STRUCTURES CLUSTER OF EXCELLENCE

UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

A friendly introduction to random spectral triples

(at finite resolution)

Young Researchers Convent, Oct. 2022

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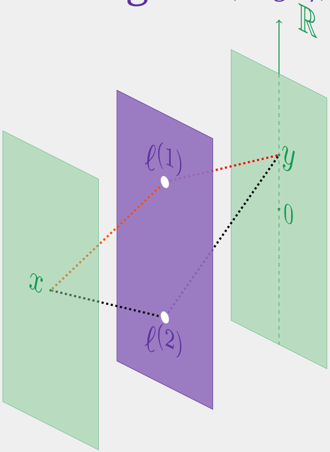
Based on:

1912.13288 [pl] ; 2007.10914 [pl] ; 2105.01025 [pl,de] ; 2111.02858 [pl,de]

pl TEAM Fundacja na Rzecz Nauki Polskiej
de ERC, indirectly & DFG-Structures Excellence Cluster

Path integrals (roughly)



double-slit experiment

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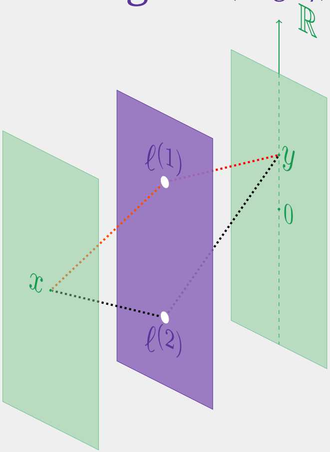
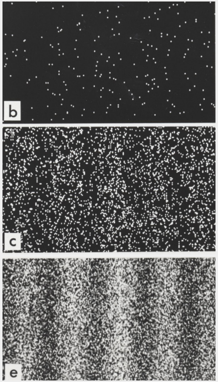


double-slit experiment

Results & Interpretation



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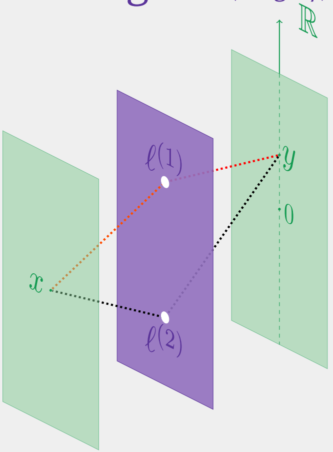

double-slit experiment

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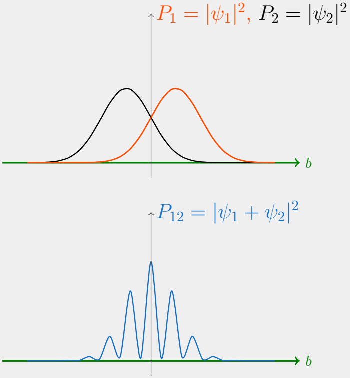
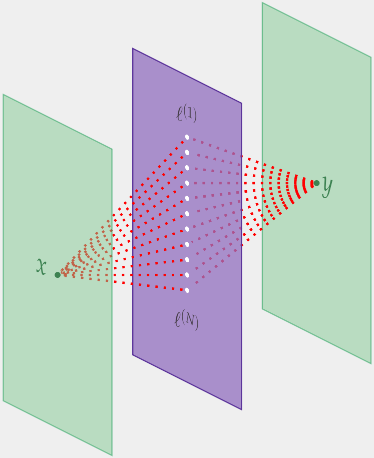
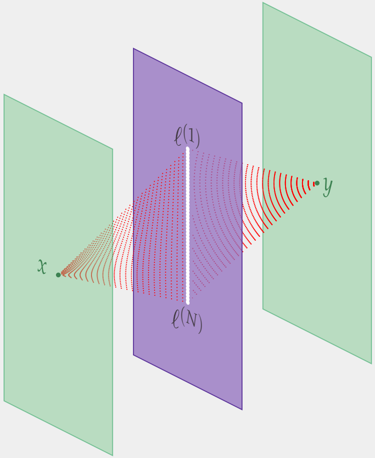


$$\psi_1, \psi_2 : \mathbb{R} \to \mathbb{C},\ P_i(y) = |\psi_i(y)|^2$$



- transition probability amplitude
  $K(y, x) = \langle y, x \rangle$ from an inner product on a Hilbert space

$$K(y, x) = \sum_{j=1}^{N} K(y, \ell^{(j)}) K(\ell^{(j)}, x)$$



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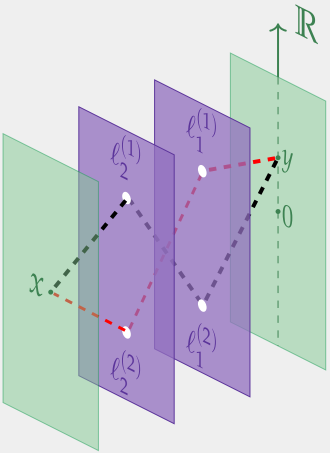
$$K(y, x) = \sum_{j=1}^{N} K(y, \ell^{(j)}) K(\ell^{(j)}, x)$$

$$\to \int K(y, \ell) K(\ell, x) \mathrm{d}\ell =: K^{\star 2}(y, x)$$

These conditions make sense if displayed in full notation.



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- ...it works for more screens...

$$K(y,x) = \sum_{i,j=1}^{2} K(y,\ell_1^{(i)})K(\ell_1^{(i)},\ell_2^{(j)})K(\ell_2^{(j)},x)$$

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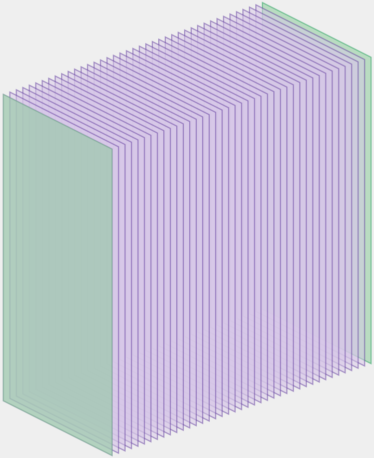

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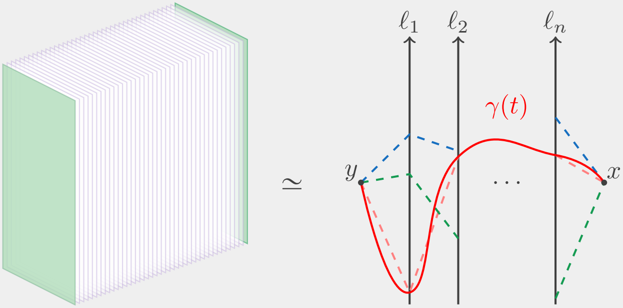
$\{\text{sum over all holes of all screens}\} \simeq \{\text{sum over all paths}\}$



$\lim_n c_n K^{\star n} = K$ has as solution a path integral

$$K(y, t_y; x, t_y) = \int_\gamma \exp\left\{\frac{\mathrm{i}}{\hbar} S[\gamma(t)]\right\} \mathrm{d}\gamma(t)$$

According to A. Zee, this is one of Feynman's *Gedankenexperimente*.

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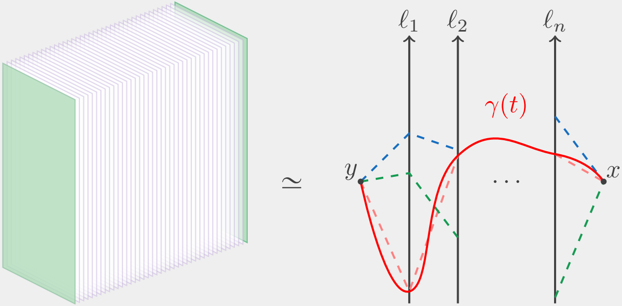

$\lim_n c_n K^{\star n} = K$ has as solution a path integral (Euclidean)

$$K_{\text{Euclidean}}(y, t_y; x, t_y) = \int_\gamma \exp\left\{ -\frac{1}{\hbar} S[\gamma(t)] \right\} \mathrm{d}\gamma(t)$$

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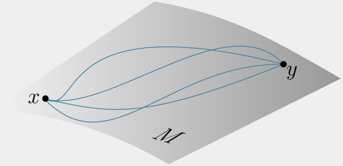
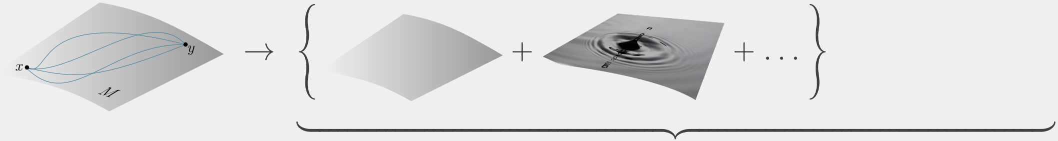
Path integrals and (Euclidean) quantum gravity



- 1st. challenge:
  - Whereas in quantum mechanics: path integrals <u>on</u> a fixed spacetime M

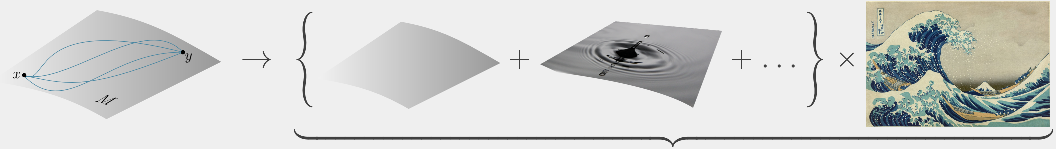
Path integrals and (Euclidean) quantum gravity



Quantum superposition of geometries

- 1st. challenge:
  - Whereas in quantum mechanics: path integrals <u>on</u> a fixed spacetime M
  - In quantum gravity: path integrals <u>of</u> space~~time~~, $Z = \int_{\text{METRIC}} \mathrm{e}^{-\frac{1}{\hbar} S_{\text{EH}}[g]} \mathrm{d}g$

Path integrals and (Euclidean) quantum gravity



Quantum superposition of geometries (small perturbations + instantons)

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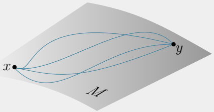
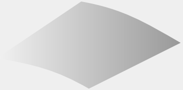
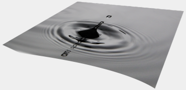
[Hokusai]

Path integrals and (Euclidean) quantum gravity



$\rightarrow \Big\{$  $+$  $+ \dots \Big\} \times$ 

Quantum superposition of geometries (small perturbations + instantons)

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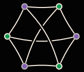
- 2nd. challenge: replace C^∞-category

discrete | algebraic

single geometry
← paradigms →

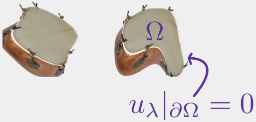
[Hokusai][Tetrahedra from Wikipedia]

↑  glue from 'traces of tensors', cf. Răzvan's talk

- the 1st theorem of the *spectral formalism* is *Weyl's law* (1911). 

The Laplace spectrum of $\Omega \subset \mathbb{R}^d$ $(\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \ldots)$ obeys

$$\#\{i : \lambda_i \leqslant \Lambda\} \sim \mathrm{vol}\,\Omega \cdot \Lambda^{d/2} + o(\Lambda^{d/2})$$



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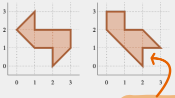

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- one cannot positively answer Marek Kac's 1966-question† knowing the spectrum, but you can 'hear the shape of Ω' knowing a *spectral triple*

[Gordon, Webb, Wolpert, *Invent. Math.* '92 after Milnor, Sunada, Bérard, …] [Connes, *JNCG* 2013]

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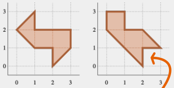

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- differential *noncommutative (nc) geometry* = nc topology + metric data

$$\begin{array}{ccc} \{\text{nice topological spaces}\} & \simeq & \{\text{unital } \textit{commutative } C^*\text{-algebras}\} \\ \downarrow & & \downarrow \\ \{\text{nice `nc topological spaces'}\} & \simeq & \{\text{unital } \sout{\textit{commutative}}\ C^*\text{-algebras}\} \end{array}$$

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- spectral triples (A, H, D) generalize spin geometry $(C^\infty(M), L^2(M, \mathbb{S}), D_M)$

[Gordon, Webb, Wolpert, *Invent. Math.* '92 after Milnor, Sunada, Bérard, …] [Connes, *JNCG* 2013]
[Gelfand, Najmark *Mat. Sbornik* '43] [Connes, *NCG* '94]

Physical motivation of spectral triples (deterministic)

- Physics ∩ Noncommutative Geometry ∋ The Standard Model from the Spectral Action

$$
\begin{gathered}
-\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4} g_s^2 f^{abc} f^{ade} g^b_\mu g^c_\nu g^d_\mu g^e_\nu + \\
\tfrac{1}{2} i g_s^2 (\bar{q}^\sigma_i \gamma^\mu q^\sigma_j) g^a_\mu + \bar{G}^a \partial^2 G^a + g_s f^{abc}\partial_\mu \bar{G}^a G^b g^c_\mu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - \\
M^2 W^+_\mu W^-_\mu - \tfrac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c_w^2} M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \\
\tfrac{1}{2}\partial_\mu H \partial_\mu H - \tfrac{1}{2} m_h^2 H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+ \phi^- - \tfrac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \\
\tfrac{1}{2c_w^2} M \phi^0 \phi^0 - \beta_h [\tfrac{2M^2}{g^2} + \tfrac{2M}{g} H + \tfrac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)] + \tfrac{2M^4}{g^2}\alpha_h - \\
i g c_w [\partial_\nu Z^0_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + \\
Z^0_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)] - i g s_w [\partial_\nu A_\mu (W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - A_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu (W^+_\nu \partial_\nu W^-_\mu - \\
W^-_\nu \partial_\nu W^+_\mu)] - \tfrac{1}{2} g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \tfrac{1}{2} g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + \\
g^2 c_w^2 (Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2 (A_\mu W^+_\mu A_\nu W^-_\nu - \\
A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w [A_\mu Z^0_\nu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - \\
2 A_\mu Z^0_\mu W^+_\nu W^-_\nu] - g\alpha [H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-] - \tfrac{1}{8} g^2 \alpha_h [H^4 + \\
(\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2 H^2] - \\
g M W^+_\mu W^-_\mu H - \tfrac{1}{2} g \tfrac{M}{c_w^2} Z^0_\mu Z^0_\mu H - \tfrac{1}{2} i g [W^+_\mu (\phi^0 \partial_\mu \phi^- - \phi^- \partial_\mu \phi^0) - \\
W^-_\mu (\phi^0 \partial_\mu \phi^+ - \phi^+ \partial_\mu \phi^0)] + \tfrac{1}{2} g [W^+_\mu (H \partial_\mu \phi^- - \phi^- \partial_\mu H) - \\
W^-_\mu (H \partial_\mu \phi^+ - \phi^+ \partial_\mu H)] + \tfrac{1}{2} g \tfrac{1}{c_w} (Z^0_\mu (H \partial_\mu \phi^0 - \phi^0 \partial_\mu H) - \\
i g \tfrac{s_w^2}{c_w} M Z^0_\mu (W^+_\mu \phi^- - W^-_\mu \phi^+) + i g s_w M A_\mu (W^+_\mu \phi^- - W^-_\mu \phi^+) - \\
i g \tfrac{1-2c_w^2}{2c_w} Z^0_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) + i g s_w A_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) - \\
\tfrac{1}{4} g^2 W^+_\mu W^-_\mu [H^2 + (\phi^0)^2 + 2\phi^+\phi^-] - \tfrac{1}{4} g^2 \tfrac{1}{c_w^2} Z^0_\mu Z^0_\mu [H^2 + (\phi^0)^2 + \\
2(2s_w^2 - 1)^2 \phi^+\phi^-] - \tfrac{1}{2} g^2 \tfrac{s_w^2}{c_w} Z^0_\mu \phi^0 (W^+_\mu \phi^- + W^-_\mu \phi^+) -
\end{gathered}
$$

$$
\begin{gathered}
\tfrac{1}{2} i g^2 \tfrac{s_w^2}{c_w} Z^0_\mu H (W^+_\mu \phi^- - W^-_\mu \phi^+) + \tfrac{1}{2} g^2 s_w A_\mu \phi^0 (W^+_\mu \phi^- + \\
W^-_\mu \phi^+) + \tfrac{1}{2} i g^2 s_w A_\mu H (W^+_\mu \phi^- - W^-_\mu \phi^+) - g^2 \tfrac{s_w}{c_w} (2c_w^2 - \\
1) Z^0_\mu A_\mu \phi^+ \phi^- - g^1 s_w^2 A_\mu A_\mu \phi^+ \phi^- - \bar{e}^\lambda (\gamma\partial + m_e^\lambda) e^\lambda - \bar{\nu}^\lambda \gamma\partial \nu^\lambda - \\
\bar{u}^\lambda_j (\gamma\partial + m_u^\lambda) u^\lambda_j - \bar{d}^\lambda_j (\gamma\partial + m_d^\lambda) d^\lambda_j + i g s_w A_\mu [-(\bar{e}^\lambda \gamma^\mu e^\lambda) + \\
\tfrac{2}{3} (\bar{u}^\lambda_j \gamma^\mu u^\lambda_j) - \tfrac{1}{3} (\bar{d}^\lambda_j \gamma^\mu d^\lambda_j)] + \tfrac{ig}{4c_w} Z^0_\mu [(\bar{\nu}^\lambda \gamma^\mu (1+\gamma^5) \nu^\lambda) + \\
(\bar{e}^\lambda \gamma^\mu (4s_w^2 - 1 - \gamma^5) e^\lambda) + (\bar{u}^\lambda_j \gamma^\mu (\tfrac{4}{3} s_w^2 - 1 - \gamma^5) u^\lambda_j) + (\bar{d}^\lambda_j \gamma^\mu (1 - \\
\tfrac{8}{3} s_w^2 - \gamma^5) d^\lambda_j)] + \tfrac{ig}{2\sqrt{2}} W^+_\mu [(\bar{\nu}^\lambda \gamma^\mu (1+\gamma^5) e^\lambda) + (\bar{u}^\lambda_j \gamma^\mu (1 + \\
\gamma^5) C_{\lambda\kappa} d^\kappa_j)] + \tfrac{ig}{2\sqrt{2}} W^-_\mu [(\bar{e}^\lambda \gamma^\mu (1+\gamma^5) \nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\lambda\kappa} \gamma^\mu (1 + \\
\gamma^5) u^\lambda_j)] + \tfrac{ig}{2\sqrt{2}} \tfrac{m_e^\lambda}{M} [-\phi^+ (\bar{\nu}^\lambda (1-\gamma^5) e^\lambda) + \phi^- (\bar{e}^\lambda (1+\gamma^5) \nu^\lambda)] - \\
\tfrac{g}{2} \tfrac{m_e^\lambda}{M} [H (\bar{e}^\lambda e^\lambda) + i\phi^0 (\bar{e}^\lambda \gamma^5 e^\lambda)] + \tfrac{ig}{2M\sqrt{2}} \phi^+ [-m_d^\kappa (\bar{u}^\lambda_j C_{\lambda\kappa} (1 - \\
\gamma^5) d^\kappa_j) + m_u^\lambda (\bar{u}^\lambda_j C_{\lambda\kappa} (1+\gamma^5) d^\kappa_j] + \tfrac{ig}{2M\sqrt{2}} \phi^- [m_d^\lambda (\bar{d}^\lambda_j C^\dagger_{\lambda\kappa} (1 + \\
\gamma^5) u^\kappa_j) - m_u^\kappa (\bar{d}^\lambda_j C^\dagger_{\lambda\kappa} (1-\gamma^5) u^\kappa_j] - \tfrac{g}{2} \tfrac{m_u^\lambda}{M} H (\bar{u}^\lambda_j u^\lambda_j) - \\
\tfrac{g}{2} \tfrac{m_d^\lambda}{M} H (\bar{d}^\lambda_j d^\lambda_j) + \tfrac{ig}{2} \tfrac{m_u^\lambda}{M} \phi^0 (\bar{u}^\lambda_j \gamma^5 u^\lambda_j) - \tfrac{ig}{2} \tfrac{m_d^\lambda}{M} \phi^0 (\bar{d}^\lambda_j \gamma^5 d^\lambda_j)
\end{gathered}
$$

...this 'fits' in

$$\mathrm{Tr}(f(D/\Lambda)) + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle$$

# of generations and $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ ⟶ NCG ⟶ *Classical* Standard Model

[Connes, Lott, *Nucl. Phys. B* '91; . . . Chamseddine, Connes, Marcolli *ATMP* '07 (Euclidean)]
[Barrett *J. Math. Phys.* '07 (Lorenzian); Connes-Chamseddine *JHEP* '12; van Suijlekom's textbook NCG∩HEP '15]

Physical motivation of spectral triples (deterministic)

- Physics ∩ Noncommutative Geometry ∋ The Standard Model from the Spectral Action

$M_3(\mathbb{C}) \ni \Upsilon_e, \Upsilon_\nu, \ldots, \Upsilon_d$

$$
D_F = \begin{pmatrix}
0 & 0 & \Upsilon_\nu^* & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_R^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Upsilon_e^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Upsilon_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \Upsilon_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^* & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^* & \otimes 1_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Upsilon_u & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Upsilon_d & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Upsilon_R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_\nu^T & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_e^T & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^T & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^T & \otimes 1_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_u & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_d & 0 & 0 &
\end{pmatrix} \in M_{96}(\mathbb{C})_{\text{s.a.}}
$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

\# of generations and $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \rightarrowtail$ NCG $\rightarrowtail$ *Classical* Standard Model

[Connes, Lott, *Nucl. Phys. B* '91; . . . Chamseddine, Connes, Marcolli *ATMP* '07 (Euclidean)]
[Barrett *J. Math. Phys.* '07 (Lorenzian); Connes-Chamseddine *JHEP* '12; van Suijlekom's textbook NCG∩HEP '15]

Towards a quantum theory of noncommutative spaces

« *The far distant goal is to set up a functional integral evaluating (...)*

observables $\mathscr{S}$ $\quad \langle \mathscr{S} \rangle = \int \mathscr{S}\, e^{-\operatorname{Tr} f(D/\Lambda) - \frac{1}{2}\langle J\psi, D\psi\rangle + \rho(e,D)} \,\mathrm{d}e\,\mathrm{d}\psi\,\mathrm{d}D \quad (*)$ »

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functional integral $\xrightarrow[\text{paradigm shift}]{}$ operator integral

$$\int_{\text{METRIC}} \mathrm{e}^{-\frac{1}{\hbar} S_{\text{EH}}[g]} \mathrm{d}g \xrightarrow{\text{Einstein-Hilbert} \longrightarrow \text{spectral}} \int_{\text{DIRAC}} \mathrm{e}^{-\frac{1}{\hbar}\mathrm{Tr} f(D)} \mathrm{d}D$$

(hard to define for manifolds)

$f : \mathbb{R} \to \mathbb{R}$ with $f(D) \to \infty$ at large argument

[∗ Eq. 1.892, Connes, Marcolli, *NCG, QFT and motives*, 2007]

$\inf_{\gamma: x \to y} \{\int_\gamma \mathrm{d}s\} =: d(x,y) = \sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| : \, ||D_M f - f D_M|| \leqslant 1 \}$ [Connes; Monge-Kantorovich]

Truncating the smooth geometries

- truncations of the algebra or of the spectrum to $\#(\text{d.o.f.}) \leqslant N$

[Barrett-Glaser; Khalkhali-Pagliaroli]

for $\mathbb{S}^1$ [Van Suijlekom]

Matrix geometries
$(M_N(\mathbb{C}), H_{\text{fin. dim.}}, D_{\text{fin. dim.}})$

smooth spin geometries
$(C^\infty(M), L^2(\mathbb{S}), D_M)$
(commutative geometry)

Operator systems
$(P_N A P_N, P_N H, P_N D P_N)$

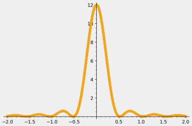
algebraic truncation

$C(\mathbb{S}^2) \twoheadrightarrow \langle$  $\rangle$

[Barrett & fuzzy literature*]

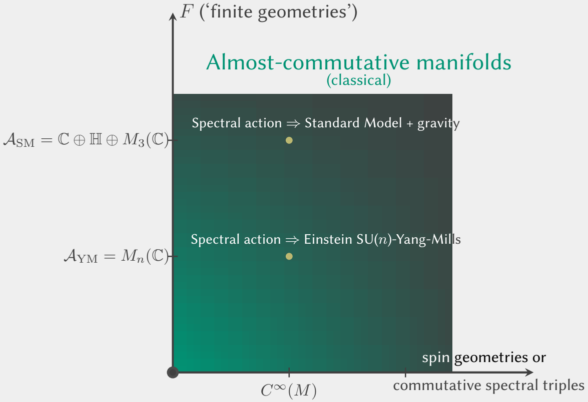
Toeplitz/Féjer truncations

[Connes-van Suijlekom]



Choi-Effros operator systems also appear in quantum information
[De las Cuevas, Netzer, arXiv:2102.04240]

Main Result (almost there)



Spectral Action

Classical

$$S(D) = \operatorname{Tr} f(D/N) \qquad \text{(bosons)}$$

$$\sim \sum_{s \in \mathrm{Sp}\,\mathrm{Dim} \cap \mathbb{R}_+} f_s N^s ⨍ |D|^{-s} + f(0)\zeta(0) \ldots$$

Quantum

$$\mathcal{Z}_{\mathrm{AC}} \stackrel{?}{=} \int_{\mathrm{Dirac}} e^{-\frac{1}{\hbar} \operatorname{Tr} f(D/N)} \mathrm{d}D$$

(hard to define for almost-comm. manifolds)

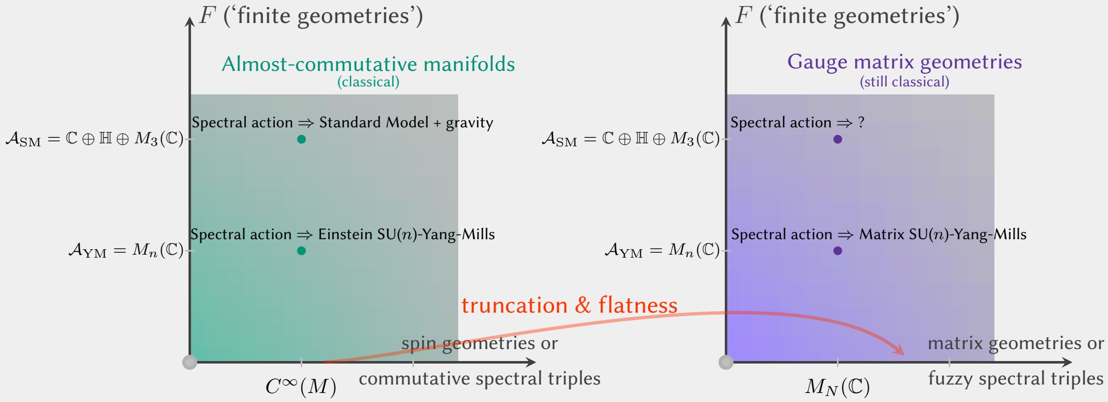
[Chamseddine, Connes, Marcolli *ATMP* '07] using heat kernel expansion, for 4-manifolds:

$$N^4 ⨍ |D|^{-4} = c_4(N)\mathrm{vol}(M)$$
[cosmological constant]

$$N^2 ⨍ |D|^{-2} = c_2(N) \int R$$
[Einstein-Hilbert]

$$\zeta_D(0) = c_0 \int (R^* R^*) + c_0' \int C^2$$
[Gauß-Bonnet + conformal gravity]

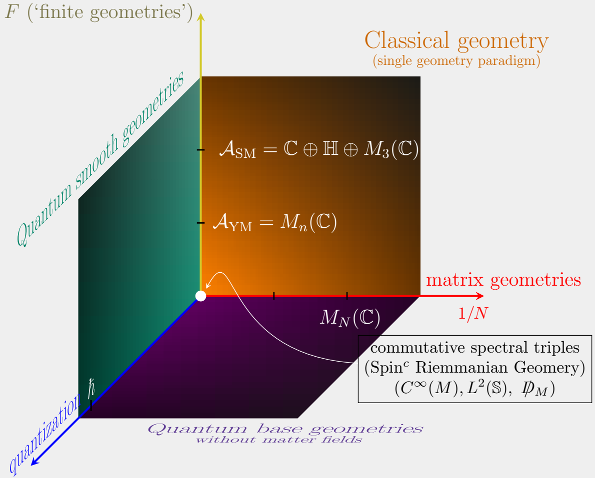
Main Result



Matrix Yang-Mills(-Higgs) functional obeying triple axioms;
its partition function is a multi-matrix model

[CP *Ann. Henri Poincaré* **23** '22 `arXiv:2105.01025`]

Landscape



Aim: Make sense of

$$\mathcal{Z} = \int_{\text{Dirac}} e^{-\operatorname{Tr}_H f(D)} \mathrm{d}D$$

- *Plane* $(\hbar, 1/N, 0)$ *of 'base geometries'*
- *Plane* $(\hbar, 0, F) = \lim_{N\to\infty}(\hbar, 1/N, F)$
- *Plane* $(0, 1/N, F) = \lim_{\hbar\to 0}(\hbar, 1/N, F)$ *of classical geometries*

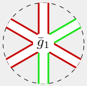
Multimatrix models with multi-traces

- A chord-diagram formula computes the spectral action (in any signature)

$$\mathcal{Z} = \int_{\text{Dirac}} e^{-\operatorname{Tr}_H f(D)} \mathrm{d}D \quad (\hbar = 1)$$

$$= \int_{M_{p,q}} e^{-N \operatorname{Tr}_N P - \operatorname{Tr}_N^{\otimes 2}(Q_{(1)} \otimes Q_{(2)})} \mathrm{d}\mathbb{X}_{\text{Leb}}$$

  - $\mathbb{X} \in M_{p,q}$ = products of $\mathfrak{su}(N)$ and $\mathcal{H}_N$
  - $\mathrm{d}\mathbb{X}_{\text{Leb}}$ is the Lebesgue measure on $M_{p,q}$
  - P, $Q_{(i)}$ in $\mathbb{C}_{\langle k \rangle} = \mathbb{C}\langle \mathbb{X} \rangle$ nc-polynomials
- $\mathcal{Z}_{\text{formal}}$ leads to colored ribbon graphs

$g_1 \operatorname{Tr}_N(ABBBAB) \quad \leftrightarrow$ 

Chord-diagram is what it sounds like: 

[CP '19, CP '21, CP '22a, CP '22b]

Multimatrix models with multi-traces

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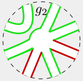
$g_2 \operatorname{Tr}_N^{\otimes 2}(AABABA \otimes AA) \quad \leftrightarrow$ 

Chord-diagram is what it sounds like: 

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Multimatrix models with multi-traces

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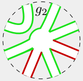
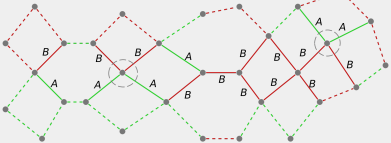


- Ribbon graphs: Enumeration of maps, here 'face-worded'



- Intersection numbers of ψ-classes

$$\sum_{a_1 + \ldots + a_n = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdot \psi_2^{a_2} \cdots \psi_n^{a_n} \prod_{j=1}^{n} \frac{(2a_j - 1)!!}{s_j^{2a_j+1}}$$

$$= \sum_{G \text{ trivalent of type } (g,n)} \frac{2^{2g-2+n}}{\#\operatorname{Aut}(G)} \prod_{e \in G} \frac{1}{s_{L(e)} + s_{R(e)}}$$

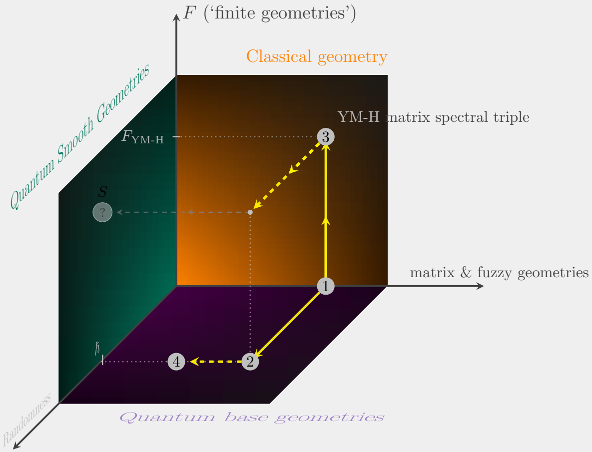
Chord-diagram is what it sounds like: 

[CP '19, CP '21, CP '22a, CP '22b]

[Brezin, Itzykson, Parisi, Zuber, *CMP* '78; Kontsevich, *CMP*, '92] ∃! such (1, 1)-graph

$G =$  $\Rightarrow s^{-3} \int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{2^1}{\#\operatorname{Aut}(G)} \frac{1}{(2s)^3} \Rightarrow \int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24}$

Conclusion: Some progress



1. Matrix Geometries [Barrett, *J. Math. Phys.* 2015]
2. Dirac ensembles [Barrett, Glaser, *J. Phys. A* 2016] and how to compute the spectral action [CP '19]
3. Gauge matrix spectral triples [CP '22a]
4. Functional Renormalisation (Wetterich equation) in formal random matrix theory [CP '21a, CP '21b] and [CP '22b] *(not this talk)*

Thanks for listening!

References: [CP 1912.13288, to appear in *J. Noncommut. Geom.*] on the spectral action, [CP *Ann. Henri Poincaré 2022*] on Yang-Mills-Higgs.
Related: [CP *Ann. Henri Poincaré 2021*] on Wetterich Eq., [CP *J. High Energ. Phys* 2021] [CP *Lett. Math. Phys.* 2022] on algebra and FRG

Conclusion

- spin $M \times \{\text{finite spectral triple}\} \equiv$ almost-commutative

  (reproduces classical Standard Model, but hard to quantize)

- *fuzzy* or *matrix* geometry $\approx$ finite spectral triple + $\mathbb{C}\ell$-action

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- fuzzy $\times$ finite $=$ gauge matrix spectral triple, it is $\mathrm{PU}(n)$-Yang-Mills(-Higgs) if the fin. geom. algebra is $M_n(\mathbb{C})$; partition func. is a k-matrix model, k large:

$$\mathcal{Z}_{\text{GAUGE MATRIX}} = \int_{\text{DIRACS}} \mathrm{e}^{-\operatorname{Tr}_H f(D)} \mathrm{d}D = \int_{\text{base}\times\text{YM}\times\text{Higgs}} \mathrm{e}^{-S_{\text{gauge}} - S_{\text{H}} - S_{\text{gauge-H}} - S} \mathrm{d}\mu_{\text{G}}(L)\, \mathrm{d}\mu_{\text{G}}(A)\, \mathrm{d}\Phi$$

with $(L, A, \phi) \in [\mathfrak{su}(N)]^{\times 4} \times [\mathcal{N}^{\text{gauge}}_{N,n}]^{\times 4} \times \mathcal{N}^{\text{Higgs}}_{N,n}$

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close relatives of $u(N) \otimes u(n)$

Thanks for listening!

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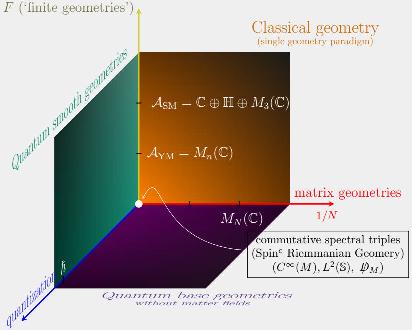
Conclusion

- small step towards [Eq. 1.892, Connes, Marcolli, *NCG, QFT and motives*, 2007]

« *The far distant goal is to set up a functional integral evaluating spectral observables* $\mathscr{S}$ $\langle \mathscr{S} \rangle = \int \mathscr{S}\, \mathrm{e}^{-\operatorname{Tr} f(D/\Lambda) - \frac{1}{2}\langle J\psi, D\psi\rangle + \rho(e,D)} \mathrm{d}e\, \mathrm{d}\psi\, \mathrm{d}D$ »



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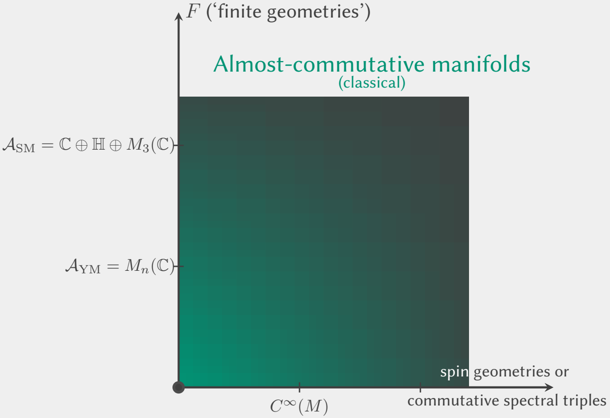
Supplement: Yang-Mills-Higgs matrix theory



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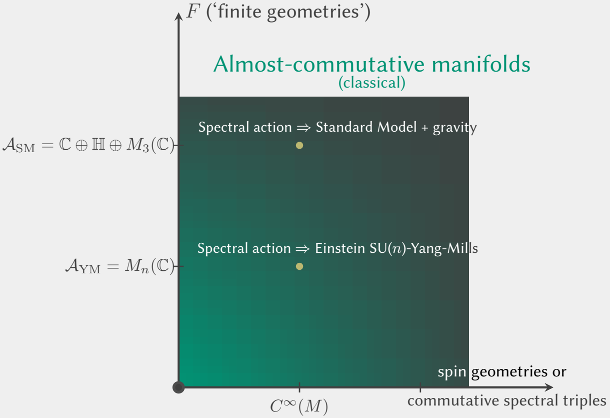

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(hard for almost-commutative manifolds)

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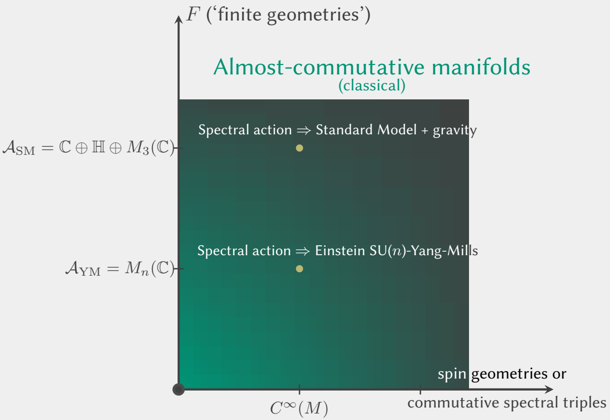

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Definition [CP' 21]. A *gauge matrix spectral triple* $G_\ell \times F$ is the spectral triple product of a matrix geometry G_ℓ with a finite geometry $F = (A_F, H_F, D_F)$, $\dim A_F < \infty$.

LEMMA-DEFINITION [CP' 21]. Consider a gauge matrix spectral triple $G_\ell \times F$ with

$$F = (M_n(\mathbb{C}), M_n(\mathbb{C}), D_F)$$

and G_ℓ Riemannian ($d = 4$) fuzzy geometry on $M_N(\mathbb{C})$, whose fluctuated Dirac op. is

$$D_\omega = \sum_{\mu=0}^{3} \overbrace{\gamma^\mu \otimes (\ell_\mu + \mathscr{a}_\mu) + \gamma^{\hat\mu} \otimes (x_\mu + \mathscr{s}_\mu)}^{D_{\text{gauge}}} + \overbrace{\gamma \otimes \Phi}^{D_{\text{Higgs}}}, \qquad \mathscr{a}_\mu = \text{‘gauge potential’}, x_\mu = \text{spin connection?}$$

The *field strength* is given by $\mathscr{F}_{\mu\nu} := [\overbrace{\ell_\mu + \mathscr{a}_\mu}^{d_\mu}, \ell_\nu + \mathscr{a}_\nu] =: [\mathsf{F}_{\mu\nu}, \ \cdot\ ]$

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Lemma. The gauge group $\mathrm{G}(\mathcal{A}) \cong \mathcal{U}(\mathcal{A})/\mathcal{U}(Z(\mathcal{A})) \cong \mathrm{PU}(N) \times \mathrm{PU}(n)$ acts as follows

$$\mathsf{F}_{\mu\nu} \mapsto \mathsf{F}^u_{\mu\nu} = u\mathsf{F}_{\mu\nu}u^* \qquad \text{for all } u \in \mathrm{G}(\mathcal{A})$$

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The proof uses [§6 of W. van Suijlekom, *Noncommutative Geometry and Particle Physics*, 2015]

Meaning	Random matrix case, flat $d = 4$ Riem. Tr = trace of ops. $M_N \otimes M_n \to M_N \otimes M_n$	Smooth operator
Derivation	$\ell_\mu = [L_\mu \otimes 1_n, \ \cdot\ ]$	∂_i
Gauge potential	$a_\mu = [A_\mu, \ \cdot\ ]$	$\mathbb{A}_i$
Covariant derivative	$d_\mu = \ell_\mu + a_\mu$	$\mathbb{D}_i = \partial_i + \mathbb{A}_i$

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Field strength	$[d_\mu, d_\nu] = \overbrace{[\ell_\mu, \ell_\nu]}^{\not\equiv 0} + [\ell_\mu, a_\nu] - [\ell_\nu, a_\mu] + [a_\mu, a_\nu]$	$[\mathbb{D}_i, \mathbb{D}_j] = \overbrace{[\partial_i, \partial_j]}^{\equiv 0} + \partial_i \mathbb{A}_j - \partial_j \mathbb{A}_i + [\mathbb{A}_i, \mathbb{A}_j]$
Yang-Mills action	$-\frac{1}{4}\mathrm{Tr}(\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu})$	$-\frac{1}{4}\int_M \mathrm{Tr}_{\mathfrak{su}(n)}(\mathbb{F}_{ij}\mathbb{F}^{ij})\mathrm{vol}$
Higgs field	Φ	h
Higgs potential	$\mathrm{Tr}(f_2\Phi^2 + \Phi^4)$	$\int_M \big(-\mu^2\lvert h\rvert^2 + \lambda\lvert h\rvert^4\big)\mathrm{vol}$
Gauge-Higgs coupling	$-\mathrm{Tr}(d_\mu \Phi d^\mu \Phi)$	$-\int_M \lvert \mathbb{D}_i h\rvert^2 \mathrm{vol}$

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Field strength	$[d_\mu, d_\nu] = \overbrace{[\ell_\mu,\ell_\nu]}^{\not\equiv 0} + [\ell_\mu, a_\nu] - [\ell_\nu, a_\mu] + [a_\mu, a_\nu]$	$[\mathbb{D}_i,\mathbb{D}_j] = \overbrace{[\partial_i,\partial_j]}^{\equiv 0} + \partial_i\mathbb{A}_j - \partial_j\mathbb{A}_i + [\mathbb{A}_i,\mathbb{A}_j]$
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Higgs field	Φ	h
Higgs potential	$\mathrm{Tr}(f_2\Phi^2 + \Phi^4)$	$\int_M \big(-\mu^2\|h\|^2 + \lambda\|h\|^4\big)\mathrm{vol}$
Gauge-Higgs coupling	$-\,\mathrm{Tr}(d_\mu\Phi d^\mu\Phi)$	$-\int_M \|\mathbb{D}_i h\|^2\mathrm{vol}$

THM. [CP '22] If the RG-flow is computable in terms of $\mathrm{U}(N)$-invariants, the algebra of Functional Renormalisation is $M_k(\mathcal{A}_{N,k},\star)$ where

$$\mathcal{A}_{N,k} = (\mathbb{C}^{(N)}_{\langle k\rangle} \otimes \mathbb{C}^{(N)}_{\langle k\rangle}) \oplus (\mathbb{C}^{(N)}_{\langle k\rangle} \boxtimes \mathbb{C}^{(N)}_{\langle k\rangle})$$

whose product in homogeneous elements reads:

$$\begin{aligned}
(U \otimes W) \star (P \otimes Q) &= PU \otimes WQ\,, \\
(U \boxtimes W) \star (P \otimes Q) &= U \boxtimes PWQ\,, \\
(U \otimes W) \star (P \boxtimes Q) &= WPU \boxtimes Q\,, \\
(U \boxtimes W) \star (P \boxtimes Q) &= \mathrm{Tr}(WP) U \boxtimes Q\,.
\end{aligned}$$

Example: a Hermitian 3-matrix model

Consider two operators $O_1 = \frac{\bar{g}_1}{2}[\mathrm{Tr}_N(\frac{A^2}{2})]^2$ and $O_2 = \bar{g}_2 \,\mathrm{Tr}_N(ABC)$. We compute $g_1 g_2^2$-coefficients:

$$\mathrm{Hess}_{I,J}\, O_1 = \delta_I^J \delta_I^A \bar{g}_1 \{ \underbrace{\mathrm{Tr}_N(A^2/2)\cdot[1_N \otimes 1_N]} + \underbrace{A \boxtimes A} \},$$

where a ‘filled ribbon’ means contracted in the one-loop graph, and ‘empty ribbon’ uncontracted.

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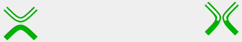

where a 'filled ribbon' means contracted in the one-loop graph, and 'empty ribbon' uncontracted.

$$\mathrm{Hess}\, O_2 = \bar{g}_2 \begin{bmatrix} 0 & C\otimes 1_N & B\otimes 1_N \\ 1_N\otimes C & 0 & A\otimes 1_N \\ 1_N \otimes B & 1_N\otimes A & 0 \end{bmatrix} \Rightarrow [\mathrm{Hess}\, O_2]^{\star 2} = \bar{g}_2^2 \begin{bmatrix} \overbrace{C\otimes C} + \overbrace{B\otimes B} & B\otimes A & C\otimes A \\ A\otimes B & A\otimes A + C\otimes C & C\otimes B \\ A\otimes C & B\otimes C & B\otimes B + A\otimes A \end{bmatrix}.$$

$$[\bar{g}_1\bar{g}_2^2]\,\mathrm{STr}\{\mathrm{Hess}\, O_1 \star [\mathrm{Hess}\, O_2]^{\star 2}\} = \mathrm{Tr}_N(A^2/2)\times\left[(\mathrm{Tr}_N\, C)^2 + (\mathrm{Tr}_N\, B)^2\right] + \mathrm{Tr}_N(ACAC + ABAB)\,.$$

These are effective vertices of the four one-loop graphs that can be formed with the contractions of (the filled ribbon half-edges of) any of {  } with any of {  }