Quantum Field Theory II

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Abstract

While for this lecture I am mostly following Mark Srednicki's book there are a few chapters where I deviate from his arguments but derive essentially the same equations. The chapter numbering corresponds to the book.

Contents

16 Loop corrections to the vertex

In the following sections we will introduce renormalization using scalar φ^2 theory in six dimensions. Technically, we have already introduced the necessary parameters which we need to renormalize amplitudes and absorb ultraviolet divergences whenever they appear in perturbative calculations. To fulfill the LSZ consistency conditions for an interacting theory in Eq.(5.27) we generalize the Lagrangian for a real scalar field to

$$\mathscr{L} = \frac{Z_\varphi}{2}\, \partial^\mu \varphi \, \partial_\mu \varphi - \frac{Z_m}{2}\, m^2 \, \varphi^2 + \frac{Z_g}{6}\, g \, \varphi^3 + Y \, \varphi \; . \tag{16.1}$$

Of the four Z and Y factors we need two to ensure $\langle 0|\varphi(x)|0\rangle = 0$ and $\langle p|\varphi(x)|0\rangle = 1$. The first condition leads to a shift in $\varphi(x)$, the second to a change of normalization. Following Eq.(9.20) we can use $Y = ig\,\Delta 0/2$ and Z_φ to ensure the correct properties of the scalar field. The remaining constants Z_m and Z_g we have to fix based on physical conditions on the measured masses and couplings.

Our line of thinking we already know from the exact propagator in Eqs.(14.1) and (14.6)

$$\begin{aligned} \mathbf{\Delta}(x_1 - x_2) &= i\, \langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle = \delta_1\delta_2 \; iW(J)\Big|_{J=0} \\ \tilde{\mathbf{\Delta}}(k^2) &= \frac{-1}{k^2 - m^2 + i\varepsilon + \Pi(k^2)} = \frac{-1}{k^2 - m^2 + i\varepsilon} + \mathcal{O}(g^2) \; . \end{aligned} \tag{16.2}$$

The field normalization Z_φ and the mass definition Z_m we can both fix using the exact propagator. The result for $\mathbf{\Delta}(k^2)$ should have a single pole at $k^2 = m^2$ with residue one. The two conditions

$$\Pi(m^2) = 0 \qquad\qquad \Pi'(m^2) = 0 \tag{16.3}$$

ensure this. Eqs.(14.37) and (14.38) give the explicit formulas for $d = 6 - \epsilon$ space-time dimensions

$$\begin{aligned} Z_\varphi &= 1 - \frac{g^2}{6(2\pi)^3} \left(\frac{1}{\epsilon} + \log \frac{\mu}{m} + \text{finite terms} \right) + \mathcal{O}(g^4) \\ Z_m &= 1 - \frac{g^2}{(2\pi)^3} \left(\frac{1}{\epsilon} + \log \frac{\mu}{m} + \text{finite terms} \right) + \mathcal{O}(g^4) \; , \end{aligned} \tag{16.4}$$

with $\alpha = g^2/(2\pi)^3$. From the four constants from Eq.(16.1) only Z_g is left to be fixed by an appropriate condition on an interaction process.

In analogy to the exact propagator we define an exact three-point vertex function, for which the original vertex $iZ_g g$ is the leading term,

$$i\mathbf{V}_3(k_1,k_2,k_3) = iZ_g g + (ig)^3 \int \frac{d^d\ell}{(2\pi)^d} \frac{i}{\ell^2 - m^2} \frac{i}{(\ell+k_1)^2 - m^2} \frac{i}{(\ell+k_1+k_2)^2 - m^2} + \mathcal{O}(g^5) \; , \tag{16.5}$$

with all incoming momenta $k_1 + k_2 + k_3 = 0$. One difference between the two formulas which will become important later is that the form of the propagator in terms of the self energy Π is true to all orders while the vertex function is defined order by order. For a scalar theory the loop integral has a particularly simple structure, because the numerator does not depend on the loop momentum ℓ. Such scalar loop integrals are important building blocks for a computation of higher order corrections and are available in the literature (if a convenient analytic formula exists) or numerically. The three point function with three external legs and three external propagators we write as:

$$\begin{aligned} C(k_1^2,k_2^2,k_3^2;m,m,m) &\equiv \int \frac{d^{6-\epsilon}\ell}{(2\pi)^{6-\epsilon}} \frac{1}{\ell^2 - m^2} \frac{1}{(\ell+k_1)^2 - m^2} \frac{1}{(\ell+k_1+k_2)^2 - m^2} \\ &= \Gamma\left(\frac{\epsilon}{2}\right) \frac{1}{2i}\, (4\pi)^{-3+\epsilon/2} m^{-\epsilon} + C_{\text{fin}}(k_1^2,k_2^2,k_3^2;m,m,m) \; . \end{aligned} \tag{16.6}$$

First, the factor $m^{-\epsilon}$ looks weird, but it simply corresponds to the non-integer dimensionality of the loop integration. Because the scalar three point function is not guaranteed to be finite we split it into a divergent part and a finite remainder. As long as all three propagators involve a mass m none of them can diverge for vanishing internal or external momenta, so the result is infrared finite.
In the ultraviolet, the loop integration diverges for $d \geq 6$. This is where dimensional regularization comes in, reducing the number of dimensions below the critical value $d = 6$. As we will see next, a pole appears in the form of $\Gamma(\epsilon/2)$. The The finite term C_{fin} cannot be expressed easily, so we leave it for numerical evaluation. If the external momenta all add to zero there is no scalar product of external momenta which we cannot write in terms of the three k_j^2.

In text books we usually compute such scalar integrals using Feynman parameters

$$\frac{1}{a_1 a_2 ... a_n} = \Gamma(n) \int_0^1 dx_1 \int_0^1 dx_2 ... \int_0^1 dx_n \frac{\delta(1 - x_1 - x_2 - ... - x_n)}{(a_1 x_1 + a_2 x_2 + ... + a_n x_n)^n} , \tag{16.7}$$

where each a_j is one of the inverse propagators. This parameterization leads us to integrals of the type

$$\begin{aligned} \int \frac{d^d\ell}{i\pi^{d/2}} \frac{(\ell^2)^R}{(\ell^2 - D)^M} &= \frac{(4\pi)^{d/2}}{i} \int \frac{d^d\ell}{(2\pi)^d} \frac{(\ell^2)^R}{(\ell^2 - D)^M} \\ &= (-1)^{R-M} \; D^{R-M+d/2} \; \frac{\Gamma(R + d/2)\Gamma(M - R - d/2)}{\Gamma(d/2)\Gamma(M)} , \end{aligned} \tag{16.8}$$

with some function D depending on the masses and external momenta. For positive integers the gamma function is defined as $\Gamma(n + 1) = n!$. Some of these gamma functions arise from the angular integration over the unit sphere while others come from the iterative solution of the $|\vec{\ell}|$ integral.
We already know from Eq.(16.6) that we are interested in the value of $\Gamma(\epsilon/2)$ for $\epsilon \to 0$, which we can compute starting with the digamma function in an appropriate representation

$$\begin{aligned} \Psi(1 + x) &= \frac{\Gamma'(1 + x)}{\Gamma(1 + x)} \\ &= -\gamma_E + \sum_{n=2}^{\infty} (-1)^n x^{n-1} \zeta_n , \end{aligned} \tag{16.9}$$

valid for $|x| < 1$. This differential equation for the gamma function itself we can integrate with the boundary condition $\Gamma(1) = 1$

$$\begin{aligned} \log \Gamma(1 + x) &= -\gamma_E x + \sum_{n=2}^{\infty} \frac{(-1)^n x^n \zeta_n}{n} + \text{const} \\ &= -\gamma_E x + \frac{x^2}{2} \zeta_2 + \mathcal{O}(x^3) . \end{aligned} \tag{16.10}$$

This result we can simply exponentiate for small values of x, giving us

$$\begin{aligned} \Gamma\left(1 + \frac{\epsilon}{2}\right) &= e^{-\gamma_E \epsilon/2} \left(1 + \frac{\zeta_2}{8}\epsilon^2 + \mathcal{O}(\epsilon^3)\right) \\ \Gamma\left(\frac{\epsilon}{2}\right) &= 2e^{-\gamma_E \epsilon/2} \left(\frac{1}{\epsilon} + \frac{\zeta_2}{8}\epsilon + \mathcal{O}(\epsilon^2)\right) \qquad \text{with} \quad \Gamma(x + 1) = x\Gamma(x) . \end{aligned} \tag{16.11}$$

Note that usually we compute the loop integrals in $d = 4 - 2\epsilon$ dimensions while for our φ^3 theory we choose $d = 6 - \epsilon$. Independent of the number of dimensions there exists a problem with Eq.(16.5): the vertex function develops a dimension which it should not have. Dimensional regularization only affects the loop or phase space integrals and should be removed via analytic continuation by the time we compute an observable. To be on the safe side we introduce a factor $\tilde{\mu}^\epsilon$ into the expression for $\mathbf{V}_3$

$$\begin{aligned}
\frac{1}{g}\mathbf{V}_3(k_1,k_2,k_3) &= Z_g + \frac{(ig)^3}{ig} i^3\,\tilde{\mu}^\epsilon\;C(k_1^2,k_2^2,k_3^2;m,m,m) + \mathcal{O}(g^5) \\
&= Z_g + ig^2\tilde{\mu}^\epsilon\;\left(\Gamma\left(\frac{\epsilon}{2}\right)\frac{1}{2i}\,(4\pi)^{-3+\epsilon/2}m^{-\epsilon} + C_{\text{fin}}\right) + \mathcal{O}(g^5) \\
&= Z_g + \frac{1}{2}\frac{g^2}{(4\pi)^3}\Gamma\left(\frac{\epsilon}{2}\right)\left(\frac{4\pi\tilde{\mu}^2}{m^2}\right)^{\epsilon/2} + \cdots \\
&= Z_g + \frac{1}{2}\frac{g^2}{(4\pi)^3}\frac{2}{\epsilon}\left(\frac{4\pi\tilde{\mu}^2}{m^2}e^{-\gamma_E}\right)^{\epsilon/2} + \cdots \\
&= Z_g + \frac{\alpha}{\epsilon}\left(\frac{\mu}{m}\right)^\epsilon + \cdots
\end{aligned} \tag{16.12}$$

In the last step we simply rescale $\tilde{\mu} \to \mu$ conveniently and introduce $\alpha = g^2/(4\pi)^3$. Terms we neglect are either of higher power in g or finite. To absorb the pole in the expression for $\mathbf{V}_3$ we can write

$$\begin{aligned}
Z_g &= 1 + \delta Z_g \\
&= 1 - \frac{\alpha}{\epsilon}\left(\frac{\mu}{m}\right)^\epsilon + \delta Z_{\text{fin}} \\
&= 1 - \frac{\alpha}{\epsilon}e^{\epsilon\log(\mu/m)} + \delta Z_{\text{fin}} \\
&= 1 - \alpha\left(\frac{1}{\epsilon} + \log\frac{\mu}{m} + \mathcal{O}(\epsilon)\right) + \delta Z_{\text{fin}}\;,
\end{aligned} \tag{16.13}$$

allowing for an arbitrary finite contribution Z_{fin}. This form gives us the vertex function to net-to-leading order

$$\mathbf{V}_3(k_1,k_2,k_3) = g + \text{finite terms}\;, \tag{16.14}$$

where the finite contributions arise from C_{fin} as well as from δZ_{fin}. Different choices for δZ_{fin} correspond to different definitions of the (measurable) coupling. In the absence of a more physically motivated definition we can for example choose the minimal scheme $\delta Z_{\text{fin}} = 0$.

Before we move on, future collider physicists might benefit from a naming convention: whenever a scale μ appears in association with an ultraviolet pole we call it the renormalization scale μ_R, if it is associated with an infrared pole we call it the factorization scale μ_F. What the appearance of such a scale says on a more fundamental level is that it is not possible to regularize a divergent loop integral without introducing any scale. We could have introduced a cutoff scale directly, and dimensional regularization at first appeared to avoid this introduction of a scale, but it turns out that the scale appears again through the back door.
The appearance of scales is an ubiquitous feature of perturbative field theory. They are artifacts due to order-by-order regularization and renormalization (or more general pole subtraction), and the dependence of observables on these scales should vanish at arbitrary loop order. This is why sometimes we can use the renormalization and factorization scale dependence to estimate the minimum theory uncertainty of a perturbative prediction.

17 Other 1PI vertices

Similar to the three-point vertex we can compute the n-point interaction. For $n > 3$ such interactions are not present at leading order in the φ^3 Lagrangian, but they of course occur at one-loop order. One example is $n = 4$ with four incoming momenta $k_1 + k - 2 + k_3 + k_4 = 0$, which we write as

$$\begin{aligned} i\mathbf{V}_4(k_1, k_2, k_3, k_4) &= 0 + (ig)^4 \frac{(-1)^4}{i^4} \int \frac{d^d\ell}{(2\pi)^d} \\ &\qquad \times \; \frac{1}{\ell^2 - m^2} \frac{1}{(\ell + k_1)^2 - m^2} \frac{1}{(\ell + k_1 + k_2)^2 - m^2} \frac{1}{(\ell + k_1 + k_2 + k_3)^2 - m^2} + ... \\ &= g^4 D(k_1, k_2, k_3, k_4; m, m, m, m, m) \\ &\quad + g^4 D(k_2, k_1, k_3, k_4; m, m, m, m, m) \\ &\quad + g^4 D(k_1, k_3, k_2, k_4; m, m, m, m, m) + \mathcal{O}(g^6) \; . \end{aligned} \tag{17.1}$$

Unlike for the three-point function there now exist non-trivial permutations of the four external momenta. As for the scalar three-point function C the infrared momentum regime is cut off by the mass m, which means the scalar four-point integral is IR finite. Comparing the power of the loop momentum ℓ in the integration measure to the four denominators, which for large internal momenta scale like $(l^2)^4$ we guess that also the ultraviolet regime is finite

$$D(k_1, k_2, k_3, k_4; m, m, m, m, m) = D_{\text{fin}}(k_1, k_2, k_3, k_4; m, m, m, m, m) \; . \tag{17.2}$$

Because of the slightly complicated structure of the $(5 + 1)$-dimensional integral we can resort to the Feynman parameterization Eq.(16.7). Four scalar propagators turn into a sum of all propagators to the fourth power. For Eq.(16.8) this means $R = 0$ and $M = 4$ with $d = 6$, so the ratio of gamma functions involved reads

$$\int \frac{d^d\ell}{i\pi^{d/2}} \frac{1}{(\ell^2 - D)^4} = (-1)^{-4} \; D^{-4+3} \; \frac{\Gamma(3)\Gamma(4-3)}{\Gamma(3)\Gamma(4)} \; , \tag{17.3}$$

for some value of D. None of these gamma functions are divergent. From Eq.(16.8) we see that for scalar theories with $R = 0$ the dangerous gamma function is $\Gamma(M - d/2)$. It leads to an ultraviolet pole corresponding only for $M \leq 3$. From the four-point vertex on all higher loop-induced vertices are UV finite and do not have to be taken care of with the help of Z factors. This is excellent news because of the four constants in the Lagrangian Eq.(16.1) none are left once the LSZ conditions, the pole of the massive propagator, and the three-point coupling are properly defined.

27 Other renormalization schemes

In the first part of his Section 27 Mark Srednicki introduces the $\overline{\text{MS}}$ renormalization scheme and computes the physical scalar mass, *i.e.* the position of the propagator pole in both schemes. The difference between the two schemes is of order α and finite.
Next, we can compute an observable in both schemes. A specific simplification is that our observable m_{pole} coincides with the prediction of the on-shell renormalization scheme. If this observable does not depend on the unphysical renormalization scale μ the $\overline{\text{MS}}$ mass needs to run. Its derivative with respect to $\log\mu^2$ is essentially called the anomalous dimension γ_m. The same argument we can apply to a transition amplitude, to find a running coupling $\alpha(\mu^2)$.

To compute the $2 \to 2$ scattering process in the $\overline{\text{MS}}$ scheme we need to combine the same building blocks we combined in Section 20 and add the correction factors in the LSZ equation. The leading oder result for the massless transition amplitude is given in Eq.(20.1)

$$\begin{aligned}\mathcal{T}_{\text{tree}} &= g^2\left(\tilde{\Delta}(s)+\tilde{\Delta}(t)+\tilde{\Delta}(u)\right)\\ &= g^2\left(\frac{1}{-s-i\varepsilon}+\frac{1}{-t+i\varepsilon}+\frac{1}{-u+i\varepsilon}\right) ,\end{aligned} \tag{27.1}$$

remembering $s+t+u=0$. The full expression given in Eqs.(20.18) and (20.19) consists of three distinct channels, corresponding to s-channel, t-channel, and u-channel exchange. They are related through crossing symmetries, so all we need to quote is for example the s-channel part, renormalized in the usual on-shell scheme

$$\begin{aligned}\mathcal{T}_s &= \mathbf{V}_3^2(s)\tilde{\mathbf{\Delta}}(s)+\mathbf{V}_{4,s}\\ &= g^2\frac{1}{-s}\left[1-\frac{11\alpha}{12}\ \log\frac{-s}{m^2}-\frac{\alpha}{2}\ \log^2\frac{t}{u}+\frac{\alpha}{12}\left(39-\pi\sqrt{3}-\pi^2\right)\right]+\mathcal{O}(\alpha^2) .\end{aligned} \tag{27.2}$$

This expression is not defined in the massless limit $m \to 0$. As we will see later, the $\log s/m^2$ can have two origins: for fixed masses and large energies s it describes an ultraviolet divergence. This one is linked to renormalization and is, as we will see, cured by the new renormalization scheme. For fixed energy s and small masses it describes an infrared pole and its discussion we will have to postpone.

Switching to the $\overline{\text{MS}}$ scheme the building blocks for the transition amplitude have different values in the massless limit:

$$\begin{aligned}\Pi(s) &= -\frac{\alpha}{12}\left(-s+6m^2\right)+\frac{\alpha}{2}\int_0^1 dxD\log\frac{D}{\mu^2}+\mathcal{O}(\alpha^2)\\ &= \frac{\alpha}{12}s-\frac{\alpha}{2}s\int_0^1 dx\ x(1-x)\log\frac{-x(1-x)s}{\mu^2}+\mathcal{O}(\alpha^2)\\ &= \frac{\alpha}{12}s-\frac{\alpha}{12}s\ \log\frac{-s}{\mu^2}-\frac{\alpha}{2}s\ \int_0^1 dx\ x(1-x)\log\left(x(1-x)\right)+\mathcal{O}(\alpha^2)\\ &= \frac{\alpha}{12}s-\frac{\alpha}{12}s\ \log\frac{-s}{\mu^2}+\frac{5\alpha}{36}s+\mathcal{O}(\alpha^2)\\ &= s\left[\frac{2\alpha}{9}-\frac{\alpha}{12}s\ \log\frac{-s}{\mu^2}+\mathcal{O}(\alpha^2)\right]\\ \tilde{\mathbf{\Delta}}(s) &= \frac{1}{-s-\Pi(s)}\\ &= \frac{1}{-s}\left(1-\frac{\Pi(s)}{s}\right)+\mathcal{O}(\alpha^2)\\ &= \frac{1}{-s}\left(1+\frac{\alpha}{12}\log\frac{-s}{\mu^2}-\frac{2\alpha}{9}\right)+\mathcal{O}(\alpha^2) .\end{aligned} \tag{27.3}$$

In a similar fashion the triple vertex function now has a result which is well defined in the massless limit, which we can read of the on-shell result just replacing m^2 with μ^2 in the logarithm

$$\mathbf{V}_3(s) = g\left(1 - \frac{\alpha}{2}\log\frac{-s}{\mu^2} + \frac{3\alpha}{2}\right) + \mathcal{O}(\alpha^2)\ . \tag{27.4}$$

The last contribution which we have to take into account to compute the transition amplitude Eq.(27.2) did not exist in the on-shell scheme. This is because $\Pi'(m^2)$ ensured that the LSZ condition on the residue of the propagator was automatically fulfilled by the definition of the counter term Z_φ.

If the LSZ condition is not automatically fulfilled, *i.e.* the fields φ do not have the correct normalization of the one-particle state, we know that we have to apply a multiplicative normalization of each field. The correction factor we can derive from the definition of the propagator in position space,

$$\mathbf{\Delta}(x_1 - x_2) = i\langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle\ . \tag{27.5}$$

The normalization of the propagators in position space and in momentum space is given by the same number. If in momentum space the residue is given by

$$R = \left[\frac{d}{dk^2}\ \frac{1}{\tilde{\mathbf{\Delta}}(k^2)}\bigg|_{k^2=m^2_\text{pole}}\right]^{-1} \neq 1\ , \tag{27.6}$$

we can ensure the proper normalization of the scalar fields by shifting $\varphi(x) \to \varphi(x)/\sqrt{R}$. This applies for example to the LSZ formula, Eq.(5.15), which requires appropriately normalized fields

$$\begin{aligned}\langle f|i\rangle &= i^{n+n'}\int d^4x_1 e^{-ik_1x_1}\left(\partial_1^2 + m^2\right)\cdots d^4x'_1 e^{-ik'_1x'_1}\left(\partial_{1'}^2 + m^2\right)\cdots\langle 0|T\varphi(x_1)\cdots\varphi(x'_1)\cdots|0\rangle \\ &\to i^{n+n'}\int d^4x_1 e^{-ik_1x_1}\left(\partial_1^2 + m^2_\text{pole}\right)\cdots d^4x'_1 e^{-ik'_1x'_1}\left(\partial_{1'}^2 + m^2_\text{pole}\right)\cdots\langle 0|T\frac{\varphi(x_1)}{\sqrt{R}}\cdots\frac{\varphi(x'_1)}{\sqrt{R}}\cdots|0\rangle\ .\end{aligned} \tag{27.7}$$

In the computation of an actual process, for example in Section 10, each of the Klein-Gordon operators gets combined with propagators from the time ordered vacuum expectation values of the scalar field. For example in Eq.(10.10) we remove all propagators from the external legs this way. Only the internal propagators are left to contribute when we for example use Feynman rules. The Klein-Gordon operator acting on a field without the appropriate field normalization will include the residue

$$\left(\partial_1^2 + m^2_\text{pole}\right)\ \mathbf{\Delta}(x_1 - x_2) = R\ \delta^4(x_1 - x_2)\ . \tag{27.8}$$

Again, this factor applies to all external fields in the LSZ equation, so we find an additional dependence of the transition matrix element on R,

$$\begin{aligned}\langle f|i\rangle &= i^{n+n'}\int d^4x_1 e^{-ik_1x_1}\left(\partial_1^2 + m^2_\text{pole}\right)\cdots d^4x'_1 e^{ik'_1x'_1}\left(\partial_{1'}^2 + m^2_\text{pole}\right)\cdots\langle 0|T\frac{\varphi(x_1)}{\sqrt{R}}\cdots\frac{\varphi(x'_1)}{\sqrt{R}}\cdots|0\rangle \\ &= R^{n+n'}\frac{1}{R^{(n+n')/2}}\ \cdots \qquad \Rightarrow \qquad \mathcal{T} = R^{(n+n')/2}\ \mathcal{T}\bigg|_\text{LSZ}\end{aligned} \tag{27.9}$$

The residue factor R we can compute from its definition

$$\begin{aligned}\frac{1}{R} &= \frac{d}{dk^2}\ \left(-k^2 + m^2 - \Pi(k^2)\right)\bigg|_{k^2=m^2_\text{pole}} \\ &= -1 - \Pi'(m^2_\text{pole}) \\ &= -\left(1 + \Pi'(m^2)\right) + \mathcal{O}(\alpha^2) \\ &= -\left[1 + \frac{\alpha}{12}\left(\log\frac{\mu^2}{m^2} + \frac{17}{3} - \pi\sqrt{3}\right)\right]\end{aligned} \tag{27.10}$$

The integral giving us the last line we quote from Srednicki's book. The fact that the residue comes out as (-1) instead of one is simply an effect of our metric in combination with the definition of the propagator following Srednicki's conventions. This field renormalization constant is the only contribution to the $\overline{\text{MS}}$ transition amplitude which includes a logarithm of the mass, *i.e.* which is not defined in the massless limit.

The result from Eqs.(27.3), (27.4), and (27.10) together with the unchanged expression for the unrenormalized 4-vertex we can combine to

$$\begin{aligned}
\mathcal{T}_s =& R(s)^2 \left(\mathbf{V}_3^2(s) \tilde{\mathbf{\Delta}}(s) + \mathbf{V}_{4,s} \right) \\
=& \left[1 - \frac{\alpha}{6} \left(\log \frac{\mu^2}{m^2} + \frac{17}{3} - \pi\sqrt{3} \right) \right] \times \\
& \left[\left(1 - \alpha \log \frac{-s}{\mu^2} + 3\alpha \right) \frac{g^2}{-s} \left(1 + \frac{\alpha}{12} \log \frac{-s}{\mu^2} - \frac{2\alpha}{9} \right) + \frac{g^2}{6} \frac{3\alpha}{s} \left(\pi^2 + \log^2 \frac{t}{u} \right) \right] + \mathcal{O}(\alpha^2) \\
=& -\frac{g^2}{s} \left[1 - \frac{11\alpha}{12} \log \frac{s}{\mu^2} - \frac{\alpha}{6} \left(\log \frac{\mu^2}{s} + \log \frac{s}{m^2} \right) + \text{constant terms} \right] + \mathcal{O}(\alpha^2) \\
=& -\frac{\alpha(4\pi)^3}{s} \left(1 - \frac{3\alpha}{4} \log \frac{s}{\mu^2} - \frac{\alpha}{6} \log \frac{s}{m^2} + \text{constant terms} \right) + \mathcal{O}(\alpha^2) \qquad \text{using} \quad \alpha = \frac{g^2}{(4\pi)^3} \; . \qquad (27.11)
\end{aligned}$$

The factor $-g^2/s$ in front is the tree level value $\mathcal{T}_{\text{tree.}s}$. In this expression we omit all constant terms because all are interested in are the two kinds of logarithms. In the massless limit the second $\log s/m^2$ still diverges. It corresponds to an infrared divergence which is physical and can be removed through a modification of our perturbative series. Properly understanding the implications of such universal infrared divergences we unfortunately have to postpone on the introduction of the DGLAP equation in QCD, which we will discuss in detail in the next semester.
The scaling logarithm $\log \mu^2/s$ is new to the $\overline{\text{MS}}$ computation — for the physical on-shell renormalization scheme the scale did not appear in the final result for the transition amplitude. To compute an observable cross section, all we have to do is add the s, t, and u-channel contributions and square Eq.(27.11), which means that somewhere within the expression for the transition amplitude the scale dependence has to vanish,

$$\frac{d\mathcal{T}_s}{d \log \mu^2} \stackrel{!}{=} 0 \; . \qquad (27.12)$$

In the massless limit, the only candidate parameter which might cancel the explicit logarithm is a scale dependence of the coupling α, so we deduce

$$\begin{aligned}
0 &= \frac{d}{d \log \mu^2} \left(\alpha - \frac{3\alpha^2}{4} \log \frac{\mu^2}{s} + \mathcal{O}(\alpha^2) \right) \\
&= \frac{d\alpha}{d \log \mu^2} + \frac{3\alpha^2}{4} + \mathcal{O}\left(\alpha \frac{d\alpha}{d \log \mu^2} \right) \\
&= \frac{d\alpha}{d \log \mu^2} \left(1 + \mathcal{O}(\alpha) \right) + \frac{3\alpha^2}{4} \\
\Leftrightarrow \qquad \frac{d\alpha}{d \log \mu^2} &= -\frac{3\alpha^2}{4} + \mathcal{O}(\alpha^3) \; . \qquad (27.13)
\end{aligned}$$

This is the renormalization group equation describing the running of the coupling constant of a φ^3 theory in six dimensions. We can expand this equation to all orders in the couplings constant,

$$\frac{d\alpha}{d \log \mu^2} = \beta = -\alpha^2 \sum_{n=0} b_n \alpha^n \qquad \text{with} \quad b_0 = \frac{3}{4} \; . \qquad (27.14)$$

The right-hand side is called the beta function (note that the value of the beta function depends on if we compute the running with respect to μ or to μ^2). This equation we can solve by integrating the coupling it from one scale μ to another, higher, scale p

$$\begin{aligned}\alpha(p^2) &= \alpha(\mu^2) + \beta \log \frac{p^2}{\mu^2} \\ &= \alpha(\mu^2) \left(1 - \alpha(\mu^2) b_0 \log \frac{p^2}{\mu^2} + \mathcal{O}(\alpha^2) \right) \\ &= \frac{\alpha(\mu^2)}{1 + \alpha(\mu^2) b_0 \log \frac{p^2}{\mu^2} + \mathcal{O}(\alpha^2)} \; . \end{aligned} \tag{27.15}$$

The scale of α in the denominator can be chosen freely within reason because a change in scale is or order α and hence absorbed in the unknown higher order terms. For $p^2 > \mu^2$ we see that the coupling decreases for larger scale choices. This feature is called asymptotic freedom and one of the most distinctive properties of QCD. It is linked to the negative sign of the beta function. In the form of Eq.(27.15) we are save because the denominator of this expression is always positive. However, eventually this will change once we include an improved description of the loop effects. In addition, the improved treatment of the running coupling goes beyond fixed-order perturbation theory and leads to scaling violations: the transition matrix element does depend on the unphysical renormalization scale.

71 The path integral for nonabelian gauge theory

The definition of the path integral in a nonabelian gauge theory like QCD has serious implications on the definition of the relevant degrees of freedom. As we will see later, we can describe these in terms of anti-commuting scalar fields. Therefore, we start this section defining Grassmann variables, *i.e.* anti-commuting numbers which obey

$$\{\psi_i, \psi_j\} = 0 \qquad \text{for} \quad i, j = 1, ..., n \; . \tag{71.1}$$

Starting from these numbers we will collect a set of useful formulas which will then allow us to solve a problem in the definition of the path integral for gluon fields.
From our experience with the harmonic oscillator (chapter 7), free scalar fields (chapter 8), interacting scalar fields (chapter 9), and photons (chapter 57) we know that path integrals essentially come down to Gaussian integrals, where the integration variables are promoted to field configurations. So let us derive the relevant formulas step by step.

1. To begin, we limit ourselves to just one such number ($n = 1$), which has a set of interesting implications:

$$\begin{aligned}
\psi^2 &= 0 \\
f(\psi) &= a + \psi b && \text{for a general power series} \\
\partial_\psi f(\psi) &= b && \text{form the usual `be } \epsilon > 0\text{' construction} \\
\int_{-\infty}^{\infty} d\psi \; f(\psi) &= b && \text{requiring linearity and shift invariance .}
\end{aligned} \tag{71.2}$$

   The value of the integral in the last line is only fixed modulo an over-all factor which we define to be unity.

2. Generalizing this case we now allow for n Grassmann variables, organized in a vector ψ. The most general power series now is

$$f(\psi) = a + b_j \; \psi_j + \frac{1}{2} c_{j_1 j_2} \; \psi_{j_1} \psi_{j_2} + \cdots + \frac{1}{n!} d_{j_1 \cdots j_n} \; \psi_{j_1} \cdots \psi_{j_n} \; , \tag{71.3}$$

   where we imply the usual summing convention. The last prefactor we can simplify using the anti-commutation properties of the fields, getting $d_{j_1 \cdots j_n} = d \epsilon_{j_1 \cdots j_n}$ in terms of the generalized totally anti-symmetric Levi-Civita tensor ϵ. The integral over the Grassmann variable space becomes

$$\int_{-\infty}^{\infty} d^n \psi \; f(\psi) = d \; . \tag{71.4}$$

3. On the way to Gauss-type integrals over Grassmann variables we next consider linear shifts or rotations

$$\psi_i = R_{ij} \, \psi'_j \qquad \text{with} \qquad \int_{-\infty}^{\infty} d^n \psi \; f(\psi) = \frac{1}{\det R} \int_{-\infty}^{\infty} d^n \psi' \; f(\psi(\psi')) \; . \tag{71.5}$$

   The corresponding formula for a vector of real numbers is

$$\int_{-\infty}^{\infty} d^n x \; f(x) = \det R \int_{-\infty}^{\infty} d^n x' \; f(x(x')) \; , \tag{71.6}$$

   so we see that the structure is very similar, with the exception that for Grassmann variables the determinant has moved into the denominator.

4. The generalization of Eq.(71.5) to quadratic forms leads to a modified Gaussian integral

$$\begin{aligned}
&\int_{-\infty}^{\infty} d^n \psi \; e^{\frac{1}{2} \psi^T M \psi} = \sqrt{\det M} \\
\text{while} \qquad &\int_{-\infty}^{\infty} d^n x \; e^{-\frac{1}{2} x^T M x} = \frac{(2\pi)^n}{\sqrt{\det M}} \; ,
\end{aligned} \tag{71.7}$$

   again close in structure to the case of real numbers x.

5. Complex Grassmann variables we can define just as one would expect

$$\psi = \frac{1}{\sqrt{2}} (\psi_1 + i\psi_2)$$
$$\bar{\psi} = \frac{1}{\sqrt{2}} (\psi_1 - i\psi_2) \; . \tag{71.8}$$

The complex integral we define either in terms of the real an imaginary parts or in terms of $d^2\psi = i d\psi d\bar{\psi}$, giving us

$$\int_{-\infty}^{\infty} d^n\psi \, d^n\bar{\psi} \; e^{\frac{1}{2}\psi^\dagger M \psi} = \det M$$
$$\text{while} \qquad \int_{-\infty}^{\infty} d^n x \, d^n \bar{x} \; e^{-\frac{1}{2}x^\dagger M x} = \frac{(2\pi)^n}{\det M} \; , \tag{71.9}$$

All relations to this point can be proven with relatively little effort, but since the structure is straightforward we can use them just as a collection of useful formulas.

6. From Eq.(71.9) we know how to get to complex scalar fields and their path integral. We start with the Lagrangian

$$\begin{aligned} \mathscr{L} &= \partial_\mu \phi^\dagger \, \partial^\mu \phi - m^2 \phi^\dagger \phi + g\Sigma \phi^\dagger \phi \\ &= -\phi^\dagger(y) \, \delta^4(x-y) \, \left(\partial_x^2 + m^2 - g\Sigma \right) \, \phi(x) \equiv -\phi^\dagger(y) \, M(x-y) \, \phi(x) \; , \end{aligned} \tag{71.10}$$

where ϕ is the usual complex scalar field (or vector of complex scalar fields) and Σ is a real, non-propagating, constant background field. Note the explicit minus sign in this definition of M corresponds to the Gaussian integral for complex numbers. The corresponding path integral is

$$Z(\phi) = \int \mathcal{D}\phi^\dagger \, \mathcal{D}\phi \; e^{i \int d^4x \, \mathscr{L}} \; , \tag{71.11}$$

modulo appropriate normalization and an adjustable phase e^i. Using the appropriate definition of a functional determinant we should be able to write

$$Z(\phi) = \int \mathcal{D}\phi^\dagger \, \mathcal{D}\phi \; e^{-i \int d^4x \, d^4y \, \phi^\dagger M \phi} \equiv \frac{2(2\pi)^n}{\det M} \tag{71.12}$$

For the functional determinant there does not exist a handy definition, so Eq.(71.12) is as good a definition as any other. The prefactor $2(2\pi)^n$ we can ignore, since the normalization of the path integral is still arbitrary. The property $\det(M_1 M_2) = (\det M_1)(\det M_2)$ which we will need later is clearly true.

From the usual argument we know that only the part of the path integral which includes external sources contributes to the transition amplitude, while the free Lagrangian gets absorbed into the normalization. In the same spirit we can separate the matrix M into the free Lagrangian and the interaction with the background field

$$\begin{aligned} M &= \delta^4(x-y) \, \left(\partial_x^2 + m^2 - g\Sigma \right) \\ &= \int d^4z \; \delta^4(x-z) \, \left[\partial_x^2 + m^2 \right] \, \left[\delta^4(z-y) - g\Delta(z-y)\Sigma(z) \right] \; , \end{aligned} \tag{71.13}$$

provided the propagator cancels the free Lagrangian contribution, $\Delta(z-y)(\partial_z^2 + m^2) = \delta^4(z-y)$. In the path integral Eq.(71.11) only the interaction term will eventually contribute, so we limit ourselves to

$$Z(\phi) = \frac{1}{\det \left(\delta^4(x-y) - g\Delta(x-y)\Sigma(x) \right)} \; . \tag{71.14}$$

If we interpret the delta distribution as unity we can evaluate the path integral following the general property $\det M = \exp \operatorname{Tr} \log M$. This way we can reproduce the known results for the transition amplitude in this model and confirm the functional determinant approach.

7. The last step in terms of complex scalar fields we should be able to define in terms of Grassmann valued fields. Naively, those would be fermions in the spinor representation of the Lorenz group, but from the above discussion it should be clear that we can as well define scalar anti-commuting fields. The net result of the switch to anti-commuting scalars will be that we can express determinants instead of inverse determinants as path integrals.

   Without an actual motivation we can start from a Lagrangian for an anti-commuting scalar field c^a with adjoint $SU(3)$ charge which couples to gluons. The change in conventions from ϕ and $\phi^\dagger$ to c and $\bar{c}$ just reflects the conventions in the literature. The color-octet fields c and $\bar{c}$ we call Fadeev-Popov ghosts. We write the kinetic and coupling terms for the ghosts independently,

$$\mathscr{L} = \partial_\mu \bar{c}^a \partial^\mu c^a - g f^{abc} A_\mu^c (\partial^\mu \bar{c}^a) c^b \; , \tag{71.15}$$

   there this form of the dimension-4 interaction term requires some justification. If we could write it in terms of the covariant derivative this would definitely be helpful, so we find

$$\begin{aligned} \mathscr{L} &= \partial_\mu \bar{c}^a \partial^\mu c^a - ig (T_A^c)^{ab} A_\mu^c (\partial^\mu \bar{c}^a) c^b \qquad\qquad \text{using } f^{abc} = i (T_A^c)^{ab} \\ &= \partial_\mu \bar{c}^a \left[\delta^{ab} \partial^\mu - ig (T_A^c)^{ab} A_\mu^c \right] c^b \\ &\equiv \partial_\mu \bar{c}^a (D^\mu)^{ab} c^b \\ &= - \bar{c}^a \partial_\mu (D^\mu)^{ab} c^b \equiv \bar{c}^a(x) M^{ab}(x-y) c^b(y) \; . \end{aligned} \tag{71.16}$$

   In the second to last step we use the definition of the covariant derivative, acting on a color-adjoint field, as defined in Eq.(69.25) of Srednicki's book. By pure analogy we include this Lagrangian in our path integral and find

$$Z(c) = \int \mathcal{D}\bar{c} \mathcal{D}c \; e^{+i \int d^4x d^4y \, \bar{c}^a M^{ab} c^b} = \det M^{ab} = \det \delta^4(x-y) \; \left[-\partial_\mu (D^\mu)^{ab} \right] \; . \tag{71.17}$$

   The factor $1/2$ in the exponent of Eq.(71.9) can be absorbed into the definition of the fields c and $\bar{c}$ because we never really use the integral measure of the path integral. Of course, all this only helps if this object $\partial_\mu (D^\mu)^{ab}$ ever appears inside a path integral and needs to be removed. So let us look at the path integral in QCD and its problems.

The trick which we apply to the photon path integral is a projection of the photon field on its transverse degrees of freedom. In other words, we only consider degrees of freedom which in momentum space fulfill $k^\mu \tilde{A}_\mu = 0$. The local $U(1)$ gauge transformation

$$A_\mu \to A_\mu - \partial_\mu \Gamma \qquad\qquad \text{or} \qquad \tilde{A}_\mu \to \tilde{A}_\mu + i k_\mu \tilde{\Gamma} \tag{71.18}$$

induces only a longitudinal shift in $\tilde{A}$, which means that the projection on transverse states is gauge invariant. In QCD the same relation reads

$$A_\mu^a \to A_\mu^a - D_\mu^{ab} \, \theta^b \; , \tag{71.19}$$

which means that gauge transformations will ruin our projection onto transverse gluons inside the path integral. What we need to do is include the gauge transformations θ^a in the path integral, knowing that we can remove them from the Lagrangian. We will see that they indeed cannot be neglected. In his Eq.(71.14) Mark Srednicki shows how in an n-dimensional double integral we can include a set of integration variables which get removed by a set of boundary conditions $G_j(x,y) = 0$:

$$Z = \int d^n x \; e^{iS(x)} = \int d^n x \, d^n y \; \delta^n(y) \; e^{iS(x)} = \int d^n x \, d^n y \; \det \left(\frac{\partial G_i}{\partial y_j} \right) \delta^n(G) \; e^{iS(x)} \; . \tag{71.20}$$

The additional determinant arises because we change the form of the boundary condition. For example in QED or QCD such functions G fix the gauge. In QED we usually use the Lorenz gauge $G \equiv \partial^\mu A_\mu = 0$. The general R_ξ gauge is a generalization of this form with its gauge transformation gives us the dependence of G^a on θ^b,

$$\begin{aligned} G^a(x) &= \partial^\mu A^a_\mu(x) - \omega^a(x) \\ &\to \partial^\mu A^a_\mu(x) - \partial^\mu D^{ab}_\mu \, \theta^b - \omega^a(x) = G^a(x) - \partial^\mu D^{ab}_\mu \, \theta^b \; . \end{aligned} \tag{71.21}$$

The additional factors ω are not necessary, but there is no reason why we should not be able to include them. The path integral version of Eq.(71.20) for gluons is

$$Z(A_\mu) = \int \mathcal{D}A \, \mathcal{D}\theta \; \det \left(\frac{\delta G}{\delta \theta} \right) \delta(G^a(x)) \; e^{i \int d^4x \left(-\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} - J^{a\mu} A^a_\mu \right)} \; . \tag{71.22}$$

From Eq.(71.21) we can compute the functional derivative

$$\frac{\delta G^a(x)}{\delta \theta^b(y)} = -\delta^4(x-y) \, \partial^\mu D^{ac}_\mu \; . \tag{71.23}$$

The Fadeev-Popov determinant is precisely what the Lagrangian of two Grassmann scalars gives us in Eq.(71.17), so we can write the gluon path integral as

$$\begin{aligned} Z(A_\mu, c, \bar{c}) &= \int \mathcal{D}A \, \mathcal{D}\theta \, \mathcal{D}\bar{c} \, \mathcal{D}c \; \delta(G^a(x)) \; e^{i \int d^4x d^4y \, \bar{c}^a M^{ab} c^b} \; e^{i \int d^4x \left(-\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} - J^{a\mu} A^a_\mu \right)} \\ &= \int \mathcal{D}A \, \mathcal{D}\theta \, \mathcal{D}\bar{c} \, \mathcal{D}c \; \delta(G^a(x)) \; e^{i \int d^4x \, \mathscr{L}} \\ \text{with} \quad \mathscr{L} &= -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + \partial_\mu \bar{c}^a \partial^\mu c^b - g f^{abc} A^c_\mu \partial_\mu \bar{c}^a c^b - J^{a\mu} A^a_\mu \; . \end{aligned} \tag{71.24}$$

Note that in the literature the explicit path integral over θ is often omitted, assuming that gauge transformations are part of the path integral over A. Because later on it will be convenient, we can add terms including ω^a to the Lagrangian at will. If we consider it an auxiliary field ω^a is fixed by the gauge condition $G^a = 0$. Such terms will not affect the active field content or the interaction, they will merely contribute to the normalization of the path integral. What will come in handy is that we can mimic kinetic terms for the gluons,

$$\omega^a \omega^a = (\partial^\mu A^a_\mu)(\partial^\nu A^a_\nu) \tag{71.25}$$

The complete gluon Lagrangian including Fadeev-Popov ghost and the gauge-fixing term now reads

$$\begin{aligned} \mathscr{L} &= -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + \partial_\mu \bar{c}^a \partial^\mu c^a - g_s f^{abc} A^c_\mu \partial^\mu \bar{c}^a c^b - \frac{1}{2\xi} (\partial^\mu A^a_\mu)(\partial^\nu A^a_\nu) - J^{a\mu} A^a_\mu \\ \text{with} \qquad F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu \; . \end{aligned} \tag{71.26}$$

The derivation of the QCD Lagrangian including Fadeev-Popov ghosts is the reason why (to the best of my knowledge) we need the path integral formalism in any field theory lecture. Without it, we would miss this part of the QCD Lagrangian. To avoid confusion, the strong coupling now consistently carries an index g_s. I should admit that I am not 100% sure about the prefactor of the ghost-ghost-gluon interaction, which is tied to the definition of the covariant derivative. Mark Srednicki defines it as $D_\mu = \partial_\mu - igA_\mu$ while my usual reference Ellis-Stirling-Webber defines $D_\mu = \partial_\mu + igA_\mu$, while Otto Nachtmann's book has the same conventions as Srednicki's. For the Feynman rules in the next chapter I refer to Otto Nachtmann's appendix as the standard reference.

72 The Feynman rules for nonabelian gauge theory

The Feynman rules for QCD we can largely read off the gluon Lagrangian in Eq.(71.26). All we need to add is a set of fermions (quarks) which live in the fundamental representation of $SU(3)$. These fermions also carry electric as well as weak $SU(2)$ charges, but as long as we are only interested in perturbative series in the strong coupling g_s we do not have to worry about those weaker interactions.
The complete QCD Lagrangian including one massless quark flavor and without the coupling to the external current is

$$\mathscr{L} = -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + i\overline{\psi} \not{D} \psi + \partial_\mu \bar{c}^a \partial^\mu c^a - g_s f^{abc} A^c_\mu \partial_\mu \bar{c}^a c^b - \frac{1}{2\xi} (\partial^\mu A^a_\mu)(\partial^\nu A^a_\nu)$$

$$\text{with} \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu$$

$$D_\mu \psi_i = \partial_\mu \psi_i - i g_s T^a_{ij} A^a_\mu \psi_j \; . \tag{72.1}$$

The interactions between quarks, gluons, and ghosts we can simply read off the respective terms in Eq.(72.1). The only object we need to study a little more carefully is the gluon propagator.

For the kinetic term of the gluon we can translate the energy-momentum tensor squared into gluons fields. For the photon we know from chapter 54 how this looks:

$$F^{\mu\nu} F_{\mu\nu} = -2 A_\mu \left(g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu \right) A_\nu \tag{72.2}$$

In the gluon case the $F_{\mu\nu} F^{\mu\nu}$ term will also give rise to three and four gluon interactions, but the structure of the propagator is the same as for the photon. Combining this term with the gauge fixing term we find

$$\begin{aligned} \mathscr{L} \supset & -\frac{1}{2} A^a_\mu \left(g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu \right) A^a_\nu - \frac{1}{2\xi} A^a_\mu \partial^\mu \partial^\nu A^a_\nu \\ = & -\frac{1}{2} A^a_\mu \left(g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu + \frac{1}{\xi} \partial^\mu \partial^\nu \right) A^a_\nu \; . \end{aligned} \tag{72.3}$$

The corresponding momentum-space gluon propagator in R_ξ gauge we can construct after replacing $\partial_\mu \to -ik_\mu$. Using the path integral this means inverting the quadratic term in $\tilde{A}$ as it appears in the quadratic term in $\tilde{J}$. We use the method of the known result, namely

$$\frac{1}{4} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} + \frac{1}{\xi} \frac{k^\mu k^\nu}{k^2} \right) \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} + \xi \frac{k^\mu k^\nu}{k^2} \right) = 1 \; . \tag{72.4}$$

Unlike in QED this does not lead to a projector in the propagator. Interestingly enough, it turns out that the form of the propagator does not depend on the sign of the $1/\xi$ term in Eq.(72.1); and I have no idea why this symmetry exists. Taking into account the over-all factor $+k^2/2$ of the Lagrangian after Fourier transform, we now

$$\tilde{\Delta}^{ab}_{\mu\nu} = \frac{\delta^{ab}}{k^2} \left(g^{\mu\nu} - (1-\xi) \frac{k^\mu k^\nu}{k^2} \right) \qquad \text{Feynman rule: } -i\tilde{\Delta}^{ab}_{\mu\nu} \tag{72.5}$$

While this propagator depends on the gauge parameter ξ, observables like cross sections will of course not. The propagator for the ghost field is the usual scalar propagator, because the Grassmann property only appears in symmetry factors,

$$\tilde{\Delta}^{ab} = \frac{\delta^{ab}}{k^2} \qquad \text{Feynman rule: } +i\tilde{\Delta}^{ab} \tag{72.6}$$

Just for completeness, we repeat the Feynman rule for a propagating quark,

$$i \frac{\not{p} + m}{p^2 - m^2} \tag{72.7}$$

Of the different interactions we also start with the $F^{a\mu\nu}F^a_{\mu\nu}$ term. In addition to the quadratic term shown in Eq.(72.3) the covariant derivative includes higher powers of the gluon field

$$\begin{aligned}\mathscr{L} &\supset -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu\right)\left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + g_s f^{ade} A^{d\mu} A^{e\nu}\right) \\ &\supset -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) g_s f^{ade} A^{d\mu} A^{e\nu} - \frac{1}{4} g_s f^{abc} A^b_\mu A^c_\nu \left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}\right) - \frac{1}{4} g_s^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} \\ &= -g_s f^{abc} (\partial_\mu A^a_\nu) A^{b\mu} A^{c\nu} - \frac{1}{4} g_s^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} \end{aligned} \tag{72.8}$$

The four terms contributing to the triple gluon Lagrangian are all identical. However, when we derive the Feynman rule for the triple gluon vertex we need to take into account all possible assignments of the external gluon indices and momenta. In terms of three incoming gluon momenta k_j we find

$$-g_s f^{a_1 a_2 a_3} \;\left[g_{\mu_1\mu_2}(k_1-k_2)_{\mu_3} + g_{\mu_2\mu_3}(k_2-k_3)_{\mu_1} + g_{\mu_3\mu_1}(k_3-k_1)_{\mu_2} \right] \;, \tag{72.9}$$

while the corresponding four-gluon expression is

$$i g_s^2 f^{a_1 a_2 a} f^{a a_3 a_4} \;\left[g_{\mu_2\mu_3} g_{\mu_1\mu_4} - g_{\mu_1\mu_3} g_{\mu_2\mu_4} + \text{cyclical permutations} \right] \;. \tag{72.10}$$

These multi-gluon interactions tend to make calculations even of simple processes very lengthy. It is highly recommended to use a tool like FORM for such computations.

The Feynman rule for the quark-quark-gluon interaction is

$$i g_s \; T^a \; \gamma_\mu \;, \tag{72.11}$$

in complete analogy to the QED case $ie\gamma_\mu$. The (3×3) color matrix $T^a (a = 1, ..., 8)$ is sandwiched between the quark indices. Similarly to the gamma matrices we usually have to evaluate traces of color matrices. This is why the index and the quadratic Casimir introduced in chapter 70 turn out useful. Two color matrices which linked by a gluon will have the same index $a_1 = a_2$.

Finally, the interaction of a gluon A^a_μ with two ghost $c^b_{\text{out}}, c^c_{\text{in}}$, includes a derivative in position space, translating into

$$g_s f^{abc} \; p^\mu_{\text{out}} \tag{72.12}$$

The momentum is defined in the direction of the outgoing ghost leg.

Because they are new in QCD we need to briefly discuss the impact of the ghosts. Whenever we encounter virtual gluons we need to check if corresponding virtual ghost diagrams exist. One example will be the gluon self energy which we will compute soon. Because of their Grassmann property ghost loops come with a symmetry factor (-1), just like closed fermion loops. The gluon propagator depends on the gauge choice:

- Feynman gauge $\xi = 1$ gives a particularly simple propagator $g^{\mu\nu}/k^2$
- Landau gauge $\xi = 0$ guarantees a transverse gluon propagator $k^\mu \tilde{\Delta}_{\mu\nu} = 0$
- Unitary gauge $\xi \to \infty$ is only useful for massive weak gauge bosons

From the QED case we know that the polarization sum for external gluons is essentially identical to the residue of the gluon propagator. If we use the simple (Feynman) form $\sum_{\text{pols}} \epsilon_\mu \epsilon^*_\nu = -g_{\mu\nu}$ we need to take into account external ghosts. Only if we project the external gluons onto the transverse degrees of freedom external ghosts do not appear. In general, we need to square the matrix elements for external gluons and ghosts independently, add them after taking into account all statistics factors, and consider the sum as the result for external physical gluons.

73 The beta function in nonabelian gauge theory

The computation of the beta function for the strong coupling g_s from one-loop vertices and propagators follows exactly the same path as the QED case. The only difference is the appearance of new diagrams in the gluon self energy, both from the gluon self coupling and from Fadeev-Popov ghosts. Before we can compute these contributions we need to define the QCD Lagrangian including its Z factors. Note that for these Z factors I am not following Mark Srednicki's conventions

$$\begin{aligned}\mathscr{L} = &-\frac{Z_A}{2} A^{a\mu} \left(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right) A^{a\nu} + \frac{1}{2\xi} A^{a\mu}\,\partial_\mu\partial_\nu A^{a\nu} \\ &- Z_{g,3} g_s f^{abc}\,\partial_\mu A^a_\nu A^{b\mu} A^{c\nu} - \frac{Z_{g,4} g_s^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} \\ &+ Z_c \partial_\mu \bar{c}^a \partial^\mu c^a - Z_{g,c} g_s f^{abc} A^a_\mu (\partial_\mu \bar{c}^b) c^c + i Z_q\, \overline{\psi} \partial\!\!\!/ \psi + Z_{g,q} g_s\, \overline{\psi} A\!\!\!/^a T^a \psi\ .\end{aligned} \tag{73.1}$$

Altogether, there are four renormalization constants for the strong coupling $Z_{g,*}$ which should lead to identical definitions $g_{s,0} \to g_s$. To write down this relation we first have to compute the mass dimension of g_s in $d = 4 - \epsilon$ dimensions. Compared to the corresponding argument for $e_0 \to e$ in QED nothing changes, so we can use it to write

$$g_{s,0} = \frac{Z_{g,q}}{Z_q \sqrt{Z_A}}\, \tilde{\mu}^{\epsilon/2}\, g_s \qquad \text{or} \qquad \alpha_{s,0} = \frac{Z_{g,q}^2}{Z_q^2 Z_A}\, \tilde{\mu}^{\epsilon}\, \alpha_s \tag{73.2}$$

If it is indeed true that the same strong coupling governs all the interactions listed in Eq.(73.1) this should hold for the bare as well as for the renormalized strong coupling,

$$\frac{Z_{g,q}}{Z_q\sqrt{Z_A}} = \frac{Z_{g,c}}{Z_c\sqrt{Z_A}} = \frac{Z_{g,3}}{Z_A^{3/2}} = \sqrt{\frac{Z_{g,4}}{Z_A^2}}\ . \tag{73.3}$$

This is the non-abelian version of the Ward identities in QED, where we found $Z_1 = Z_2$, or $Z_{g,q} = Z_q$ in QCD conventions. We will find that this relation is not true for QCD. This means that while for QED we would only have needed to compute Z_3 for the running of the coupling constant, in QCD we need to compute all three renormalization constants involved.

The easiest Z factor we need to compute is the quark self energy correction. In Feynman gauge the gluon propagator becomes $\Delta^{ab}_{\mu\nu} = \delta^{ab} g_{\mu\nu}/k^2$, which is identical to the photon propagator in the same gauge. The only modification is the additional color factor. Two color matrices T^a linked by a gluon have the same index, so for external quarks j and i we find

$$\left(T^a T^a\right)_{ij} = C(R)\,\delta_{ij}\ . \tag{73.4}$$

Adding the color factor to the QED result from Eq.(62.34) gives

$$Z_q = 1 - \frac{g_s^2}{8\pi^2}\, C(R) \left(\frac{1}{\epsilon} + \text{finite}\right) + \mathcal{O}(g_s^4)\ . \tag{73.5}$$

Part of the one-loop correction to the quark-quark-gluon vertex we can also read off the QED result. Here, the series of color matrices T^a is evaluated along the fermion line, with the additional condition that a gluon propagator identifies the two indices. This means that for an external gluon with color index a attached between the external quarks j and i we need to compute

$$\begin{aligned}
\left(T^bT^aT^b\right)_{ij} &= \left(T^b\left(T^bT^a+[T^a,T^b]\right)\right)_{ij} \\
&= \left(T^b\left(T^bT^a+if^{abc}T^c\right)\right)_{ij} \\
&= \left(C(R)T^a+\frac{i}{2}f^{abc}[T^b,T^c]\right)_{ij} \\
&= \left(C(R)T^a-\frac{1}{2}f^{abc}f^{bcd}T^d\right)_{ij} \\
&= \left(C(R)T^a+\frac{1}{2}(T^a_A)^{bc}(T^d_A)^{bc}T^d\right)_{ij} \\
&= \left(C(R)T^a-\frac{1}{2}\,\mathrm{Tr}(T^a_AT^d_A)\,T^d\right)_{ij} = \left(C(R)-\frac{T(A)}{2}\right)T^a_{ij} = \left(\frac{4}{3}-\frac{3}{2}\right)T^a_{ij}\;. \qquad (73.6)
\end{aligned}$$

Applying this factor to Eq.(62.50) we find

$$Z^{\text{abelian}}_{g,q} = 1-\frac{g_s^2}{8\pi^2}\left(C_R-\frac{T(A)}{2}\right)\left(\frac{1}{\epsilon}+\text{finite}\right)+\mathcal{O}(g_s^4)\;. \qquad (73.7)$$

Note that the quark self energy and the abelian vertex corrections do not share the same color factor. The second Feynman diagram we need to compute for $Z_{g,q}$ is the external gluon attached to the gluon inside the loop. We only compute the color factor for this diagram

$$\begin{aligned}
f^{abc}\left(T^bT^c\right)_{ij} &= \frac{1}{2}f^{abc}\left([T^b,T^c]\right)_{ij} \\
&= \frac{i}{2}f^{abc}f^{bcd}\,T^d_{ij} \\
&= -\frac{i}{2}(T^a_A)^{bc}(T^d_A)^{bc}\,T^d_{ij} = -\frac{i}{2}T(A)\,T^a_{ij} \qquad (73.8)
\end{aligned}$$

The contribution of the three-gluon vertex to $Z_{g,q}$ altogether gives

$$\begin{aligned}
Z^{\text{non-abelian}}_{g,q} &= 1-\frac{g_s^2}{8\pi^2}\frac{3T(A)}{2}\left(\frac{1}{\epsilon}+\text{finite}\right)+\mathcal{O}(g_s^4) \\
Z_{g,q} = Z^{\text{abelian}}_{g,q}+Z^{\text{non-abelian}}_{g,q} &= 1-\frac{g_s^2}{8\pi^2}\left(C_R+T(A)\right)\left(\frac{1}{\epsilon}+\text{finite}\right)+\mathcal{O}(g_s^4)\;. \qquad (73.9)
\end{aligned}$$

Finally, we need to compute the one-loop correction to the gluon propagator. Four diagrams contribute at one loop:

1. the (abelian) quark loop we can generalize from the QED case — remember the factor (-1) for the closed fermion loop. Its color factor is $\mathrm{Tr}(T^aT^b) = T(R)\delta^{ab} = \delta^{ab}/2$.
2. the gluon loop with two three-gluon vertices gives a color factor $f^{acd}f^{bcd} = -T(A)\delta^{ab}$. Remember that it requires a phase space factor $1/2$ for two identical particles running in the loop.
3. the one-point diagram with one four-gluon interaction would only contribute for a massive particle in the loop. For a massless gluon it vanishes based on symmetry arguments.
4. the ghost loop looks like a purely scalar integral at first, but the interaction gives a momentum dependence in the numerator. Its color factor is the same as for the gluon loop. For a closed loop of anti-commuting field we again have to add a factor (-1).

Adding all contributions gives

$$Z_A = 1+\frac{g_s^2}{8\pi^2}\left(\frac{5}{3}T(A)-\frac{4}{3}T(R)\right)\left(\frac{1}{\epsilon}+\text{finite}\right)+\mathcal{O}(g_s^4)\;. \qquad (73.10)$$

We can insert Eq.(73.5), (73.9) and (73.10) into Eq.(73.2) and find

$$\begin{aligned}\log \alpha_{s,0} &= \log \frac{Z_{g,q}^2}{Z_q^2 Z_A} + \epsilon \log \tilde{\mu} + \log \alpha_s + \cdots \\ &= \frac{g_s^2}{8\pi^2 \epsilon} \left(-2C(R) - 2T(A) + 2C(R) - \frac{5}{3} T(A) + \frac{4}{3} T(R) \right) + \epsilon \log \tilde{\mu} + \log \alpha_s + \cdots \\ &= \frac{\alpha_s}{2\pi\epsilon} \left(-\frac{11}{3} T(A) + \frac{4}{3} T(R) \right) + \epsilon \log \tilde{\mu} + \log \alpha_s + \cdots \end{aligned} \tag{73.11}$$

The dots stand for higher order corrections and finite terms. Starting from the condition that $\alpha_{s,0}$ should not depend on the scale $\tilde{\mu}$ we can use Eq.(66.9) with an additional square in the scale logarithm to compute the running of the strong coupling constant

$$\begin{aligned}\beta \equiv \frac{d\alpha_s}{d \log \tilde{\mu}^2} &= \frac{\alpha_s^2}{2} \frac{d}{d\alpha_s} \frac{\alpha_s}{2\pi} \left(-\frac{11}{3} T(A) + \frac{4}{3} T(R) \right) \\ &= \frac{\alpha_s^2}{4\pi} \left(-\frac{11}{3} T(A) + \frac{4}{3} T(R) \right) . \end{aligned} \tag{73.12}$$

In general, the beta function of the strong coupling constant is given by a series in α_s, so in general we write

$$\beta = \frac{\alpha_s^2}{12\pi} \sum_{\text{colored states}} D_j T_j = -\alpha_s^2 \sum_{n=0} b_n \alpha_s^2 \; , \tag{73.13}$$

with $T_j = T(R) = 1/2$ for particles in the fundamental representation of $SU(3)$ and $T_j = T(A) = 3$ for particles in the adjoint representation. The factors D_j are -11 for a the gluon, +4 for a Dirac fermion or quark, +2 for a Majorana fermion, +1 for a complex scalar, and +1/2 for a real scalar. In this form we can compute the running of α_s in essentially any renormalizable theory. For the Standard Model with six quarks we find

$$b_0 = \frac{1}{4\pi} \left(11 - 6 \times \frac{2}{3} \right) > 0 \; . \tag{73.14}$$

The positive sign of b_0 or the negative sign of β imply that the strong coupling decreases for larger energy scales or smaller distances. Just like the ϕ^3 theory in six dimensions, which we discussed in chapter 27, QCD is asymptotically free.

Following our argument from chapter 27 we can use the beta function to relate the strong coupling at one energy scale to the strong coupling at another energy scale. In Eq.(27.15) of these notes we solve the one-loop renormalization group equation. For QED we have learned in Eq.(62.10) that we can resum photon self energy diagrams when they are one-particle irreducible, so we find

$$\begin{aligned}\alpha_s(\mu_2^2) &= \alpha_s(\mu_1^2) \left(1 - \alpha_s \, b_0 \; \log \frac{\mu_2^2}{\mu_1^2} + \mathcal{O}(\alpha_s^2) \right) \\ &\to \frac{\alpha_s(\mu_1^2)}{1 + \alpha_s \, b_0 \; \log \dfrac{\mu_2^2}{\mu_1^2} + \mathcal{O}(\alpha_s^2)} \; . \end{aligned} \tag{73.15}$$

Of course, to fixed order in perturbation theory the two expressions are identical. As long as $\mu_2 > \mu_1$ asymptotic freedom $b_0 > 0$ ensures that the strong coupling is well defined everywhere. On the other hand, towards small scales $\mu_2 < \mu_1$ this is not true any longer. We can define a low-scale reference value $\mu_2 = \Lambda_{\text{QCD}}$ where the denominator of Eq.(73.13) becomes zero. This is the Landau pole of the strong coupling. If we want to compute this pole it makes sense to identify the scale of the strong coupling in the denominator with μ_1, so at one loop order we find

$$
\begin{aligned}
1 + \alpha_s(\mu_1^2)\, b_0 \, \log \frac{\Lambda_{\text{QCD}}^2}{\mu_1^2} = 0 \qquad \Leftrightarrow \qquad \log \frac{\Lambda_{\text{QCD}}^2}{\mu_1^2} = -\frac{1}{\alpha_s(\mu_1^2) b_0} \\
\alpha_s(\mu_2^2) &= \frac{\alpha_s(\mu_1^2)}{1 + \alpha_s(\mu_1^2)\, b_0 \left(\log \frac{\mu_2^2}{\Lambda_{\text{QCD}}^2} + \log \frac{\Lambda_{\text{QCD}}^2}{\mu_1^2} \right)} \\
&= \frac{\alpha_s(\mu_1^2)}{\alpha_s(\mu_1^2)\, b_0 \log \frac{\mu_2^2}{\Lambda_{\text{QCD}}^2}} \\
&= \frac{1}{b_0 \log \frac{\mu_2^2}{\Lambda_{\text{QCD}}^2}} \, . \qquad (73.16)
\end{aligned}
$$

This scheme can be generalized to any order in perturbative QCD and is not that different from the Thomson limit renormalization scheme of QED, except that with the introduction of Λ_{QCD} we are choosing a reference point which is particularly hard to compute perturbatively. One thing that is interesting in the way we introduce Λ_{QCD} is the fact that we introduce a scale into our theory without ever setting it. All we did was renormalize a coupling which becomes strong at large energies and search for the mass scale of this strong interaction. This trick is called dimensional transmutation.

Running couplings and resummation

In the last chapter we have introduced the running strong coupling, not really linked to physics. To interpret the effect of a running coupling we look for an observable which depends on just one energy scale. One simple example which includes α_s at least in the one-loop corrections is the R parameter in QED and QCD

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_{\text{quarks}} Q_q^2 = \frac{11N_c}{9} \; . \tag{73.17}$$

The numerical value at leading order assumes five quarks. Including higher order corrections we can express the result in a power series in the renormalized strong coupling α_s. The only physical scale is the energy of the e^+e^- system p^2. The strong coupling we can in principle evaluate at any scale μ^2. The parameter R has no mass dimension, so the perturbative series in α_s including all scale dependences reads

$$R\left(\frac{p^2}{\mu^2}, \alpha_s\right) = \sum_{n=0} r_n \left(\frac{p^2}{\mu^2}\right) \alpha_s^n(\mu^2) \qquad\qquad r_0 = \frac{11N_c}{9} \; . \tag{73.18}$$

Because R is an observable it cannot depend on any artificial scale choices μ. Writing this dependence as a total derivative and setting it to zero we find an equation which would be called a Callan-Symanzik equation if instead of the running coupling we had included a running mass

$$\begin{aligned}
0 \stackrel{!}{=} \frac{d}{d\log\mu^2} R\left(\frac{p^2}{\mu^2}, \alpha_s(\mu^2)\right) &= \mu^2 \left[\frac{\partial}{\partial\mu^2} + \frac{\partial\alpha_s}{\partial\mu^2}\frac{\partial}{\partial\alpha_s}\right] R\left(\frac{p^2}{\mu^2}, \alpha_s\right) \\
&= \left[\mu^2\frac{\partial}{\partial\mu^2} + \beta\,\frac{\partial}{\partial\alpha_s}\right] \sum_{n=0} r_n \left(\frac{p^2}{\mu^2}\right) \alpha_s^n && \text{for the two arguments of } R(\mu^2, \alpha_s) \\
&= \sum_{n=1} \mu^2 \frac{\partial r_n}{\partial\mu^2}\,\alpha_s^n + \sum_{n=1} \beta\, r_n\, n\alpha_s^{n-1} && \text{with} \quad r_0 = \frac{11N_c}{9} = \text{const} \\
&= \mu^2 \sum_{n=1} \frac{\partial r_n}{\partial\mu^2}\,\alpha_s^n - \sum_{n=1}\sum_{m=0} n r_n\, \alpha_s^{n+m+1}\, b_m && \text{with} \quad \beta = -\alpha_s^2 \sum_{m=0} b_m \alpha_s^m \\
&= \mu^2 \frac{\partial r_1}{\partial\mu^2}\,\alpha_s + \left(\mu^2\frac{\partial r_2}{\partial\mu^2} - r_1 b_0\right)\alpha_s^2 + \left(\mu^2\frac{\partial r_3}{\partial\mu^2} - r_1 b_1 - 2r_2 b_0\right)\alpha_s^3 + \mathcal{O}(\alpha_s^4) \; .
\end{aligned} \tag{73.19}$$

This series has to vanish in each order of perturbation theory. The non-trivial structure, namely the mix of r_n derivatives and the perturbative terms in the β function we can read off the α_s^3 term in Eq.(73.19): first, we have the appropriate NNNLO corrections r_3. Next, we have one loop in the gluon propagator b_0 and two loops for example in the vertex r_2. And finally, we need the two-loop diagram for the gluon propagator b_1 and a one-loop vertex correction r_1. The kind-of Callan-Symanzik equation Eq.(73.19) requires

$$\begin{aligned}
\frac{\partial r_1}{\partial\log\mu^2} &= 0 \\
\frac{\partial r_2}{\partial\log\mu^2} &= r_1 b_0 \\
\frac{\partial r_3}{\partial\log\mu^2} &= r_1 b_1 + 2r_2 b_0 \\
&\cdots
\end{aligned} \tag{73.20}$$

There will be integration constants c_n which are independent of μ^2. Their p^2 dependence has to cancel the mass units inside $\log \mu^2$, so we find

$$\begin{aligned}
r_0 &= c_0 = \frac{11 N_c}{9} \\
r_1 &= c_1 \\
r_2 &= c_2 + r_1 b_0 \log \frac{\mu^2}{p^2} = c_2 + c_1 b_0 \log \frac{\mu^2}{p^2} \\
r_3 &= \int d \log \frac{\mu'^2}{p^2} \left(c_1 b_1 + 2 \left(c_2 + c_1 b_0 \log \frac{\mu'^2}{p^2} \right) b_0 \right) = c_3 + (c_1 b_1 + 2 c_2 b_0) \log \frac{\mu^2}{p^2} + c_1 b_0^2 \log^2 \frac{\mu^2}{p^2} \\
\cdots
\end{aligned} \tag{73.21}$$

This chain of r_n values looks like we should interpret the apparent fixed-order perturbative series for R in Eq.(73.18) as a series which implicitly includes terms of to the order $\alpha_s \times \left(\alpha_s \log \mu^2/p^2 \right)^{n-1}$. They become problematic if we evaluate the strong coupling at scales far away from the generic scale of R, *i.e.* $\log \mu^2/p^2 \gtrsim 1/\alpha_s \sim 10$,

Instead of the series in r_n we can use the conditions in Eq.(73.21) to express R in terms of the c_n and collect the logarithms appearing with each c_n. The geometric series we then resum to

$$\begin{aligned}
R = \sum_n r_n \left(\frac{p^2}{\mu^2} \right) \; \alpha_s^n(\mu^2) &= c_0 + c_1 \left(1 + \alpha_s b_0 \log \frac{\mu^2}{p^2} + \alpha_s^2 b_0^2 \log^2 \frac{\mu^2}{p^2} + \cdots \right) \alpha_s(\mu^2) \\
&\quad + c_2 \left(1 + 2 \alpha_s b_0 \log \frac{\mu^2}{p^2} + \cdots \right) \alpha_s^2(\mu^2) + \cdots \\
&= c_0 + c_1 \frac{\alpha_s(\mu^2)}{1 - \alpha_s b_0 \log \dfrac{\mu^2}{p^2}} + c_2 \left(\frac{\alpha_s(\mu^2)}{1 - \alpha_s b_0 \log \dfrac{\mu^2}{p^2}} \right)^2 + \cdots \\
&\equiv \sum c_n \; \alpha_s^n(p^2) \; .
\end{aligned} \tag{73.22}$$

In the last step we use what we know about the running coupling from Eq.(73.15). Note that in contrast to the r_n integration constants the c_n are by definition independent of μ^2/p^2.
This re-organization of the perturbation series for R can be interpreted as re-summing all logarithms of the kind $\log \mu^2/p^2$ and absorbing them into the running strong coupling, now evaluated at the scale p^2. All scale dependence in the dimensionless observable R is moved into α_s. In Eq.(73.22) we also see that this series in c_n will never lead to a scale-invariant result when we include a finite order in perturbation theory.

Some higher-order factors c_n are known, for example inserting $N_c = 3$ and five quark flavors just as we assume in Eq.(73.17)

$$R = \frac{11}{3} \; \left(1 + \frac{\alpha_s}{\pi} + 1.4 \left(\frac{\alpha_s}{\pi} \right)^2 - 12 \left(\frac{\alpha_s}{\pi} \right)^3 + \mathcal{O} \left(\frac{\alpha_s}{\pi} \right)^4 \right) \; . \tag{73.23}$$

This alternating series with increasing perturbative prefactors seems to indicate the asymptotic instead of convergent behavior of perturbative QCD. At the bottom mass scale the relevant coupling factor is only $\alpha_s(m_b^2)/\pi \sim 1/14$, so a further increase of the c_n would become dangerous. However, a detailed look into the calculation shows that the dominant contributions to c_n arise from the analytic continuation of logarithms, *i.e.* large finite terms for example from $\text{Re}(\log^2(-E^2)) = \log^2 E^2 + \pi^2$. In the literature such π^2 terms arising from the analytic continuation of loop integrals are often phrased in terms of $\zeta_2 = \pi^2/6$.