuzzy $\times$ finite = gauge matrix spectra triple, it is $\mathrm{PU}(n)$-Yang-Mills(-Higgs) if the fin. geom. algebra is $M_n(\mathbb{C})$; partition func. is a k-matrix model, k large.

$$\mathcal{Z}_{\text{GAUGE MATRIX}} = \int_{\text{DIRACS}} \mathrm{e}^{-\operatorname{Tr}_H f(D)} \mathrm{d}D = \int_{\text{base}\times\text{YM}\times\text{Higgs}} \mathrm{e}^{-S_{\text{gauge}}-S_{\text{H}}-S_{\text{gauge-H}}-S} \mathrm{d}\mu_{\mathrm{G}}(L)\, \mathrm{d}\mu_{\mathrm{G}}(A)\, \mathrm{d}\Phi$$

with $(\mathrm{L}, \mathrm{A}, \phi) \in [\mathfrak{su}(N)]^{\times 4} \times [\mathcal{N}^{\text{gauge}}_{N,n}]^{\times 4} \times \mathcal{N}^{\text{Higgs}}_{N,n}$

- small step towards [Eq. 1.892, Connes Marcolli, *NCG, QFT and motives*, 2007]

  « *The far distant goal is to set up a functional integral evaluating spectral observables* $\mathscr{S}$ $\quad \langle \mathscr{S} \rangle = \int \mathscr{S}\, \mathrm{e}^{-\operatorname{Tr} f(D/\Lambda) - \frac{1}{2}\langle J\psi, D\psi\rangle + \rho(e,D)} \mathrm{d}e\, \mathrm{d}\psi\, \mathrm{d}D$ »

Gaußians

close relatives of

$u(N)$

$\otimes\, u(n)$

Conclusion

- spectral triple ≡ spin manifold, after relaxing commutativity
- spin $M \times$ {finite spectral triple} ≡ almost-commutative (reproduces classical Standard Model, but hard to quantize)
- *fuzzy* or *matrix* geometry ≈ finite spectral triple + $\mathbb{C}\ell$-action; [CP 19] computes spectral action
- fuzzy × finite = gauge matrix spectra triple, it is $\mathrm{PU}(n)$-Yang-Mills(-Higgs) if the fin. geom. algebra is $M_n(\mathbb{C})$; partition func. is a k-matrix model, k large.

$$\mathcal{Z}_{\text{GAUGE MATRIX}} = \int_{\text{DIRACS}} e^{-\operatorname{Tr}_H f(D)} \mathrm{d}D = \int_{\text{base}\times\text{YM}\times\text{Higgs}} e^{-S_{\text{gauge}}-S_{\text{H}}-S_{\text{gauge-H}}-S_{\Phi}} \mathrm{d}\mu_G(L)\, \mathrm{d}\mu_G(A)\, \mathrm{d}\Phi$$

with $(\mathsf{L},\mathsf{A},\phi) \in [\mathfrak{su}(N)]^{\times 4} \times [\mathcal{N}_{N,n}^{\text{gauge}}]^{\times 4} \times \mathcal{N}_{N,n}^{\text{Higgs}}$

- small step towards [Eq. 1.892, Connes Marcolli, *NCG, QFT and motives*, 2007]

« *The far distant goal is to set up a functional integral evaluating spectral observables* $\mathcal{S}$ $\quad \langle \mathcal{S} \rangle = \int \mathcal{S}\, e^{-\operatorname{Tr} f(D/\Lambda) - \frac{1}{2}\langle J\psi, D\psi\rangle + \rho(e,D)} \mathrm{d}e\, \mathrm{d}\psi\, \mathrm{d}D$ »

dziękuję* bardzo za uwagę (i.e. thanks)

*też organizatorom!

References: [CP 1912.13288] [CP 2007.10914] [CP 2102.06999] [CP 2105.01025] [CP 2111.02858]