Extra Dimensions at Colliders

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You are looking at a rough script following my attempt to give an introduction into the phenomenology of extra dimensions to our SUPA graduate students. It covers the very basics of large and warped extra dimensions from an LHC phenomenologist's point of view. Moreover, I added a section on UV completions of models with large extra dimensions, because I got interested in this and even wrote a paper on it. Please notice that there are of course mistakes and typos all over the place, because I am also just learning the topic myself...

Contents

I. THE STANDARD MODEL AS AN EFFECTIVE THEORY

Before we can even start talking about physics beyond the Standard Model, we have to define what we mean by the Standard Model. For those of you who attended Graham Kribs' journal club — remember the discussion between him and Thomas Binoth. There are different ways we can look at the Standard Model, regardless of it great success over by now many decades of measurements. Both approaches have basic structures in common:

– a gauge theory with the group structure $SU(3) \otimes SU(2) \otimes U(1)$

– massive electroweak gauge bosons masses through spontaneous symmetry breaking (Higgs mechanism with $v = 246$ GeV and m_H unknown)

– Dirac fermions in the usual doublets and with masses equal to their Yukawa couplings

There are two philosophies behind writing down a Lagrangian for this model:

1. write a renormalizable Lagrangian with all dimension–4 operators consistent with the particle content and all symmetries
2. write a general effective–theory Lagrangian with these particles and all symmetries. Higher–dimension operators will appear and have to be suppressed by some scale Λ

The difference between these two approaches are higher–dimensional operators, operators which have an explicit suppression by a large mass. If this scale Λ is large enough, we might never see the difference between the two approaches in high–energy experiments. If Λ is smaller, we might see the Standard Model break down as an effective theory, for example at the LHC, and we should be able to determine its ultraviolet completion.

A. Experimental hints

LEP and Tevatron

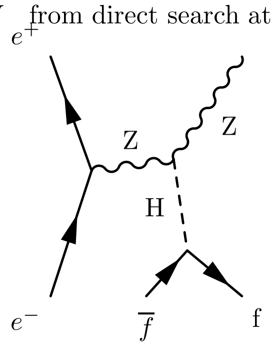
LEP(2) and Tevatron experiments have for many years tested the Standard Model to energy scales of $100 \cdots 500$ GeV. All their results are in perfect agreement with the Standard Model, apart from that fact that we could have seen direct evidence for the Higgs boson:

– electroweak gauge bosons discovered with masses $m_W \sim 80$ GeV, $m_Z \sim 91$ GeV
no anomalous W, Z decays

– 6 quarks found, $m_t \sim 172$ GeV
typical decay $t \to b\,W^+$ observed, no anomalous decays

– leptons, including τ as expected.

– electroweak precision data with global fit:
$m_H \sim 110$ GeV   best value
$m_H \lessapprox 250$ GeV   1 σ bound
$m_H > 114$ GeV   from direct search at LEP2



where $f\overline{f}$ are mostly $H \to b\bar{b}$ and $Z \to l^+l^-, \nu\overline{\nu}$.

The possible problem with the electroweak precision data is the quality of the global fit. Its best χ^2 value is poor. A reason might be that some for example b observables might be inconsistent, but we do not know $\Rightarrow$ $\boxed{\text{not conclusive}}$

Muon anomalous magnetic moment (g-2)

The anomalous magnetic moment of the muon is one of the best–measured parameters in high–energy physics, even though most of the physics which goes into its determination we would call low–energy physics nowadays. Unfortunately, the Brookhaven experiment, which recently delivered the best available measurement, has been shut down. If you want to know more about this observable — Dominik Stöckinger recently wrote a great review on it.

The measured value of the anomalous magnetic moment of μ is

$$a_\mu^{\rm exp} = \frac{1}{2}(g-2)_\mu^{\rm exp} = (11659208 \pm 6) \cdot 10^{-10} \tag{1}$$

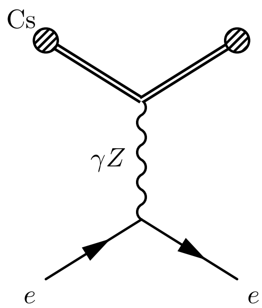
while the Standard Model predictions range between two different approaches

$$\begin{aligned} a_\mu^{\rm exp} - a_\mu^{\rm SM} &= (31.7 \pm 9.5) \cdot 10^{-10} \qquad &: \; 3.3\sigma \\ a_\mu^{\rm exp} - a_\mu^{\rm SM} &= (20.2 \pm 9.0) \cdot 10^{-10} \qquad &: \; 2.1\sigma \end{aligned} \tag{2}$$

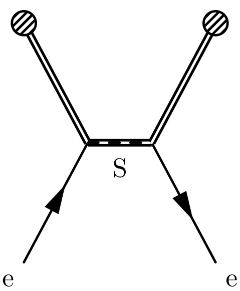
The general agreement in high–energy physics is: a 5 σ deviation from the background is called a discovery, everything else is either a rumor or a hint or a matter of taste $\Rightarrow$ $\boxed{\text{not conclusive}}$

Atomic parity violation

We know that electroweak gauge bosons have couplings which distinguish between the chirality of fermions in the $ff'W$ and ffZ vertices. In the interaction between the nucleus and the electrons in an atom, this interaction leads to parity violation, which can be measured:



Beyond the Standard Model we can look for so–called leptoquarks, scalars or vectors which carry baryon and lepton number and occur in the crossed channel compared to the usual γ, Z exchange:



Again, this experiment in Boulder was terminated with around 2 σ discrepancy between the Standard Model prediction and the final measurement $\Rightarrow$ $\boxed{\text{not conclusive}}$

Cosmology

The last Nobel prize went to studies of the cosmic microwave background. The most recent WMAP data (combined with large–scale structure measurements) confirms conclusively the existence of cold dark matter in the universe:

$$\Omega_{\rm DM} h^2 = 0.094 \ldots 0.129 \tag{3}$$

where $\Omega = 1$ is critical density for flat universe and $h = H_0/100$ km/s/Mpc $\sim$ 0.7 is just a c–number connected to the Hubble constant $H_0 \sim 73$ km/s/Mpc. Different measurements determine the matter content of the universe (averaged over all distances) to:

- baryon density $\Omega_b\, h^2 = 0.024 \pm 0.001$
- matter density $\Omega_m\, h^2 = 0.14 \pm 0.02$

Error estimates are mixture of serious studies, chemistry and miracles. Such a dark matter particle is not part of the Standard Model. Most generally, it could be a stable particle with electroweak interactions and a mass around 200 GeV. Unfortunately, we have not observed such a particle in direct of indirect searches for dark matter $\Rightarrow$ $\boxed{\text{conclusive!}}$

Flavor Physics

Flavor physics has been a major effort over the last decade — in particular once we put together B and K physics, neutrino physics, and low–energy searches for example for proton decay or neutrinoless double–beta decay. Unfortunately, all these measurements have revealed little but the existence of a finite neutrino mass of which we still do not know the over–all scale:

- proton decay not observed
- flavor changing neutral current not observed
- neutrinoless double–β decay not observed
- no unexplained effects in B physics
- no unexplained effects in K physics
- ....

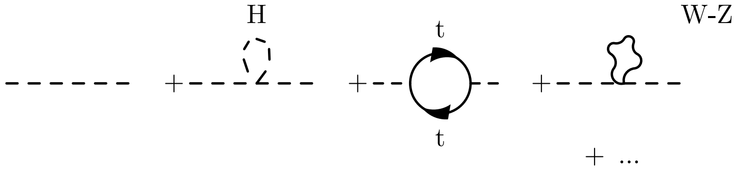
If we want to phrase it positively, we conclude that we have not found anything and not gained any clues for physics beyond the Standard Model $\Rightarrow$ $\boxed{\text{unfortunately conclusive}}$

B. Theoretical hints

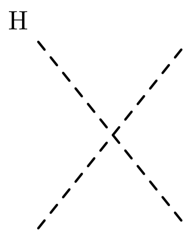
Let us start from assuming that the Standard Model is a renormalizable theory (it has no inverse powers of mass in the Lagrangian). This means that it does not have a built–in energy scale where it breaks down. An exception is gravity, because we know that at energies above the Planck scale 10^{19} GeV gravitational interactions become strong and our world should be described by some combination of the Standard Model and quantum gravity. All current observables probe scales $E \lessapprox 100$ TeV $= 10^5$ GeV so we can ignore Planck–scale or quantum gravity effects for now.

Standard Model beyond tree-level

At next–to–leading order, the (bare) leading order Higgs mass gets corrected by loops involving Standard Model particles:



We can for example compute the 4–point Higgs loop with the coupling:



$$= -\frac{3}{4} i g^2 \frac{m_H^2}{m_W^2}$$

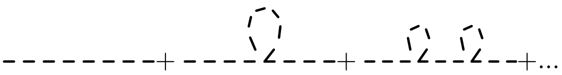
The amplitude for this diagram is given in terms of the 4–momentum q in the loop and in terms of the cutoff scale Λ. Note that the Standard Model with only dimension–4 operators does not offer an interpretation for such a scale, so at the end of the argument we have to perform the limit $\Lambda \to \infty$. Introducing a cutoff scale and sending it to infinity is certainly a physical regularization scheme:

$$
\begin{aligned}
\mathcal{M}_H^2 &= \int^\Lambda \frac{d^4q}{(2\pi)^4} \left(-\frac{3}{4} i ... \right) \frac{1}{q^2 - m_H^2} \\
&= \int \frac{d^4q}{(2\pi)^4} \left(\frac{1}{q^2 - m_H^2} - \frac{1}{q^2 - \Lambda^2} \right) \qquad \text{Pauli–Villars regularization} \\
&= (m - H^2 - \Lambda^2) \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 - m_H^2)(q^2 - \Lambda^2)} \\
&= (m_H^2 - \Lambda^2) \int \frac{d^4q}{(2\pi)^4} \int_0^1 dx \int_0^1 dy \frac{2\delta(1-x-y)}{\left[(q^2 - m_H^2)x + (q^2 - \Lambda^2)y\right]^2} \\
&= 2(m_H^2 - \Lambda^2) \int_0^1 dx \int_0^1 dy \; \delta(1-x-y) \int \frac{d^4q}{(2\pi)^4} \frac{1}{\left[q^2 - x m_H^2 - y\Lambda^2\right]^2} \\
&\sim 2(m_H^2 - \Lambda^2) \int_0^1 dx \int_0^1 dy \; \delta(1-x-y) \frac{i}{16\pi^2} \\
&= -\frac{2i\Lambda^2}{16\pi^2} \left(-\frac{3}{4} i g^2 \frac{m_H^2}{m_W^2} \right) \qquad \text{now with couplings} \\
&= -\frac{3}{32\pi^2} g^2 \frac{m_H^2}{m_W^2} \Lambda^2
\end{aligned}
\tag{4}
$$

where in the first line we have set $(-3/4i...) = 1$ for simplicity. The Pauli–Villars regularization using a cutoff scale Λ works as:

$$
\frac{1}{q^2 - m_H^2} - \frac{1}{q^2 - \Lambda^2} = \begin{cases} \dfrac{1}{q^2 - m^2} & q^2 \ll \Lambda^2 \\ \dfrac{1}{q^2 - q^2} & q^2 \gg \Lambda^2 \end{cases}
\tag{5}
$$

How does this affect the mass?



$$
\begin{aligned}
=& \frac{1}{q^2 - m_H^2} + \frac{1}{q^2 - m_H^2} \mathcal{M}_H^2 \frac{1}{q^2 - m_H^2} + \frac{1}{q^2 - m_H^2} \mathcal{M}_H^2 \frac{1}{q^2 - m_H^2} \mathcal{M}_H^2 \frac{1}{q^2 - m_H^2} + ... \\
=& \frac{1}{q^2 - m_H^2} \sum_{n=0}^{\infty} \left(\mathcal{M}_H^2 \frac{1}{q^2 - m_H^2} \right)^n = \frac{1}{q^2 - m_H^2} \frac{1}{1 - \mathcal{M}_H^2 \dfrac{1}{m^2 - m_H^2}} = \frac{1}{q^2 - m_H^2 - \mathcal{M}_H^2}
\end{aligned}
\tag{6}
$$

This means, the next–to–order contributions shift the leading–order unrenormalized mass $m_{H,b}^2$ to the unrenormalized next–to–leading order value $m_{H,b}^2 + \mathcal{M}_H^2$. We could calculate all Standard–Model corrections proportional to Λ^2 and obtain:

$$
m_H^2 = m_{H,b}^2 + \frac{3g^2}{32\pi^2} \frac{\Lambda^2}{m_W^2} \left[m_H^2 + 2m_W^2 + m_Z^2 - \frac{4n_f}{3} m_t^2 \right]
\tag{7}
$$

This form is dictated by the fact that the Higgs couples to every Standard–Model particle proportional to its mass. This formula means that the unrenormalized Higgs mass will always be driven to the cutoff Λ of the Standard Model, unless we do something about it.

The naive solution $m_H^2 + 2m_W^2 + m_Z^2 - 4n_f m_t^2/3 = 0$ is called Veltman's condition, but it is of course only 1–loop solution. Moreover, talk to Martin Schmaltz about it and watch his (correct) rant about different particles in the loop behaving differently in the Pauli–Villars regularization.

Dimensional regularization

In modern calculations we usually use dimensional regularization, $d^4q \to d^Nq$. In his QCD book Rick Field gives the formula for the relevant Feynman integral:

$$\int \frac{d^Nq}{(2\pi)^N} \frac{(q^2)^R}{(q^2-m^2)^M} = \frac{i(-)^{R-M}}{(16\pi^2)^{N/4}} (m^2)^{R-M+N/2} \frac{\Gamma(R+N/2)\Gamma(M-R-N/2)}{\Gamma(N/2)\Gamma(M)} \tag{8}$$

With $R=0, M=1, N=4-2\varepsilon$ we find

$$\begin{aligned}\int \frac{d^Nq}{(2\pi)^N} \frac{(q^2)^R}{(q^2-m^2)^M} &= -\frac{i}{(16\pi^2)^{N/4}} (m_H^2)^{-1+N/2} \frac{\Gamma(N/2)\Gamma(1-N/2)}{\Gamma(N/2)\Gamma(1)} \\ &= -\frac{i}{(16\pi^2)^{1-\varepsilon/2}} m_H^{2-2\varepsilon} \frac{\Gamma(\varepsilon)}{\varepsilon-1} \\ &= -\frac{i}{(16\pi^2)^{1-\varepsilon/2}} m_H^{2-2\varepsilon} \frac{e^{-\gamma_E\varepsilon}}{\varepsilon-1} \left(\frac{1}{\varepsilon}+\frac{\zeta_2}{2}\varepsilon+...\right)\end{aligned}$$

We can use a simple trick $x^\varepsilon = \exp(\log x^\varepsilon) = \exp(\varepsilon \log x) = 1 + \varepsilon \log x + \varepsilon^2/2 \log^2 x + ...$ to compute the limit $\varepsilon \to 0$

$$\int \frac{d^Nq}{(2\pi)^N} \frac{(q^2)^R}{(q^2-m^2)^M} = \frac{i}{16\pi^2} m_H^2 \left(\frac{1}{\varepsilon} + \mathcal{O}(\varepsilon^0)\right) \tag{9}$$

The next–to–leading order contribution to the Higgs mass now looks like $m_H^2/\varepsilon + \mathcal{O}(\varepsilon^0)$ and will be removed by for example on–shell or $\overline{\text{MS}}$ renormalization. The problem with this magical vanishing of the quadratic divergence is that dimensional regularization $(4-2\varepsilon)$ is not a physical regularization scheme, as far as we can see...

Numerical results

We can quantify the level of fine tuning, which would be required to remove the huge next–to–leading order contributions using a counter term.

$$m_H^2 = m_{H,b}^2 + \mathcal{M}_H^2 - \delta m_H^2 \tag{10}$$

which using for example $\Lambda = 10$ TeV implies

$$\delta m_H^2 \sim \mathcal{M}_H^2 = \begin{cases} -\frac{3}{8\pi^2}\lambda_t^2\Lambda^2 \sim -(2\text{ TeV})^2 & t\text{ loop} \\ \frac{1}{16\pi^2} g^2\Lambda^2 \sim (100\text{ TeV})^2 & W\text{ loop} \\ \frac{1}{16\pi^2}\lambda^2\Lambda^2 \sim (500\text{ TeV})^2 & H\text{ loop}\end{cases} \tag{11}$$

For a varying cutoff scale Λ we find:

$$m_H = m_{H,b}^2 - \delta m_H^2 + \begin{cases} (-100+10+5)\ (200\text{ GeV})^2 & \text{for } \Lambda = 10\text{ TeV} \\ (-10000+1000+500)\ (200\text{ GeV})^2 & \text{for } \Lambda = 100\text{ TeV} \\ ... \end{cases} \tag{12}$$

To summarize our arguments for physics beyond the Standard Model, before we discuss possible scenarios, here is the short list:

1. the experimental reason to believe in BSM physics is dark matter (or the experience that until now every increase in energy has brought us in new physics). Any new 100 GeV WIMP can do the job
2. the theoretical reason to believe in BSM physics is the lack of stability of fundamental scalar masses in perturbative field theory

⇒ So what could this new physics at the TeV scale be?

supersymmetry	cancel Λ^2 terms
little Higgs (bosonic supersymmetry)	cancel Λ^2 terms
composite-Higgs models: technicolor, topcolor, ...	cut off integral
extra dimensions	$\Lambda_{\text{Planck}} \to$ TeV
...	...

II. FUNDAMENTAL PLANCK SCALE IN (4+N) DIMENSIONS

A. Einstein–Hilbert action and proper time

To understand the trick of extra dimensions, we have to generalize the Einstein–Hilbert action to $(4+n) > 4$ dimensions.
Let's first remember/learn what this action means (following *e.g.* Peacock's book):

$$S = -\frac{1}{2} \int d^4x \sqrt{-g} M_\star^2 \, R \tag{13}$$

The root $\sqrt{-g}$ can also be written as $\sqrt{|g|}$, remembering that we are using the metric $\eta_{\mu\nu} = (+,-,-,-)$.
Start with the relativistic distance between two space–time points

$$ds^2 = \eta_{\mu\nu} x^\mu x^\nu \tag{14}$$

where we call

$$\begin{aligned} & ds^2 < 0, \ \text{spacelike} \\ & ds^2 = 0, \ \text{lightlike} \\ & ds^2 > 0, \ \text{timelike(the only allowed for massive particles)} \end{aligned} \tag{15}$$

To get a feeling for what ds^2 means let's integrate it along a path in space–time:

– (trivial case of) object at rest: $\Delta x^\mu = (\Delta t, 0, 0, 0)$

$$\int d^4x \sqrt{ds^2} = \int \sqrt{\eta_{\mu\nu} x^\mu x^\nu} = \int \sqrt{(dt)^2} = \int dt = \Delta t \tag{16}$$

which is just the time felt by this observer at rest

– moving along $\hat{x}$ direction

$$\begin{aligned} \int d^4x \sqrt{ds^2} &= \int \sqrt{(dt)^2 - (dx)^2} && \text{invariant under Lorentz trafos, go check...} \\ &= \int \sqrt{(dt)^2} && \text{in proper Lorentz frame} \\ &= \Delta t && \text{again time felt by the resting observer} \end{aligned} \tag{17}$$

⇒ definition of proper time:

$$d\tau = \sqrt{\eta_{\mu\nu} x^\mu x^\nu} \tag{18}$$

B. Free fall, metric and Christoffel symbols

Start with the equations describing a freely falling object

$$\begin{aligned} \frac{d^2 x^\mu}{d\tau^2} &= 0 && \text{equation of motion for trajectory } x^\mu \\ d\tau^2 &= \eta_{\mu\nu} dx^\mu dx^\nu && \text{just definition of } d\tau^2 \\ dx^\mu &= \frac{\partial x^\mu}{\partial y^\nu} dy^\nu && \text{coordinate transformation} \end{aligned} \tag{19}$$

This means for the proper time

$$\begin{aligned} d\tau^2 &= \eta_{\mu\nu} dx^\mu dx^\nu \\ &= \eta_{\mu\nu} \frac{\partial x^\mu}{\partial y^\rho} dy^\rho \frac{\partial x^\nu}{\partial y^\sigma} dy^\sigma \\ &= \left(\eta_{\mu\nu} \frac{\partial x^\mu}{\partial y^\rho} \frac{\partial x^\nu}{\partial y^\sigma} \right) dy^\rho dy^\sigma \equiv g_{\rho\sigma} dy^\rho dy^\sigma \end{aligned} \tag{20}$$

→ definition of <u>general metric tensor</u> and its transformation law

$$\boxed{g_{\rho\sigma} = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial y^\rho} \frac{\partial x^\nu}{\partial y^\sigma}} \tag{21}$$

Similarly, we can transform the equation of motion (just sketched here)

$$\begin{aligned}
\frac{dx^\mu}{d\tau} &= \frac{\partial x^\mu}{\partial y^\nu} \frac{\partial y^\nu}{d\tau} \\
0 \equiv \frac{d^2 x^\mu}{d\tau^2} &= \frac{d}{d\tau} \frac{\partial x^\mu}{\partial y^\nu} \cdot \frac{\partial y^\nu}{d\tau} + \frac{\partial x^\mu}{\partial y^\nu} \frac{\partial^2 y^\nu}{d\tau^2} \\
&= \frac{dy^\nu}{d\tau} \frac{\partial x^\mu}{\partial y^\nu \partial y^\rho} \frac{\partial y^\rho}{d\tau} + \frac{\partial x^\mu}{\partial y^\nu} \frac{\partial^2 y^\nu}{d\tau^2} \\
\Leftrightarrow 0 \equiv \frac{d^2 y^\nu}{d\tau^2} &+ \Gamma^\nu_{\rho\sigma} \frac{dy^\rho}{d\tau} \frac{dy^\sigma}{d\tau}
\end{aligned} \tag{22}$$

→ definition of Christoffel symbol

$$\boxed{\Gamma^\nu_{\rho\sigma} = \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial^2 x^\mu}{\partial y^\rho \partial y^\sigma}} \tag{23}$$

which together with the metric tensor completely describes the kinematics in general relativity. The next question would be — can we express for example the Christoffel symbol in terms of the metric?
Compute

$$\frac{\partial g_{\rho\sigma}}{\partial y^\lambda} = \frac{\partial}{\partial y^\lambda} \left(\eta_{\mu\nu} \frac{\partial x^\mu}{\partial y^\rho} \frac{\partial x^\nu}{\partial y^\sigma} \right) = \Gamma^\mu_{\rho\lambda} g_{\sigma\mu} + \Gamma^\mu_{\sigma\lambda} g_{\rho\mu} \tag{24}$$

which by exchanging indices can be combined to

$$\frac{\partial g_{\mu\nu}}{\partial y^\lambda} + \frac{\partial g_{\mu\lambda}}{\partial y^\nu} - \frac{\partial g_{\nu\lambda}}{\partial y^\mu} = 2\Gamma^\rho_{\nu\lambda} g_{\rho\mu} \tag{25}$$

using $g_{\rho\mu} g^{\mu\rho} = 1$.
In other words, we can express the Christoffel symbols in terms of the metric tensors (and its derivatives):

$$\boxed{\Gamma^\rho_{\nu\lambda} = \frac{1}{2} g^{\rho\mu} \left(\frac{\partial g_{\mu\nu}}{\partial y^\lambda} + \frac{\partial g_{\mu\lambda}}{\partial y^\nu} - \frac{\partial g_{\nu\lambda}}{\partial y^\mu} \right)} \tag{26}$$

If we now have a guess, we would think that we can write the action for general relativity in terms of the metric tensor.

C. Simple invariant Lagrangian

Again, I will just sketch how we can write down a simple Lagrangian for general relativity. The transformation of the metric tensor from one coordinate system x into another one x' reads:

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial^2 x^\sigma}{\partial x'^\nu} \, g_{\rho\sigma} \tag{27}$$

which fixes the Jacobian

$$\det g'_{\mu\nu} = \left| \frac{\partial x}{\partial x'} \right|^2 \det g_{\mu\nu} \tag{28}$$

which we can use to compensate the Jacobian from the integration measure

$$d^4 x'^\mu = \left| \frac{\partial x'}{\partial x} \right| \, d^4 x^\mu \tag{29}$$

to build a very simple Lorentz–invariant Lagrangian

$$\boxed{\int d^4x \, \sqrt{-g} \, \rho = \text{const}} \tag{30}$$

for any Lorentz–scalar density ρ.
What kind of scalar — made out of $g_{\mu\nu}$ and possibly $\Gamma^\mu_{\rho\sigma}$ — can we use to for example include sources (particles) in this action? A guess would be to find a tensor made out of second derivatives to be useful in a field equation and gives the right special relativistic limit. Luckily, there is a unique tensor which serves this purpose (following Peacock's book): the Riemann tensor

$$R^\mu_{\alpha\beta\gamma} \equiv \frac{d\Gamma^\mu_{\alpha\gamma}}{dx^\beta} - \frac{d\Gamma^\mu_{\alpha\beta}}{dx^\gamma} + \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\gamma\alpha} - \Gamma^\mu_{\sigma\gamma}\Gamma^\sigma_{\beta\alpha} \tag{31}$$

which can be contracted to give the Ricci tensor

$$R_{\alpha\beta} \equiv R^\mu_{\alpha\beta\mu} \tag{32}$$

and the Ricci scalar

$$R \equiv g^{\alpha\beta} R_{\alpha\beta} \equiv R^\mu_\mu \tag{33}$$

which is precisely the scalar we are looking for to put into our action!
Before constructing the action we should check the mass units:

$$\begin{aligned} [d^4x] &= m^{-4} \\ [g] &= m^0 \\ [\Gamma] &= m \\ [R] &= m^2 \end{aligned} \tag{34}$$

which means that $[d^4x\sqrt{-g}R] = m^{-2}$. So we are not quite there yet, the mass unit of the action is still wrong. We have to introduce a fundamental mass parameter into general relativity which we call $M_\star$. We arrive at:

$$\boxed{S = -\frac{1}{2} \int d^4x\sqrt{-g} M_\star^2 \, R} \tag{35}$$

with a conventional numerical c-number factor in front. Since we are still talking about 4 dimensions we can identify $M_\star \equiv M_{\text{Planck}}$, with the Planck mass measured through the gravitational coupling G_N.

D. Now we are ready!

For the rest of the discussion of large flat extra dimensions I will largely follow Graham Kribs' hep-ph/0605325, both in logic and notation. Our first task is to write down the Einstein–Hilbert action in (n+4) dimensions, to see how extra space dimensions actually solve the hierarchy problem. Inserting the Ricci scalar — derived from the Riemann tensor as unique a building bock for Einstein's field equations — into our old action we obtain the correct $(4+n)$–dimensional action as

$$S_{\text{bulk}} = -\frac{1}{2} \int d^{4+n}x \, \sqrt{-g^{(4+n)}} \, M_\star^{n+2} \, R^{(4+n)} \tag{36}$$

where bulk means that this action governs our $(4+n)$–dimensional space, to be distinguished from 'brane', which refers to a 4–dimensional subspace where all or some of our Standard Model field live. The increased power of the Planck mass $M_\star$ is again chosen to correct the over–all mass dimension. The mass dimension of the $(4+n)$–dimensional Ricci scalar is the same as it's 4–dimensional counterpart, because it is created by the number of space–time derivatives.
This formula still has to be filled with physics content, *i.e.* we have to define $g^{(4+n)}$ and $R^{(4+n)}$. Of course, we have to distinguish the different Dirac indices we are talking about. In general, the usual space–time vector get extended to $x_A = \{x_\mu, y_j\}$ where the usual Greek indices run from $\mu = 0 \cdots 3$ and the additional Roman indices run fill the

remaining n components. I will try to stick to running the Roman indices as $j = 4 \cdots (4+n)$. The capital Roman index describes the bulk and runs from $A = 0 \cdots (4+n)$. The bulk metric can be written as

$$\begin{aligned} ds^2 &= g^{(4+n)}_{MN} dx^M dx^N & & M,N = 0,...,n+4-1 \\ &= g^{(4)}_{\mu\nu} dx^\mu dx^\nu - dx^j dx_j & & j = 4,...,n+4-1 \\ &= (\eta_{\mu\nu} + h_{\mu\nu}) \, dx^\mu dx^\nu - dx^M dx_M & & \text{allowing for a 4–dimensional graviton} \\ &= (\eta_{\mu\nu} + h_{\mu\nu}) \, dx^\mu dx^\nu - r^2 d\Omega_{(n)} & & \text{after compactifying } j \text{ on a torus} \end{aligned} \tag{37}$$

Note that this simple model requires the extra dimensions to be flat (compactifying on a torus does not mean we bend them, it is just another way of referring to periodic boundary conditions). The power of r arises because ds^2 is bilinear in the space–time vector. At this point, I should reiterate the specific requirements we have to make on the extra dimensions to make the following argument.

– if we write the split $(4+n)$–dimensional metric tensor it looks like $g^{(4+n)} = g^{(4)} \otimes (-1)$, as long as we assume that the extra dimensions are flat. We cannot allow any sources (particles or stars or black holes) off our Standard Model brane. Sources on our brane will of course affect the bulk, but we will discuss later how a mathematically infinitely narrow brane is unrealistic. So we might imagine looking at a slightly wider brane and ignore the bulk region close to the brane, so we can assume that the extra dimensions are indeed flat. For the Einstein–Hilbert action this means $\sqrt{g^{(4+n)}} = \sqrt{g^{(4)}}$, as long as the extra dimensions are perpendicular to our (3+1)–dimensional brane.

– the special geometry of the extra dimensions allows us to rewrite and if possible integrate out all additional dimensions, as long as we assume that the extra dimensions are orthogonal to our brane, as suggested by the diagonal metric tensor:

$$\int d^{4+n}x = \int d^4x \, r^n \, d\Omega_{(n)} = (2\pi r)^n \int d^4x \equiv V_{\text{torus}} \int d^4x \tag{38}$$

– from looking at Riemann's tensor you can guess that flat, orthogonal, extra dimensions without any sources will hardly affect the Ricci scalar. More specifically, the Ricci scalars in 4 and $(4+n)$ dimensions can be linked through Einstein's field equations, which we have not talked about yet. In the absence of matter (which we are assuming for the extra dimensions) they read

$$R_{jk} - \frac{1}{n+2} g_{jk} R = 0 \tag{39}$$

which after contracting with g^{jk} requires $R = 0$. In other words, the extra–dimensional part of R is zero, or $R^{(4+n)} = R^{(4)}$.

We can combine these pieces and simplify the higher–dimensional bulk action

$$\begin{aligned} S_{\text{bulk}} &= -\frac{1}{2} M_\star^{n+2} \int d^{4+n}x \, \sqrt{-g^{(4+n)}} \, R^{(4+n)} \\ &= -\frac{1}{2} M_\star^{n+2} \, (2\pi r)^n \int d^4x \, \sqrt{-g^{(4)}} \, R^{(4)} \\ &\equiv -\frac{1}{2} M_{\text{Planck}}^2 \int d^4x \sqrt{-g^{(4)}} \, R^{(4)} \end{aligned} \tag{40}$$

In the last line we have matched the two theories, *i.e.* we have assumed that from a 4–dimensional point of view the actions have to be identical, as long as we do probe high enough energy scales to observe quantum–gravity effects. This leads us to the basis of extra dimensions as a solution to the hierarchy problem: the 4–dimensional Planck scale M_{Planck} which we measure on our brane/in our world is not the fundamental scale of gravity. It is merely a derived parameter which depends on the fundamental $(4+n)$–dimensional Planck scale and on the geometry of the extra dimensions, in the simplest case the compactification radius of the n–dimensional torus. Matching the two theories gives

$$\boxed{M_{\text{Planck}} = M_\star \, (2\pi r M_\star)^{n/2}} \tag{41}$$

The derived 4–dimensional Planck scale is indeed measured to be around 10^{19} GeV. If we can assume that the correction factor $(2\pi r M_\star)^n$ is large we can postulate that the fundamental Planck scale $M_\star$ is not much larger than 1 TeV. In that case the cutoff of our field theory is not much above the expected Higgs boson mass and there is no problem with the stability of the two scales m_H and $M_\star$, which we introduced as the hierarchy problem.
Assuming $M_\star = 1$ TeV we can solve the equation above for r for a given number of extra dimensions and obtain the compactification radius:

n	r
1	10^{12} m
2	10^{-3} m
3	10^{-8} m
...	...
6	10^{-11} m

Obviously, the case $n = 1$ is dangerous, because gravity gets modified at large distances. For larger values of n we have to test Newtonian gravity at small distances, which is harder. However, small distances just means larger energies, and we might be able to find cosmological or collider observables which are sensitive to such effects. Graham in his overview discusses quite a few of them.

III. GRAVITONS IN FLAT (4+N) DIMENSIONS

Note that at this point we have not talked about particles in the theory. Let's still assume that Standard Model fields do not propagates in more than 4 dimensions. All we postulate is a continuation of Newtonian gravity into $(4 + n)$ dimensions.
Let's consider two masses on our Standard Model brane. For large distances $r' \gg r$ the two masses are far enough apart that the curled–up extra dimensions will not be resolved. Again, we can think about large distances as small energies, which means that our test energy $1/r'$ is too small to see effects coming in at much larger energies $1/r$. Which means that for $r' \gg r$ we observe ordinary 4–dimensional Newtonian gravity.
Probing smaller distances r' (or higher energies $1/r'$) the 4–dimensional distance will fit into the extended extra dimensions, which means that gravity propagates into all $(4 + n)$ dimensions and the volume integral over the n–dimensional torus is cut off by a $(4 + n)$–dimensional sphere with radius r':

$$V(r') = \begin{cases} -G_N \dfrac{m_1 m_2}{r'} \qquad r' \gg r \qquad \text{(4–dim theory at small energies)} \\ -G_N^{(4+n)} \dfrac{m_1 m_2}{r'} \sim -G_N \dfrac{M^2_{\text{Planck}}}{M^2_\star} \dfrac{m_1 m_2}{r'} = -G_N \, (2\pi M_\star)^n \, \dfrac{m_1 m_2}{r'^{1-n}} \qquad r' \ll r \end{cases} \tag{42}$$

tp: this argument sound totally convincing, but I think the power of r' should actually be $1+n$, have to check, damnit! For the 4–dimensional theory Newton's constant is defined as $G_N = 1/(16\pi M^2_{\text{Planck}})$. Modulo c-number pre-factors it is obvious that the fundamental Planck scale in the bulk is given by $G_N^{(4+n)} = 1/(16\pi M_\star^2)$. Such a modification of Newtonian gravity can be tested experimentally without even looking at the details of a model!

A. Propagating an extra–dimensional graviton

Now we understand why large extra space dimensions solve the hierarchy problem. We also know how Newtonian gravitation is modified. However, if $M_\star \sim 1$ TeV and we can probe these scales at colliders, we have to understand quantum gravity effects at colliders. Which takes us to Kaluza–Klein effective theories.
First, we expand the $(4+n)$–dimensional metric around the flat metric η_{MN}, treating the resulting $(n+4)$–dimensional graviton field h_{MN} as a small perturbation:

$$\begin{aligned} ds^2 &= g^{(4+n)}_{MN} dx^M dx^N \qquad\qquad M, N = 0, ..., 3+n \\ &= \left(\eta_{MN} + \frac{1}{M_\star^{n/2+1}} h_{MN} \right) dx^M dx^N \end{aligned} \tag{43}$$

The factor $1/M_\star^{n/2+1}$ fixes the mass unit of the graviton to $[h] = m^{1+n/2}$. In general, for a boson in $(4+n)$-dimensions we would expect to be able to write down a squared mass term (corresponding to the Klein–Gordon equation) in the Lagrangian, which means $[d^{4+n}x\, m^2\, SS] = m^{-4-n}\, m^2\, m^{2(1+n/2)} = m^0$ for the correct mass dimension of the bosonic field.
At this stage we would not get around computing Ricci tensor/scalar (which I will of course not do in this lecture), to express the left–hand side of Einstein's equation $R_{AB} - g_{AB}R/(2+n)$ in terms of the graviton field h_{AB} and its derivatives. Here is how it would be done. We start with the definitions in $(n+4)$ dimensions and a short-hand notation of derivative with respect to x_C:

$$\begin{aligned}
R_{AB} &= R^M_{ABM} = \frac{d\Gamma^M_{AM}}{dx^B} - \frac{d\Gamma^M_{AB}}{dx^M} + \Gamma^M_{SB}\Gamma^S_{MA} - \Gamma^S_{SM}\Gamma^S_{BA} \\
R &= g^{AB}R_{AB} \\
\Gamma^M_{AB} &= \frac{1}{2} g^{MS}\left(\frac{\partial g_{SA}}{\partial x^B} + \frac{\partial g_{SB}}{\partial x^A} - \frac{\partial g_{AB}}{\partial x^S}\right) \\
&= \frac{1}{2}\left(\eta^{MS} - \frac{1}{M_\star} h^{MS}\right)\left(\partial_B h_{SA} + \partial_A h_{SB} - \partial_s h_{AB}\right)
\end{aligned} \tag{44}$$

Note that the definition of R_{AB} includes total derivatives with respect to x. Just to give you an idea we can compute the partial derivative of the Christoffel symbols:

$$\begin{aligned}
\partial_B \Gamma^M_{AM} &= \frac{1}{2}\partial_B\left(\eta^{MS} - \frac{1}{M_\star^{1+n/2}} h^{MS}\right)\left(\partial_M h_{SA} + \partial_A h_{SM} - \partial_S h_{AM}\right) \\
&+ \frac{1}{2}\left(\eta^{MS} - \frac{1}{M_\star^{1+n/2}} h^{MS}\right)\left(\partial_B\partial_M h_{SA} + \partial_B\partial_A h_{SM} - \partial_B\partial_S h_{AM}\right) \\
&= -\frac{1}{2M_\star^{1+n/2}}\partial_B h^{MS}\left(\partial_M h_{SA} + \partial_A h_{SM} - \partial_S h_{AM}\right) \\
&+ \frac{1}{2}\left(\partial_B\partial^S h_{SA} + \partial_B\partial_A h^S_S - \partial_B\partial_S h^S_A\right) \\
&- \frac{1}{2M_\star^{1+n/2}} h^{MS}\left(\partial_B\partial_M h_{SA} + \partial_B\partial_A h_{SM} - \partial_B\partial_S h_{AM}\right)
\end{aligned} \tag{45}$$

$$\begin{aligned}
\partial_M \Gamma^M_{AB} &= \frac{1}{2}\partial_M\left(\eta^{MS} - \frac{1}{M_\star^{1+n/2}} h^{MS}\right)\left(\partial_B h_{SA} + \partial_A h_{SB} - \partial_S h_{AB}\right) \\
&+ \frac{1}{2}\left(\eta^{MS} - \frac{1}{M_\star^{1+n/2}} h^{MS}\right)\left(\partial_M\partial_B h_{SA} + \partial_M\partial_A h_{SB} - \partial_M\partial_S h_{AB}\right) \\
&= -\frac{1}{2M_\star^{1+n/2}}\partial_M h^{MS}\left(\partial_B h_{SM} + \partial_A h_{SB} - \partial_S h_{AB}\right) \\
&+ \frac{1}{2}\left(\partial^S\partial_B h_{SA} + \partial^S\partial_A h_{SB} - \partial^S\partial_S h_{AB}\right) \\
&- \frac{1}{2M_\star^{1+n/2}} h^{MS}\left(\partial_M\partial_B h_{SA} + \partial_M\partial_A h_{SB} - \partial_M\partial_S h_{AB}\right)
\end{aligned} \tag{46}$$

The other kind of terms in R_{AB} yields similar terms of the kind:

$$\Gamma^M_{SB}\Gamma^S_{MA} = \frac{1}{4}\left(\eta - \frac{1}{M_\star^{1+n/2}}...h\right)(\partial h + ...)\left(\eta - \frac{1}{M_\star^{1+n/2}}...h\right)(\partial\eta + ...) \tag{47}$$

Combining all these terms should (according to Graham, who cites another paper by Gian Giudice, Ricardo Rattazzi and James Wells) give the final result for the left–hand side of Einstein's equations:

$$\begin{aligned}
&R_{AB} - \frac{1}{n+2} g_{AB} R \\
&= \frac{1}{M_\star^{1+n/2}}\left[\Box h_{AB} - \partial_A\partial^C h_{CB} - \partial_B\partial^C h_{CA} + \partial_A\partial^B h^C_C - \eta_{AB}\Box h^C_C + \eta_{AB}\partial^C\partial^D h_{CD}\right]
\end{aligned} \tag{48}$$

for a general metric/graviton in $(4+n)$ dimensions. The d'Alembert operator is defined as $\Box = \partial_C\partial^C$.

B. Brane matter and bulk gravitons

The right–hand side of Einstein's equations is given by the energy–momentum tensor, again normalized to the proper mass dimension. The energy–momentum tensor can be computed from the Lagrangian for the respective theory using a function derivative with respect to the metric. We will give examples later, at this point all we need to know is that T_{AB} is a function which includes all particle fields which live on our brane (and possibly in the bulk):

$$\boxed{R_{AB} - \frac{1}{2+n} g_{AB} R = -\frac{T_{AB}}{M_\star^{2+n}}} \tag{49}$$

The usual 4–dimensional $T_{\mu\nu}$ we have to generalize to $(4+n)$ dimensions. Obviously, the tensor rank (size of the matrix) increases from 4×4 to $(4+n)\times(4+n)$. Moreover, each entry now has dimension m^{4+n}, which requires the proper normalization using the usual (only) mass scale $M_\star$.
Our model is still assuming that all Standard Model fields are confined to our brane. In that case we can simplify the $(4+n)$–dimensional energy–momentum tensor:

– all matter is localized to $y_j = 0$, which means all entries in the energy–momentum tensor have to be localized:

$$\boxed{T_{AB}(x;y) = T_{AB}(x)\delta^{(n)}(y)} \tag{50}$$

note that this $\delta(y)$ is dodgy from a Heisenberg uncertainty point of view, because we pretend to know exactly the location of a matter particle in the extra dimension. Which means we know nothing about its momentum into this direction. However, the momentum we also localize, so that scattering processes with exclusively Standard Model particles observe 4–momentum conservation on the brane. The solution to this problem is to postulate a small finite width of the Standard Model brane. As far as this size is larger than the inverse energy scale we are probing our system with, this approximation is not a problem. Naturally, a size $1/M_\star$ will be fine, because above this scale we will not be able to compute anything with our effective Kaluza–Klein theory anyhow.

– finite entries into the energy–momentum tensor only appear on the brane, which together with the last point gives us the form of the energy momentum tensor as it appears on the right–hand side of Einstein's equations:

$$T_{AB}(x;y) = \eta^\mu_A \, \eta^\nu_B \, T_{\mu\nu}(x)\, \delta^{(n)}(y) = \begin{pmatrix} T_{\mu\nu}(x)\,\delta^{(n)}(y) & 0 \\ 0 & 0 \end{pmatrix} \tag{51}$$

Of course, the x appearing in the argument of our energy–momentum tensor do not have to be only space coordinates. It means that all arguments of $T_{\mu\nu}$ are localized to the Standard Model brane. As our usual check we look at the mass dimensions of our different objects: if $[T_{AB}] = m^{4+n}$ and $[\delta^{(n)}(y)] = m^n$ (remember how is cancels integrations) then $[T_{\mu\nu}] = m^4$, as expected.

The set of $(n+4)^2$ Einstein equations now splits into homogeneous equations for the bulk (including the bulk–brane mixing indices) and into a regular inhomogeneous equation for the brane:

$$\begin{aligned}
R_{\mu\nu} - \frac{1}{2+n} g_{\mu\nu} R &= -\frac{T_{\mu\nu}}{M_\star^{2+n}} && \text{is a 4-dimensional theory!} \\
R_{\mu k} - \frac{1}{2+n} g_{\mu k} R &= 0 \\
R_{jk} - \frac{1}{2+n} g_{jk} R &= 0 && \text{the condition for solving the hierarchy problem } R^{(4)} \equiv R^{(4+n)}
\end{aligned} \tag{52}$$

Just as before, we postulate periodic boundary conditions in all extra dimensions, with a compactification radius r

$$x_M = (x_\mu; y_i) \qquad i \geq 1 \qquad y_i \equiv y_i + 2\pi r \tag{53}$$

which means we can write $h_{AB}(x_M)$ as a Fourier series in the extra dimensions:

$$h_{AB}(x;y) = \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_j=-\infty}^{\infty} \frac{h^{(m)}_{AB}(x)}{\sqrt{(2\pi r)^n}} e^{i\frac{m_j y_j}{r}} \tag{54}$$

Note that we are now evaluating the graviton in a mixed position space x_μ and (Fourier–) momentum space ($y_j \mapsto m_j$). We can rewrite the left–hand side of the Einstein equations in this mixed space. We already phrased it in terms of the $(4+n)$-dimensional graviton field h_{AB} and its derivatives. Take for example the first (d'Alembert) term:

$$R_{AB} - \frac{1}{2+n} g_{AB} R \sim \frac{1}{M_\star^{1+n/2}} \left(\Box h_{AB} + \cdots \right) \equiv -\frac{T_{AB}}{M_\star^{2+n}} = -\frac{\eta_A^\mu \eta_B^\nu T_{\mu\nu} \delta^{(n)}(y)}{M_\star^{2+n}} \tag{55}$$

The d'Alembert term can be written in its Fourier components

$$\begin{aligned}
\Box h_{AB} &= \sum_{m_j} \frac{1}{(2\pi r)^{n/2}} \partial_C \partial^C \left[h_{AB}^{(m)}(x) \; e^{i(m\cdot y)/r} \right] \\
&= \sum_{m_j} \frac{1}{(2\pi r)^{n/2}} \partial_C \left[\left(\delta_\mu^C h_{AB}^{(m)}(x) + h_{AB}^{(m)}(x) \frac{i m_j}{r} \delta_j^C \right) \; e^{i(m\cdot y)/r} \right] \\
&= \sum_{m_j} \frac{1}{(2\pi r)^{n/2}} \left[\left(\partial_\mu \partial^\mu h_{AB}^{(m)}(x) + 0 \right) + \left(\delta_\mu^C h_{AB}^{(m)}(x) + h_{AB}^{(m)}(x) \frac{i m_j}{r} \delta_j^C \right) \frac{i m_k}{r} \eta_{Ck} \right] e^{i(m\cdot y)/r} \\
&= \sum_{m_j} \frac{1}{(2\pi r)^{n/2}} \left[\Box h_{AB}^{(m)}(x) - h_{AB}^{(m)}(x) \frac{m^j m_j}{r^2} \right] e^{i(m\cdot y)/r} \\
&= \sum_{m_j} \frac{1}{(2\pi r)^{n/2}} \; e^{i(m\cdot y)/r} \left(\Box + \hat{k}^2 \right) h_{AB}^{(m)}(x)
\end{aligned} \tag{56}$$

with $\hat{k} \equiv m_j/r$ and $\hat{k}^2 \equiv \sum |m_j/r|^2$. The d'Alembert box operator acting on $h_{AB}^{(m)}(x)$ is just $\partial_\mu \partial^\mu$. This Fourier transform works the same way for $h_{\mu\nu}, h_{\mu j}, h_{jk}$.
The right-hand side of Einstein's equations are either zero

$$\sum_{m_j} \frac{1}{(2\pi r)^n} 0 \cdot e^{i(m\cdot y)/r} = 0 \tag{57}$$

or functions of the 4-dimensional variables

$$\sum_{m_j} \frac{1}{(2\pi r)^n} \; f(x) \cdot e^{i(m\cdot y)/r}. \tag{58}$$

Even though the graviton fields $h_{AB}^{(m)}(x)$ are not yet the physical fields we will define in a minute, we already see the structure of the equation of motion for all fields involved: they include a quadratic term $(\Box^2 + \hat{k}^2) h_{AB}(x) = ...$ which means that they have masses m_j/r, where m_j are integers. Kaluza-Klein gravitons have a massless ground state $m_j = 0$ and excited states labeled by $|\overrightarrow{m}_j|$.

C. Kaluza–Klein towers

The detailed form of the physical graviton fields is not particularly important. Their precise definition can be found in the Giudice-Rattazzi-Wells paper, their counting of degrees of freedom in Graham's lectures. The Einstein equations in the most convenient form look like:

$$\begin{aligned}
&(\Box + \hat{k}^2) \; G_{\mu\nu}^{(k)} &&= \frac{1}{M_{\text{Planck}}} \left[-T_{\mu\nu} + \left(\frac{\partial_\mu \partial_\nu}{\hat{m}^2} + \eta_{\mu\nu} \right) \frac{T_\lambda^\lambda}{3} \right] && \text{massive graviton} \\
&(\Box + \hat{k}^2) \; H^{(\overrightarrow{k})} &&= \frac{1}{2 M_{\text{Planck}}} \sqrt{\frac{3(n-1)}{n+2}} T_\mu^\mu && \text{scalar, includes radion} \\
&(\Box + \hat{k}^2) \; V_{\mu j}^{(k)} &&= 0 && \text{graviscalars} \\
&(\Box + \hat{k}^2) \; S_{jk}^{(k)} &&= 0 && \text{massive graviphotons}
\end{aligned} \tag{59}$$

The structure of these equation reveals a few particularities: the fields $V_{\mu j}^{(k)}$ and $S_{jk}^{(k)}$ do not couple to the Standard model, because in the presence of a general energy-momentum tensor they still behave like free massless fields. The

massive gravitons $G^{(k)}_{\mu\nu}$ couple to the Standard Model. Their Fourier coordinate only appears as a mass term $\hat{k}^2$ and in the coupling to the trace of the energy-momentum tensor. This means their couplings are level–degenerate and their masses and couplings depend only on the length, but not on the orientation of the vector m_j.
I can only quote the properties of conformally invariant theories, where $T^\mu_\mu = 0$. For such massless theories we find

$$\boxed{(\Box + \hat{k}^2)\; G^{(k)}_{\mu\nu} = -\frac{T^{\mu\nu}}{M_{\text{Planck}}}} \tag{60}$$

which describes physical gravitons at the LHC, produced by quark or gluon interactions and either vanishing or decaying to leptons.
The scalar mode $H^{(\vec{k})}$ plays a special role. Its massless mode is called a radion and corresponds to a fluctuation of the volume of the compactified extra-dimension. We assume that the compactification radius r is somehow stabilized, and such mechanism gives mass to the radion. More importantly, the radion only couples to a massive theory, so it is not surprising that as a scalar with no Standard Model charge it will mix with a Higgs boson without very drastic effects.
Before we discuss the coupling of gravitons to Standard Model particles we introduce a mechanism for summing over the Kaluza-Klein levels. The mass splitting between the KK states is given by $1/r$ which translates into ($M_\star = 1$ TeV as before):

$$\delta m \sim \frac{1}{r} = 2\pi M_\star \left(\frac{M_\star}{M_{\text{Planck}}}\right)^{2/n} = \begin{cases} 0.003 \text{ eV} & (n=2) \\ 0.1 \text{ MeV} & (n=4) \\ 0.05 \text{ GeV} & (n=6) \end{cases} \tag{61}$$

On the scale of modern light-energy experiments, this mass splitting is tiny, $0.003 \text{ eV} \ll 0.1 \text{ MeV} \ll 0.05 \text{ GeV} \ll m_Z$. This means, at colliders we will be confronted with towers composed out of a huge number of tightly spaces massive gravitons with identical couplings to Standard–Model particles. Instead of summing for example over all gravitons radiated off an LHC process, we can integrate over a continuous graviton mass space.
For n dimensions we want to compute the numbers of gravitons with masses between $|k|$ and $|k+dk|$. k represents the number of gravitons in the compactified n dimensions. In other words, we need to integrate an n-dimensional sphere:

$$dN = S_{n-1}\; |k|^{n-1}\; d|k| \tag{62}$$

with the area of an n-sphere

$$S_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{63}$$

This density in terms of states we can translate into a mass density kernel, using

$$\frac{dm}{d|k|} = \frac{1}{r} \quad \Rightarrow \quad dN = S_{n-1}\, r^n\, m^{n-1}\, dm \quad \Rightarrow \quad dN = S_{n-1}\, \frac{1}{(2\pi M_\star)^n} \left(\frac{M_{\text{Planck}}}{M_\star}\right)^2 m^{n-1}\, dm \tag{64}$$

We will later think of this distribution as a kernel in for example final-state phase space integrals. Two properties of this distribution dN can be easily read off. The integral is IR finite

$$\begin{aligned} \int dN = \int_0^\mu S_{n-1}\, \frac{1}{(2\pi M_\star)^n} \left(\frac{M_{\text{Planck}}}{M_\star}\right)^2 m^{n-1}\, dm &= S_{n-1}\, \frac{1}{(2\pi M_\star)^n} \left(\frac{M_{\text{Planck}}}{M_\star}\right)^2 \left.\frac{m^n}{n}\right|_0^\mu \\ &= S_{n-1}\, \frac{1}{(2\pi M_\star)^n} \left(\frac{M_{\text{Planck}}}{M_\star}\right)^2 \frac{\mu^n}{n} \end{aligned} \tag{65}$$

and in the UV it is strongly peaked, the stronger the larger n. Note that the limit between the well-defined IR tail and the sharp UV peak is based on the specific of the model, in our case on the existence of only one compactification length scale.

D. Graviton Feynman rules

To compute graviton production cross sections like $pp \to$ KK+jet or $pp \to$ KK $\to \mu^+\mu^-$ we need to couple the gravitons to the Standard model, i.e. write a proper Lagrangian.
Start from the general relativity Lagrangian in $(4+n)$ dimensions

$$S = -\frac{1}{2}\int d^{4+n}x\,\sqrt{|g|}\,M_\star^{n+2}\,R = \int d^{4+n}\mathcal{L}^{(4+n)} \tag{66}$$

which we can express in term of the graviton field h_{AB} (still in (4+n) dimensions)

$$\boxed{\mathcal{L} = -\frac{1}{2}h^{AB}\Box h_{AB} + \frac{1}{2}h^A_A\Box h^B_B - h^{AB}\partial_A\partial_B h^C_C + h^{AB}\partial_A\partial_C h^C_B - \frac{1}{M_\star^{1+n/2}}h^{AB}T_{AB}} \tag{67}$$

The last term corresponds to the right–hand side of the Einstein equations. Instead of deriving this coupling term we can at least check its consistency: if the equations for pure Newtonian gravity give

$$R_{AB} - \frac{1}{n+2}\,g_{AB}\,R = 0 \tag{68}$$

then the inhomogeneous term on the right–hand side

$$R_{AB} - \frac{1}{n+2}\,g_{AB}\,R = -\frac{T_{AB}}{M_\star^{2+n}} \tag{69}$$

has to correspond to a term in in the Lagrangian, given by the usual Euler–Lagrange equations:

$$\frac{\partial\mathcal{L}_{\text{add}}}{\partial h^{AB}} = \frac{\partial}{\partial h_{AB}}\left(-\frac{1}{M_\star^{1+n/2}}h^{AB}T_{AB}\right) = -\frac{T_{AB}}{M_\star^{1+n/2}} \tag{70}$$

Note the mismatch in powers of $M_\star$. We have used a different normalization for Einstein's equation; for $\mathcal{L}$ the proper mass unit is indeed $[1/M_\star^{2+n}\,h^{AB}\,T_{AB}] = m^{-1+n/2}\,m^{1+n/2}\,m^{4+n} = m^{4+n}$.
The tensor graviton field h_{AB} we as usually Fourier transform and express in the more appropriate 4-dimensional Kaluza–Klein fields. Let's assume a massless Standard Model, or in other words we are going to study QED and QCD with KK gravitons. This is appropriate for LHC or linear–collider observables, as long as we stay away from top–quark production. In that case all we are left with is:

$$\begin{aligned}\mathcal{L} = -\sum\Big[&-\frac{1}{2}G^{\mu\nu}(\Box+m^2)G_{\mu\nu} + \frac{1}{2}G^\mu_\mu(\Box+m^2)G^\nu_\nu - G^{\mu\nu}\partial_\mu\partial_\nu G^\lambda_\lambda + G^{\mu\nu}\partial_\mu\partial_\lambda G^\lambda_\nu \\ &- \frac{1}{M_{\text{Planck}}}\,G^{\mu\nu}T_{\mu\nu}\Big]\end{aligned} \tag{71}$$

Two things we observe:

1. the graviton spin–2 propagator is a mess
2. the interaction with massless Standard Model particles is easy

Next, we have to compute the energy–momentum tensor, for example for QED:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\,\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \\ &= -\frac{1}{2}\,\partial_\mu A_\nu\,\partial^\mu A^\nu + \frac{1}{2}\,\partial_\mu A_\nu\,\partial^\nu A^\mu\end{aligned} \tag{72}$$

which means

$$\frac{\partial\mathcal{L}}{\partial A_\nu} = 0 \qquad\qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2}\partial^\mu A^\nu + \frac{1}{2}\partial^\nu A^\mu \tag{73}$$

and gives for the Euler–Lagrange equations (Maxwell equation):

$$0 = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = -\frac{1}{2}\partial_\mu\partial^\mu A^\nu + \partial^\nu(\partial_\mu A^\mu) = -\frac{1}{2}\left(\Box A^\nu - \partial^\nu(\partial_\mu A^\mu)\right) \tag{74}$$

Remember the link with Noether's theorem and the conserved current

$$\partial_\mu j_\mu(x) = 0 \qquad \text{with} \qquad j^\mu = \frac{\partial}{\partial(\partial_\mu A_\nu)}\delta A_\nu \tag{75}$$

Similarly, define the energy–momentum tensor:

$$\begin{aligned}
T^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)}\partial^\nu A_\rho - \mathcal{L}\eta^{\mu\nu} \\
&= \left(-\frac{1}{2}\partial^\mu A^\rho + \frac{1}{2}\partial^\rho A^\mu\right)\partial^\nu A_\rho + \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\eta^{\mu\nu} \\
&= -\frac{1}{2}\left(\partial^\mu A^\rho - \partial^\rho A^\mu\right)\partial^\nu A_\rho + \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\eta^{\mu\nu} \\
&= -\frac{1}{2}F^{\mu\rho}\partial^\nu A_\rho + \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\eta^{\mu\nu}
\end{aligned} \tag{76}$$

From this we can (following Richard Ball's lecture) compute the symmetric energy–momentum tensor

$$\boxed{T^{\mu\nu} = -F^{\mu\rho}F^{\nu}_{\rho} + \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\;\eta^{\mu\nu}} \tag{77}$$

Including fermions we find the energy–momentum tensor for complete massless QED:

$$\begin{aligned}
\frac{-1}{M_{\text{Planck}}}T_{\mu\nu}G^{\mu\nu} = \frac{-1}{M_{\text{Planck}}}\Big[&\frac{i}{4}\overline{\Psi}\left(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu\right)\Psi - \frac{i}{4}\left(\partial_\mu\overline{\Psi}\gamma_\nu + \partial_\nu\overline{\Psi}\gamma_\mu\right)\Psi \\
&+ \frac{1}{2}e_Q\overline{\Psi}\left(\gamma_\mu A_\nu + \gamma_\nu A_\mu\right)\Psi + F_{\mu\rho}F^{\rho}_{\nu} + \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\Big]G^{\mu\nu}
\end{aligned} \tag{78}$$

To obtain the Feynman rules we have to just extract the terms proportional to the relevant external fields:

$$\begin{aligned}
&f(k_1) - f(k_2) - G_{\mu\nu} && -\frac{i}{4M_{\text{Planck}}}\left(W_{\mu\nu} + W_{\nu\mu}\right) \\
& && \text{with} \quad W_{\mu\nu} = (k_1 + k_2)_\mu \gamma_\nu \\
&f(k_1) - f(k_2) - A_\sigma - G_{\mu\nu} && -\frac{i}{2M_{\text{Planck}}}e_Q\left(X_{\mu\nu\sigma} + X_{\nu\mu\sigma}\right) \\
& && \text{with} \quad X_{\mu\nu\sigma} = \gamma_\mu\eta_{\nu\sigma} \\
&A_\rho(k_1) - A_\sigma(k_2) - G_{\mu\nu} && -\frac{i}{M_{\text{Planck}}}\left(W_{\mu\nu\rho\sigma} + W_{\nu\mu\rho\sigma}\right) \\
& && \text{with} \quad W_{\mu\nu\rho\sigma} = \frac{1}{2}\eta_{\mu\nu}\left(k_{1\sigma}k_{2\rho} - (k_1\cdot k_2)\eta_{\rho\sigma} + ...\right)
\end{aligned} \tag{79}$$

The same thing we can do for QCD (gluon with Dirac and SU(3) indices) to be able to compute LHC cross sections:

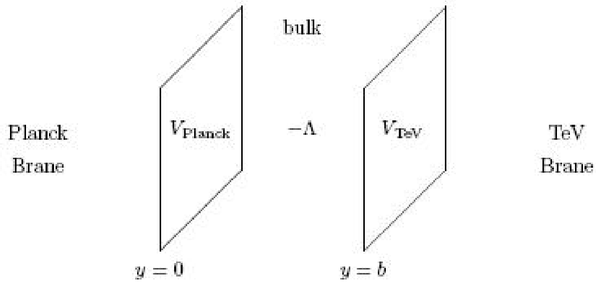
$$\begin{aligned}
&f - f - g^a_\sigma - G_{\mu\nu} && -\frac{i}{2M_{\text{Planck}}}g_S T^a\;\left(X_{\mu\nu\sigma} + X_{\nu\mu\sigma}\right) \\
&g^a_\rho - g^b_\sigma - G_{\mu\nu} && -\frac{i}{M_{\text{Planck}}}g^{ab}\left(W_{\mu\nu\rho\sigma} + W_{\nu\mu\rho\sigma}\right)
\end{aligned} \tag{80}$$

plus a $gggG$ vertex due to gluon self coupling...

E. ADD gravitons at the LHC

Flat and large extra dimensions are often named ADD after the early papers by Nima Arkani-Hamed, Savas Dimopoulos and Gia Dvali. We can compute the rate for real graviton emission at the LHC $pp \to KK$+jet on the parton level, using the Feynman rules derived above. It is two–step procedure, first computing the rate for the radiation of one KK state and then adding the entire KK tower:

$$\begin{aligned}
d\sigma^{\text{one graviton}} &= |< f, G|T^{\mu\nu}h_{\mu\nu}|p_1, p_2 >|^2\,(2\pi)^4\,\delta^4(p_i - p_f)\frac{d\Phi_f}{F(p_1,p_2)} \\
d\sigma^{\text{KK tower}} &= d\sigma^{\text{one graviton}}\frac{S_{\delta-1}\,m^{n-1}\,dm}{(2\pi M_\star)^n}\left(\frac{M_{\text{Planck}}}{M_\star}\right)^2
\end{aligned} \tag{81}$$



The (huge) factor $M_{\rm Planck}^2$ from the KK tower summation gets absorbed into the matrix element square, i.e the effective coupling we see after adding the tower is $1/M_\star \sim 1/$ TeV instead of $1/M_{\rm Planck}$, because of the integration over the all states in the KK tower!
Virtual s–channel gravitons can be observed in $q\bar{q} \to \mu^+\mu^-$ and $gg \to \mu^+\mu^-$ processes. The amplitude reads

$$\begin{aligned}\mathcal{A} &= \frac{1}{M_{\rm Planck}^2}\sum\left(T_{\mu\nu}\frac{P_{\mu\nu\alpha\beta}}{s-m_{\rm KK}^2}T_{\alpha\beta}+\frac{n-1}{3(n+2)}\frac{T_\mu^\mu T_\nu^\nu}{s-m_{KK}^2}\right)\\ &= \frac{1}{M_{\rm Planck}^2}\sum\left(T_{\mu\nu}\frac{P_{\mu\nu\alpha\beta}}{s-m_{\rm KK}^2}T_{\alpha\beta}\right) \qquad \text{for massless particles}\\ &= \frac{1}{M_{\rm Planck}^2}\sum\left(T_{\mu\nu}\frac{\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\nu}\eta_{\alpha\beta}+\eta_{\mu\nu}\eta_{\alpha\beta}/3}{2(s-m_{\rm KK}^2)}T_{\alpha\beta}\right) \qquad \text{leading in } 1/m_{\rm KK}\\ &= \frac{1}{M_{\rm Planck}^2}\sum\left(\frac{1}{s-m_{\rm KK}^2}\frac{1}{2}\left(T_{\mu\nu}T^{\mu\nu}+T_{\mu\nu}T^{\nu\mu}-0\right)\right)\end{aligned} \tag{82}$$

Because the KK tower couples universally to Standard Model particles, the virtual–graviton amplitude is simply a sum over propagators, in our case in the s channel

$$\boxed{\mathcal{A} = \frac{1}{M_{\rm Planck}^2}T_{\mu\nu}T^{\mu\nu}\sum\frac{1}{s-m_{\rm KK}^2}} \tag{83}$$

Again we integrate over the KK tower, up to a cutoff in the $m_{\rm KK}$ integral Λ and obtain a general dimension–8 operator

$$\mathcal{A} \sim \frac{S_{\delta-1}}{2}\frac{\Lambda^{n-2}}{M_\star^{n+2}} \tag{84}$$

This is not good because of the powers of the unknown cutoff in the numerator. A good effective theory should not give cross section predictions which basically require knowledge of the UV completion of the theory to produce sensible results. We can make such assumptions for example completing our KK theory with open or closed string resonances. Or we simply observe that gravity might be non-perturbatively UV–save and use this behavior to compute well–defined LHC cross sections. But we are still thinking about how to solve this...

IV. WARPED EXTRA DIMENSIONS

Briefly after the flat (ADD) models, another way of solving the hierarchy problem was suggested by Lisa Randall and Raman Sundrum. Again it makes use of one extra dimension, but one which is specifically not flat. This finite extra dimension is bounded by two branes, on one of which we exist with all Standard Model particles (RS-I).
Strictly speaking, we compactify our 5th dimension on a S^1/Z_2 orbifold. S^1 is simply a circle (just like the torus in ADD), which is equivalent to periodic boundary conditions. S^1/Z_2 means we map one half of this circle on the other, so we really only have half a circle with no periodic boundary conditions, but two different branes at $y=0$ and $y=b$, y being the additional space coordinate x^4.
I will skip everything that has to do with the cosmological constant and the Planck brane and focus on the hierarchy problem, *i.e.* $m_H \ll M_{\rm Planck}$ on the TeV brane. Instead, we will focus on the TeV brane with its effective 4–dimensional gravitons and their Feynman rules

A. Newtonian gravity in a warped extra dimension

Nobody can stop us from postulating a 5–dimensional metric:

$$\boxed{ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2} \quad \Leftrightarrow \quad g_{AB} = \begin{pmatrix} e^{-2k|y|}\eta_{\mu\nu} & 0 \\ 0 & \eta_{jk} \end{pmatrix} \tag{85}$$

The metric in 4 orthogonal directions to y depends on $|y|$. The absolute value appearing in $|y|$ corresponds to the Z_2 (orbifolding) as S^1/Z_2. When looking at our (3+1)–dimensional brane we can take into account the warp factor $e^{-2k|y|}$ in two ways (with some caveats):

1. use $g_{\mu\nu} = \eta_{\mu\nu}e^{-2k|y|}$ everywhere, which is a pain but possible
2. replace x^μ in 5 dimensions by effective coordinates $e^{-k|y|}d\tilde{x}^\mu$ and $g_{\mu\nu}$ by $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ (tilde means 4–dimensional variables)

The second vision means we shrink our effective 4–dimensional metric along y and forget about the curved space, because the warp factor does not depend on x^μ. The general–relativity action for Newtonian gravity we can write in terms of the 5–dimensional fundamental Planck scale M_{RS}. In our hand–waving argument we have to transform the 5–dimensional Ricci scalar. Just looking at the mass dimensions we see that R has mass dimension 2, or by looking at the definition of R x dimension (-2). This suggests that the 4–dimensional Ricci scalar $\tilde{R}$ which we see in 4 dimensions should roughly scale like $x^{-2} \sim \tilde{x}^{-2}\exp(+2k|y|)$, leading us to a wild guess $R \sim \tilde{R}\exp(+2k|y|)$. The formula for the action with separated x and y integrals we start from already includes the effective 4–dimensional coordinates:

$$\begin{aligned}
S &= -\frac{1}{2}\int_0^b dy \int d^4\tilde{x}\, e^{-4k|y|} R\, M_{\text{RS}}^3 \\
&\sim -\frac{M_{\text{RS}}^3}{2}\int_0^b dy \int d^4\tilde{x}\, e^{-4k|y|}\ \tilde{R}\, e^{2k|y|} \\
&= -\frac{M_{\text{RS}}^3}{2}\int_0^b dy\, e^{-2k|y|} \int d^4\tilde{x}\, \tilde{R} \\
&= -\frac{M_{\text{RS}}^3}{2}\left(-\frac{1}{2k}e^{-2kb} + \frac{1}{2k}\right)\int d^4\tilde{x}\ \tilde{R} && \text{obviously } b > 0 \\
&= -\frac{M_{\text{RS}}^3}{4k}\left(1 - e^{-2kb}\right)\int d^4\tilde{x}\ \tilde{R} \\
&\sim -\frac{M_{\text{RS}}^3}{4k}\int d^4\tilde{x}\tilde{R} && \text{assume } kb \gg 1 \text{, for reasons seen later} \\
&\equiv -\frac{M_{\text{Planck}}^2}{2}\int d^4\tilde{x}\ \tilde{R} && \text{the usual dimensional–analysis matching} \Rightarrow
\end{aligned} \tag{86}$$

The naive matching with 4–dimensional Newtonian gravity (in this case just naive dimensional analysis) means $M_{\text{Planck}}^2 \sim M_{\text{RS}}^3/(2k)$. This does not solve the hierarchy problem because it looks like $M_{\text{RS}} \sim k \sim M_{\text{Planck}} \sim 10^{19}$ GeV is the most reasonable solution.
Fortunately, this is not the whole story. Consider the Standard Model Lagrangian on the TeV brane ($y = b$) in the $\tilde{x}^\mu$ coordinates, *i.e.* with a warp factor. If we want to solve the hierarchy problem, the scalar Higgs part is crucial:

$$\begin{aligned}
S_{\text{SM}} &= \int d^4\tilde{x}\, e^{-4kb}\, \mathcal{L}_{\text{SM}} \\
&= \int d^4\tilde{x}\, e^{-4kb}\, \left[(D_\mu H)^\dagger(D^\mu H) - \lambda(H^\dagger H - v^2)^2 + ...\right]
\end{aligned} \tag{87}$$

From the Higgs–mass term we see that we can rescale all Standard Model fields — in that case H as well as v — by the warp factor on the TeV brane $\exp(-kb)$. The same we have to do for the space coordinate, as described above and for gauge fields which appear in the covariant derivative. To get rid of the entire pre-factor we have to absorb four powers of the $\exp(-kb)$ in each term of the Standard Model Lagrangian.

If we only consider contributions to $\mathcal{L}_{\rm SM}$ of mass dimension 4, we can simply rescale all SM fields according to their mass dimension:

$$\begin{aligned} \tilde{H} &= e^{-kb} H && \text{scalars} \\ \tilde{A}_\mu &= e^{-kb} A_\mu && \text{or } \tilde{D}_\mu = e^{-kb} D_\mu \\ \tilde{\Psi} &= e^{-3kb/2} \Psi && \text{fermions} \end{aligned} \tag{88}$$

which also means for all masses

$$\begin{aligned} \tilde{m} &= e^{-kb} m \\ \tilde{v} &= e^{-kb} v \end{aligned} \tag{89}$$

while the Yukawa couplings as dimensionless parameters in the Lagrangian are not affected. If we assume $kb \sim 35$ we find

$$\boxed{\tilde{v} \sim 0.1\, e^{-kb}\; M_{\rm Planck} \sim 0.1 \text{ TeV}} \tag{90}$$

Note that the derived Planck scale $M_{\rm Planck}$ is still large. To solve the hierarchy problem we have shifted all dimensionful parameters, including the Higgs mass by the warp factor $e^{-k|y|} = e^{-kb}$. The fundamental Higgs mass and the fundamental Planck mass are of the same order, only the derived Higgs mass (and all mass scales on the TeV brane) appears smaller, because of the warped geometry in the 5th dimension. In contrast, on the Planck brane at, where the warp factor is $\exp(-k|y|) = 1$, nothing has happened.

B. Gravitons in a warped dimension

Before we introduce metric fluctuations (gravitons) into our RS model, it turns out to be useful to rewrite the metric by rescaling the 5th dimension $y \to z$ to be able to write the metric as:

$$\boxed{ds^2 = e^{-A(z)} \left(g_{\mu\nu} dx^\mu dx^\nu - dz^2 \right)} \tag{91}$$

To simplify things we assume for the following brief discussion $y > 0$. This is obviously justified, as long as we limit our interest to the TeV brane. First, we define $A(z) = 2ky$ and rewrite the metric using the ansatz:

$$e^{-2ky} = e^{-A(z)} = \frac{1}{(1+kz)^2} \qquad \Leftrightarrow \qquad A(z) = 2\log(k|z|+1) \tag{92}$$

The Planck brane at $y = 0$ sits at $z = 0$. Assuming $k > 0$ we find that $y > 0$ corresponds to $z > 0$. To check if we indeed obtain the correct metric, we start from the two variables being connected as:

$$\begin{aligned} & y = \frac{1}{k} \log(1+kz) \qquad \Leftrightarrow \qquad z = \frac{1}{k}\left(e^{ky} - 1\right) \qquad \Rightarrow \qquad \frac{dz}{dy} = e^{ky} \\ \Rightarrow \qquad & dy = e^{-ky}\, dz = e^{-A(z)/2}\, dz \end{aligned} \tag{93}$$

and indeed find the correct pre-factor of dz^2.
To introduce tensor gravitons we write the relevant part of the metric:

$$ds^2 = e^{-A(z)} \left(\eta_{\mu\nu} \,+\, h_{\mu\nu}(x,z)\, dx^\mu dx^\nu \,-\, dz^2 \right) \tag{94}$$

The left-hand side of Einstein's equations we know is $G_{AB} = R_{AB} - R\, g_{AB}/(n+2)$. Including a finite warp factor $A = 2ky \neq 0$ gives rise to an additional term, which in our case ($g_{AB} = e^{-A} \eta_{AB}$) reads:

$$\begin{aligned} \delta G_{AB} &= \frac{2+n}{2} \left[\frac{1}{2} \partial_A A \partial_B A + \partial_A \partial_B A + \eta_{AB} \left(\partial_C \partial^C A - \frac{1+n}{4} \partial_C A \partial^C A \right) \right] \\ &= \frac{3}{2} \left[\frac{1}{2} \partial_A A \partial_B A + \partial_A \partial_B A + \eta_{AB} \left(\partial_C \partial^C A - \frac{1}{2} \partial_C A \partial^C A \right) \right] && d = 5 \\ &= \begin{cases} \frac{3}{2} \left(\frac{1}{2} A'^2 + A'' - A'' + \frac{1}{2} A'^2 \right) = \frac{3}{2} A'^2 & G_{55} \\ \frac{3}{2} \eta_{\mu\nu} \left(A'' - \frac{1}{2} A'^2 \right) & G_{\mu\nu} \end{cases} \end{aligned} \tag{95}$$

Combined with some source–free right–hand side of Einstein's equations just proportional to the cosmological constant, this gives us the proper description of our two branes. As a matter of fact, the G_{55} equation is already solved by our ansatz for $A(z)$.
Instead of looking at the branes in 5–dimensional space, we use the formula to write down the the effects of introducing a graviton perturbation on the TeV brane. Csaba Csaki leaves calculating the additional contribution δG_{AB} introduced by $A \neq 0$ in $g_{AB} = e^{-A}(\eta_{AB} + h_{\mu\nu})$ in the gauge $h^\mu_\mu = 0 = \partial_\mu h^\mu_\nu$ as a fairly involved exercise, and I will do the same thing. The Einstein equations without sources become:

$$-\frac{1}{2}\partial_C\partial^C h_{\mu\nu} + \frac{2+n}{4}\,\partial^C A\,\partial_C h_{\mu\nu} = 0 \tag{96}$$

They have a linear term which does not look at all like an equation of motion and which we therefore do not like. We can get rid of it rescaling (as usual) $h_{\mu\nu} = e^{(2+n)/4}\widetilde{h}_{\mu\nu}$, according to the bosonic mass dimension $[h] = m^{1+n/2}$. This gives

$$\begin{aligned} -\frac{1}{2}\partial^C\partial_C\widetilde{h}_{\mu\nu} + \left(\frac{(2+n)^2}{32}\partial^C A\,\partial_C A - \frac{2+n}{8}\partial_C A\,\partial^C A\right)\widetilde{h}_{\mu\nu} = 0 \\ -\frac{1}{2}\partial^C\partial_C\widetilde{h}_{\mu\nu} + \left(\frac{9}{32}A'^2 - \frac{3}{8}A''\right)\widetilde{h}_{\mu\nu} = 0 \qquad \text{using} A = A(z) \end{aligned} \tag{97}$$

as the equation of the motion for the rescaled graviton field $\widetilde{h}_{\mu\nu}$. We can solve this equation of the motion for $\widetilde{h}_{\mu\nu}(x,z)$ by separating variables:

$$\widetilde{h}_{\mu\nu}(x,z) = \widehat{h}_{\mu\nu}(x)\;\Phi(z) \tag{98}$$

which yields

$$\begin{aligned} 0 &\equiv -\partial^C\partial_C\left(\widehat{h}_{\mu\nu}(x)\,\Phi(z)\right) + \left(\frac{9}{32}A'^2 - \frac{3}{8}A''\right)\widehat{h}_{\mu\nu}(x)\,\Phi(z) \\ &= -\left(\partial^C\partial_C\widehat{h}_{\mu\nu}(x)\right)\Phi(z) - \widehat{h}_{\mu\nu}(x)\left(\partial_z^2\Phi(z)\right) + \left(\frac{9}{32}A'^2 - \frac{3}{8}A''\right)\widehat{h}_{\mu\nu}(x)\Phi(z) \end{aligned} \tag{99}$$

If we simply give a mass to the tensor graviton $\widehat{h}_{\mu\nu}$ using the ansatz

$$\partial_\mu\partial^\mu\widehat{h}_{\mu\nu} = m^2\widehat{h}_{\mu\nu} \tag{100}$$

we can plug this into the equation of motion and get an equation out of which $\widehat{h}_{\mu\nu}$ drops out trivially:

$$\begin{aligned} -m^2\widehat{h}_{\mu\nu}\,\Phi - \left(\partial_z^2\Phi\right)\widehat{h}_{\mu\nu} + \left(\frac{9}{16}A'^2 - \frac{3}{4}A''\right)\widehat{h}_{\mu\nu}\,\Phi = 0 \\ \Leftrightarrow \qquad -\left(\partial_z^2\Phi\right) + \left(\frac{9}{16}A'^2 - \frac{3}{4}A''\right)\Phi = m^2\Phi \end{aligned} \tag{101}$$

This is a Schrödinger–type equation of Φ, with a potential term:

$$V(z) = \frac{9}{16}A'^2 - \frac{3}{4}A'' \tag{102}$$

Given the form $A(z) = 2\log(k|z|+1)$, we can compute the potential

$$\begin{aligned} z>0 \qquad & A' = \frac{2}{kz+1}k = \frac{2k}{kz+1} = \frac{2k}{k|z|+1} \qquad \Rightarrow \qquad A'^2 = \frac{4k^2}{(k|z|+1)^2} \\ & A'' = 2k\cdot\frac{-1}{(kz+1)^2}\cdot k = -\frac{2k^2}{(k|z|+1)^2} \\ z<0 \qquad & A' = \frac{2}{-kz+1}(-k) = \frac{-2k}{k|z|+1} \qquad \Rightarrow \qquad A'^2 = \frac{4k^2}{(k|z|+1)^2} \\ & A'' = -2k\cdot\frac{1}{(-kz+1)^2}\cdot(-k) = \frac{2k^2}{(k|z|+1)^2} \end{aligned} \tag{103}$$

For the potential on our brane this means ($z > 0$):

$$V(z) = \frac{9}{16}\frac{4k^2}{(k|z|+1)^2} + \frac{3}{4}\frac{2k^2}{(k|z|+1)^2} = \frac{15}{4}\frac{k^2}{(k|z|+1)^2} \tag{104}$$

First, we have a zero mode which solves the equation:

$$\begin{aligned} -\partial_z^2 \Phi^{(0)} + V(z)\Phi^{(0)} &= 0 \\ \Rightarrow \qquad \Phi^{(0)}(z) &= e^{-3A(z)/4} \\ \Rightarrow \qquad h^0_{\mu\nu} &= e^{+3A/4}\, \widetilde{h}^{(0)}_{\mu\nu} = e^{+3A/4}\, \hat{h}^{(0)}_{\mu\nu}\, \Phi^{(0)} = \hat{h}^{(0)}_{\mu\nu}(x) \end{aligned} \tag{105}$$

Rewriting $z \to y$, we find $\Phi^{(0)}(y) = e^{-3k|y|/4} = e^{-3kb/4}$ on our TeV brane. Indeed, gravity on the TeV brane is weak because of the exponentially suppressed wave–function overlap.

Using the form of $V(z)$ we can compute the masses of the KK gravitons on the TeV brane

$$-\partial_z^2\Phi + \frac{15}{4}\frac{k^2}{(k|z|+1)^2}\Phi = m^2\Phi \tag{106}$$

The boundary conditions on the brane are given by the orbifold identification $y \to -y$ which requires for ($z > 0$)

$$\begin{aligned} 0 \equiv -\partial_z h_{\mu\nu} = \partial_z (e^{+3A/4}\hat{h}_{\mu\nu}\Phi) &= \left(\frac{3}{4}A'\Phi + \partial_z\Phi\right) e^{3A/4}\hat{h}_{\mu\nu} \\ &= \left(\frac{3}{2}\frac{k}{kz+1}\Phi + \partial_z\Phi\right) e^{3A/4}\hat{h}_{\mu\nu} \end{aligned} \tag{107}$$

which implies

$$\partial_z^2\Phi = -\frac{3}{2}k\Phi\Big|_{\text{Planck}} \qquad \partial_z^2\Phi = -\frac{3}{2}\frac{k}{kz+1}\Phi\Big|_{\text{TeV}} \tag{108}$$

With these boundary conditions the solution of the equation of motion can be expressed in terms of Bessel functions, which are numbered by an index which corresponds to the mass introduced above:

$$\Phi_m(z) = \frac{1}{\sqrt{kz+1}}\left[a_m\, Y_2\left(m\left(z+\frac{1}{k}\right)\right) + b_m\, J_2\left(m\left(z+\frac{1}{k}\right)\right)\right] \tag{109}$$

More importantly, the masses of these modes are given in terms of the roots of the Bessel function

$$\boxed{m_j = x_j\, k\, e^{-kb}} \qquad \begin{aligned} &\text{with } J_1(x_j) = 0 \\ &\text{or } x_j = 3.8, 7.0, 10.2, 16.5, ..\text{ for } j = 1,2,3,4... \end{aligned} \tag{110}$$

This means that the KK excitations in the RS I model with one warped extra dimensions are not quite equally spaced. To compute the mass values we remember that we can choose $kb \sim 35$ and $k \sim M_{\text{Planck}}$ to solve the hierarchy problem: $ke^{-kb} \sim$ TeV. In other words, the KK gravitons in the warped model have TeV–scale masses and mass differences. Obviously, this is phenomenologically very different for the large (ADD) extra dimensions. For warped extra dimensions we will not produce a tightly spaced KK tower, but for example distinct heavy s–channel excitations. One advantage of such a scenario is that we can measure things like the KK masses and spins at colliders directly.

To answer the question if we can measure these properties we have to compute the coupling strength of KK gravitons to matter, like quarks or gluons or electrons as the initial state in collider experiments. Remember that in the ADD case we had found tiny Planck–suppressed couplings for each individual KK graviton, which corresponded to an inverse–TeV–scale coupling once we integrated over the KK tower. For the warped model the relative coupling strengths on the Planck brane and on the TeV brane are approximately given by the ration of the wave function overlaps. While the zero–mode graviton has to be strongly localized on the Planck brane, to explain the weakness of Newtonian graviton the TeV brane, the KK gravitons do not have strongly peaked wave functions in the additional dimension. Hence, the ratio of wave functions becomes (assuming that the Bessel functions with their normalized arguments will not make a big difference):

$$\boxed{\frac{\Phi(z)\big|_{\text{TeV}}}{\Phi(z)\big|_{\text{Planck}}} \sim \frac{\sqrt{kz+1}\big|_{\text{Planck}}}{\sqrt{kz+1}\big|_{\text{TeV}}} \sim \frac{1}{e^{kb/2}}} \tag{111}$$

The coupling of the KK states is given by the left–hand side of Einstein's equations which enters the Lagrangian just as for the large extra dimensions. We have to distinguish between the flat zero mode with un-suppressed wave function overlap and the KK modes with the wave function normalization $\sim 1/\sqrt{kz+1}$:

$$\mathcal{L} \sim \frac{1}{M_{\text{Planck}}} T^{\mu\nu} h^{(0)}_{\mu\nu} + \frac{1}{M_{\text{Planck}} e^{-kb}} T^{\mu\nu} \sum h^{(m)}_{\mu\nu} \tag{112}$$

This means that the Randall–Sundrum–KK gravitons indeed couple with TeV scale gravitational strength and can be produced at colliders in sufficient numbers, provided they are not too heavy. Similarly to the flat extra dimensions, the couplings of the different KK excitations are (approximately) universal.

V. ULTRAVIOLET COMPLETIONS

In this addendum I will briefly describe the problem how to formulate an ultraviolet completion of extra–dimensional models. For example in ADD models the LHC can explicitely probe energy ranges above M_{Planck}, either in real graviton emission or in virtual graviton exchange. As we saw in the last sections, real graviton emission as well as virtual graviton exchange is only suppressed by powers of $M_\star$, after we integrate over the entire KK tower.
Strictly speaking, this statement is not correct. When we for example write down the higher–dimensional operator arising from s–channel graviton exchange, it will come with powers of M_{Planck} in the denominator, due to the graviton couplings. In addition, it will have powers of the ultraviolet cutoff Λ of the KK integration in the numerator, and the two of them only cancel if we assume $\Lambda = M_{\text{Planck}}$. This is motivated by the conservative estimate that for energies above M_{Planck} our KK effective theory does not describe the graviton exchange correctly and that setting all contributions arising from the ultraviolet completion of our theory to zero will be on the safe side for LHC predictions. If we knew the structure of the ultraviolet completion of the KK effective theory, which would need to be something like a quantum theory of gravity, we could compute these contributions and take them into account for the LHC cross section prediction.

A. String theory

One possible ultraviolet completion of gravity could be string theory. The effects of such a hypothetical UV completion are nicely computed in a classical paper by Maxim Perelstein and others (hep-ph/0001166): in general, we can compute for example the scattering $q\bar{q} \to \mu^+\mu^-$ without using Feynman rules, but will nevertheless arrive at the Standard–Model result as the leading term. In addition, string theory predicts a common form factor for all different helicity amplitudes contributing to this process. This form factor is essentially the Veneziano amplitude and includes the inverse string scale $\alpha' = 1/M_S^2$. While we do not exactly know the size of this scale, for extra–dimensional models it has to be between the well–tested weak scale $v = 246$ GeV and $M_\star$. Perelstein and collaborators compute this Veneziano form factor for the process $e^+e^- \to \gamma\gamma$, which is equivalent to $gg \to \mu^+\mu^-$, and expand it in powers of α':

$$\frac{\Gamma(1-\alpha' s)\,\Gamma(1-\alpha' t)}{\Gamma(1-\alpha'(s+t))} = \frac{\Gamma(1-s/M_S^2)\,\Gamma(1-t/M_S^2)}{\Gamma(1-(s+t)/M_S^2)} = 1 - \frac{\pi^2}{6}\,\frac{st}{M_S^4} + \mathcal{O}\left(M_S^{-6}\right) \tag{113}$$

The parameters s and t are the usual Mandelstam variables in the $(2 \to 2)$ process. This form of the string corrections corresponds to our KK effective field theory, modulo a normalization factor which relates the two mass scales M_S and M_{Planck}. Hence, this series in M_S is not what we are interested as the UV completion of our theory.
The string theory approach becomes more interesting at higher energies. The Veneziano form factor we gave above is proportional to $\Gamma(1 - s/M_S^2)$, which has poles for negative integer arguments $1 - s/M_S^2 = -(n+1)$ for $n = 1, 2, ...$ These poles lie at $s = nM_S^2$, which tells us that the string resonances in the s channels have to appear as $1/(1 - nM_S^2)$ in the transition amplitude. Starting from the energy threshold M_S our UV completion consists of real particles of mass $\sqrt{n}M_S$ appearing in our amplitude. This is the kind of UV completion we are looking for and which we can base cross–section calculations on.
Note that scattering partons with energies above the fundamental Planck scale probes the trans–Planckian regime of our theory of gravity without necessarily producing black–hole solutions. Black holes can occur in colliders, but they require the two partons to scatter at very high energies while at the same time getting closer than the Schwarzschild radius. The Schwarzschild radius r_h depends on the collider energy, and the production cross section of a black hole is essentially the geometric factor πr_h^2, provided the two beam collide with a small enough impact parameter. The question how these black holes can then be detected depends largely on the question is we actually produce a thermalized black hole, which would just decay to may particles via Hawking radiation, whereas otherwise the signature would look very similar to an old–fashioned contact–interaction.

B. Fixed–point gravity

According to a classical paper by Weinberg, another UV completion of gravity could be described by the possible existence of a gravitational fixed point. Such a fixed point would not be a unique feature of extra–dimensional models, but in contrast to the four–dimensional case the LHC could observe it in such models with a low fundamental Planck scale. In other words, we can simply generalize a well-established field of gravitational research.
The starting point for our argument is a renormalization group analysis of the effective action of gravity, *i.e.* the generalization of the Einstein–Hilbert action to scale–dependent parameters:

$$\frac{1}{16\pi G_k}\int\sqrt{|g|}\left(\Lambda_k+R+\mathcal{O}(R^2)\right)+S_{\text{matter},k}+S_{\text{gf},k}+S_{\text{ghosts},k} \tag{114}$$

The first term in the action is the cosmological constant, the second term is the Ricci scalar describing free gravity, and the remaining terms are the Standard–Model action without any gravity terms. Because of the Ricci scalar's mass dimension two, higher powers of R correspond to higher orders in $1/M_\star$, the only scale present in the gravitational part of the action. We will briefly discuss the limitations of perturbative gravitation later.
The index k refers to an energy scale at which we evaluate these parameters, for example the gravitational coupling $G\sim 1/M_\star^2$. Scale–dependent parameters are of course nothing new, we know for example how to evaluate the strong coupling $\alpha_s(\mu_R)$ at proper values of the renormalization scale. What we need to know to evolve our theory from one scale to another is the renormalization group equation for the gravitational coupling. Since we know the mass dimension of the gravitational coupling constant we can use a renormalization scale μ to define its dimensionless version in $(4+n)$ dimensions and add the usual renormalization constant in front:

$$g(\mu)=G\,\mu^{2+n}\qquad\longrightarrow\qquad Z(\mu)^{-1}\,G\,\mu^{2+n} \tag{115}$$

As the anomalous dimension of any field or Lagrangian parameter we refer to the quantum (or renormalization) contribution to the classical mass dimension of the bare field or parameter appearing in the Lagrangian. In this case the anomalous dimension of the gravitational coupling is $\eta=-d\log Z/d\log\mu=-1/Z\,dZ/d\log\mu$. In terms of this anomalous dimension, which in general will be a function of μ, we can write down a renormalization group equation for $g(\mu)$:

$$\begin{aligned}\frac{dg}{d\log\mu}&=\frac{d}{d\log\mu}\left(\frac{1}{Z}\,G\,\mu^{2+n}\right)=G\left(-\frac{1}{Z^2}\frac{dZ}{d\log\mu}\,\mu^{2+n}+\frac{1}{Z}\,\mu\,\frac{d\mu^{2+n}}{d\mu}\right)\\&=\frac{1}{Z}\,G\,\mu^{2+n}\left(-\frac{1}{Z}\frac{dZ}{d\log\mu}+(2+n)\right)=(\eta+n+2)\;g\end{aligned} \tag{116}$$

This equation can have two fixed points. First, vanishing values of g are stable with respect to scale variations, which means the running of the gravitational coupling has a fixed point at $g=0$. A fixed point at the trivial value $g=0$ we call a Gaussian fixed point. This fixed point exists for any value of η and describes the usual regime of Einstein–Hilbert gravity we know.
Let's assume that $\eta>0$, so that the change of $g(\mu)$ with μ has a positive sign. This means that for large positive and negative values of $\log\mu$ there could be another fixed point for a finite values $g=g_*$, where

$$\boxed{\eta(\mu)=-(2+n)}\qquad\qquad\boxed{G\sim\frac{g_*}{\mu^{2+n}}} \tag{117}$$

The scale factor valid around the fixed–point regime implies that for small scales the dimensionful gravitational coupling would become large, while for large scale it would be suppressed by a scale factor $1/\mu^{2+n}$. Note that in this argument we have omitted constant terms in the solution of the differential equation, so that we should not claim that the gravitational coupling vanishes at large scales.

The system we really need to solve is a coupled set of differential equations including the renormalization group equation for $\eta(\mu)$ and for the cosmological constant $\lambda(\mu)=\Lambda(\mu)/\mu^2$. However, for example in the papers by Martin Reuter or Daniel Litim we see that the general pattern of the non–Gaussian fixed point does not change, and indeed the gravitational coupling will be asymptotically free, *i.e.* become small in the ultraviolet. The expressions for the physical observables in the UV fixed point in the literature are

$$\begin{aligned}\lambda_*&=\frac{D^2-D-4-\sqrt{2D(D^2-D-4)}}{2(D-4)(D-1)}\\g_*&=\Gamma\left(\frac{D}{2}+2\right)\,(4\pi)^{D/2-1}\,\frac{\left(\sqrt{D^2-D-4}-\sqrt{2D}\right)^2}{2(D-4)^2(D+1)^2}\qquad\qquad D=4+n\end{aligned} \tag{118}$$

Note that this UV behavior of the gravitational couplings is exactly the opposite of what we usually think of when we are concerned with gravity becoming a strongly interacting theory at the Planck scale. Weak gravity in the ultraviolet we can think of as asymptotically free gravity.
In the usual sense we consider a theory renormalizable if in the far ultraviolet its coupling strength becomes infinitely small. Weinberg's approach, on which this study of the UV behavior of gravity builds, is to generalize the concept of renormalizability to theories with a finite UV limit of the coupling. Because of the vanishing of G in the UV, we still expect no unphysical UV divergences in such a theory. This of course does not mean that gravity will be a perturbatively renormalizable field theory — it cannot, because it has a coupling constant with an inverse mass dimension, but ultraviolet safety is a useful extension of the usual perturbative renormalizability condition which has been proven to hold for Yang–Mills theories.

Unfortunately, in the trans–Planckian energy regime we cannot write a perturbative series for example in R. However, the existence of a non–trivial fixed point for the gravitational coupling has been shown including higher–order corrections in the Einstein–Hilbert action up to $\sqrt{g}R^8$ and including a coupling to matter fields. While this ordering scheme in powers of R is of course not well defined once we are looking at energies beyond $M_\star$, there is no good reason for this fixed–point behavior to change at some arbitrary higher power of R. Moreover, it is interesting to notice that our fixed–point theory naively appears to break down if we include the R^2 term in the action. This term leads to propagators of the mass dimension $1/p^4$, which can be considered sub–leading remainders of the sum of two propagators with the leading behavior $1/p^2$, provided one of these propagators appears with a negative sign. Such particles are usually referred to as ghosts and are unphysical degrees of freedom. They should not appear in our theory! On the other hand, work by Gomez and Weinberg gives us reasons to believe that such ghost contributions vanish after taking into account all orders in R. Because one should not trust the perturbative expansion of the effective action in R this is a particularly welcome result, increasing our trust in the stable fixed–point behavior which has until now appeared order by order in R.

The next question is: how do we include the leading effects of this fixed–point behavior in our LHC calculations without having to take into account the running of all parameters with the scales for example present in the virtual–graviton propagator. The obvious way is to include a running gravitational coupling, following the paper by JoAnne Hewett and Tom Rizzo. The coupling of the integrated KK–graviton tower to the energy–momentum tensor, given by $1/M_\star^{2+n}$, is simply modified by a form factor

$$\frac{1}{M_\star^{2+n}} \longrightarrow \frac{1}{M_\star^{2+n}} \left[1 + \left(\frac{\mu}{aM_\star}\right)^{2+n}\right]^{-1} \tag{119}$$

with a fudge factor $a \sim 1$. For large scales μ this form factor becomes smaller, (hopefully) regularizing the LHC cross section prediction in the ultraviolet.

When writing down the integral over the virtual KK graviton propagator $1/(s - m_{\text{KK}}^2)$ we see that there are two integrals, one over the KK tower m_{KK} and one over the partonic center-of-mass energy $\sqrt{s}$ or over the parton momentum fractions $(x_1 x_2)$. In the form–factor approach the authors choose $\mu = \sqrt{s}$, which regularizes this dimension of this integral, but not the other. The m_{KK} integral they still have to cut off and integrate into the form $1/M_\star^{2+n}$ instead of the integrand's single–graviton coupling $1/M_{\text{Planck}}^{2+n}$.

Separating the two integrations gives us another handle at the beneficial effects of the renormalization–group running of gravity. In the energy range $\sqrt{s} < M_\star$ we can clearly identify the IR and the UV regime of the m_{KK} integral. The transition between these two regimes should take part around $\Lambda_{\text{trans}} \equiv M_\star$, the only scale known to the theory. Because we do not know the exact matching behavior we simply assume a sudden change between the regimes at $m_{\text{KK}} = \Lambda_{\text{trans}}$. From QCD studies we expect the dominant difference between the IR and UV regimes to be the anomalous dimension of the gravitational coupling and of the graviton field. The scalar graviton propagator then becomes:

$$P(s, m_{\text{KK}}) = \begin{cases} \dfrac{1}{s + m_{\text{KK}}^2} & m < \Lambda_{\text{trans}} \qquad \text{(IR)} \\[2ex] \dfrac{M_\star^{n+2}}{(s + m_{\text{KK}}^2)^{n/2+2}} & m > \Lambda_{\text{trans}} \qquad \text{(UV)} \end{cases} \tag{120}$$

In the regime $\sqrt{s} < M_\star$ this change in the anomalous dimension indeed regularizes the m_{KK} integration in the virtual–graviton amplitude. Compared to a simple cut off at Λ the effective dimension-8 operator describing virtual graviton exchange in the production process $gg \to \mu^+\mu^-$ shifts from

$$\begin{aligned}
\mathcal{S} &= \frac{S_{n-1}}{M_\star^{2+n}} \int_0^\Lambda dm \; m^{n-1} \, P(s,m) \\
&= \frac{S_{n-1}}{M_\star^4} \, \frac{1}{n-2} \, \left(\frac{\Lambda}{M_\star} \right)^{n-2} \, \left[1 + \mathcal{O}\left(\frac{s}{\Lambda^2} \right) \right] \\
&\to \frac{S_{n-1}}{M_\star^4} \, \frac{1}{n-2} \, \left(\frac{\Lambda_\text{match}}{M_\star} \right)^{n-2} \, \left(1 + \frac{n-2}{4} \right) \left[1 + \mathcal{O}\left(\frac{s}{\Lambda^2} \right) \right]
\end{aligned} \tag{121}$$

We can refer to the cut-off result as the IR contribution of the integral, and it is indeed proportional to the cut-off Λ in the numerator. For the combined fixed–point IR and UV integral this dependence is replaced by a power dependence on the matching scale, which for good reasons we assume to be $M_\star$. The IR part of the integral is of course independent of the UV completion and a function of the number of extra dimensions n. In this specific case the UV part of the integral turns out to be independent of n, except for the geometry factor S_{n-1}. We see that for larger values of $n > 5$ the UV contribution can be numerically dominant when computing LHC signal rates.
In the remaining part of the integration region $\sqrt{s} > M_\star$ we probe gravity clearly beyond the Planck scale. This also means that in the LHC scattering amplitude the fundamental Planck scale should only appear as a coupling, but not as a dynamic mass scale. Modulo a c-number normalization we simply estimate any gravity–induced operator by factors of $\sqrt{s}$ to get the correct mass dimension. The matching around $\sqrt{s} \sim M_\star$ is unfortunately not determined: either we fix the matching scale to $M_\star$ and adjust the prefactor of the high–$\sqrt{s}$ contribution to match the well–known low–$\sqrt{s}$ solution; or we fix the normalization of both parts and compute the matching scale. Interestingly, the latter gives a matching scale more than a factor two below $M_\star$...to be continued....?

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Literature: There is a huge number of papers available on extra dimensions, most notably a huge number of great original papers. Here, I would like to list some more pedagogical reviews which I read to prepare this lecture and which I can recommend to everybody who is interested in deepening their knowledge (ordered by appearance in the lecture):

– a very good and seriously complete review on dark matter is the one by Bertone, Hooper, Silk (hep-ph/0404175). Dan also wrote a popular book on the same topic, you can find it on Amazon

– the argument about the Higgs–mass divergence at one loop you can find in Martin Schmaltz' hep-ph/0210415

– a great collection of loop formulas and a great appendix including integrals is Rick Field's book 'Applications of Perturbative QCD'

– most of this lecture is based on Graham Kribs' TASI lecture (hep-ph/0605325)

– more very useful TASI lectures you can find by Csaba Csaki (hep-ph/0404096 and hep-ph/0510275)

– and by Raman Sundrum (hep-th/0508134)

– for the more formally interested, there is a great introduction by Gregory Gabadadze (hep-ph/0308112)

– the as far as I am concerned best paper written on extra dimensions is Gian Giudice, Riccardo Rattazzi and James Wells' hep-ph/9811291

– starting from some ideas on $n = 1$ we have tried to review the LHC prospects for ADD models in hep-ph/0408320