HELAS: HELicity Amplitude Subroutines for Feynman diagram evaluations

K. Hagiwara[1], J. Kanzaki[1], H. Murayama[2] and I. Watanabe[3]

Version 3.0 (1 Jan 2000)

[1] KEK, Tsukuba, Ibaraki 305-0801, Japan
[2] Department of Physics, University of California
Berkeley, CA 94720, USA
[3] Akita Keizaihoka University Junior College
Akita, Japan

Contents

Chapter 0

Introduction to HELAS ver. 3.0

Chapter 1

Introduction

HELAS [1] [2] (HELicity Amplitudes Subroutines) is a set of FORTRAN77 subroutines which enable us to compute the helicity amplitudes of an arbitrary tree-level Feynman diagram with a simple sequence of CALL SUBROUTINE statements.

It is easy to write down a FORTRAN program to calculate the helicity amplitudes of a given process by calling HELAS subroutines. For instance, the helicity amplitudes of the process $W^+W^- \to t\bar{t}$ can be evaluated by the following program with just 11 lines. First, the two incoming (W^+ and W^-) and the two outgoing (t and $\bar{t}$) particle wavefunctions are calculated by calling the following 4 subroutines:

```
CALL VXXXXX(PWM,WMASS,NHWM,-1 , WM)
CALL VXXXXX(PWP,WMASS,NHWP,-1 , WP)
CALL OXXXXX(PT ,TMASS,NHT ,+1 , FO)
CALL IXXXXX(PTB,TMASS,NHTB,-1 , FI)
```

Second, the 4 Feynman diagrams of Fig. 1 (see page 9) are calculated with the following 6 lines:

```
CALL J3XXXX(FI,FO,GAU,GZU,ZMASS,ZWIDTH , J3)
CALL VVVXXX(WP,WM,J3,GW , AMPS)
CALL FVIXXX(FI,WM,GWF,0.,0. , FVI)
CALL IOVXXX(FVI,FO,WP,GWF , AMPT)
CALL HIOXXX(FI,FO,GCHT,HMASS,HWIDTH , HTT)
CALL VVSXXX(WM,WP,HTT,GWWH , AMPH)
```

Finally, the helicity amplitudes are obtained by adding the above sub-amplitudes

```
AMP = AMPS + AMPT + AMPH.
```

The meaning of each line will become clear in the next chapter (Sect. 2.3).

Even the program to compute the helicity amplitudes of the process $e^-e^+ \to e^-\bar{\nu}_e W^+ Z$, which has 80 Feynman diagrams, has only 65 lines of CALL sentences (see Appendix B.6 for a sample program). This compactness of the helicity amplitude programs is the main advantage of using HELAS.

Another advantage of the HELAS system is that it is very easy to allow external heavy particles to decay into light quarks and leptons without loosing the spin correlation. This is achieved simply by replacing the relevant external wavefunction subroutine by a sequence

[1] This name has nothing to do with the ancient Greek '$E\lambda\lambda\alpha\varsigma$' since Greek is completely Greek to us.

[2] [helású] means *to decrease* in Japanese. Does HELAS decrease your tasks?

of HELAS subroutines describing the decay chain. Then the above program can be used as a kernel for all branching processes without modification. For instance, it is straightforward to extend the above program to calculate the amplitude for the sequential process $W^+W^- \to t\bar{t}$; $t \to bW^+; \bar{t} \to \bar{b}W^-; W^+ \to u\bar{d}; W^- \to \tau^-\bar{\nu}_\tau; \tau^- \to \nu_\tau a_1; a_1 \to \pi\pi\pi$. Correlation among the 9 external particles ($b, \bar{b}, u, \bar{d}, \nu_\tau, \bar{\tau}_\tau$, and three π's) is automatically kept at all kinematical configurations. Similarly, by replacing an initial state wavefunction by a sequence of subroutines describing an initial state splitting, the program can calculate the essential part of the fusion process $e^+e^- \to \nu_e\bar{\nu}_e t\bar{t}$.

The cross sections are then calculated by linking the above amplitude program with HELAS and a main program which generates the four-momenta and helicities of external particles. For each phase space point and for each helicity combination, the amplitude program calculates the helicity amplitude by calling HELAS. The main program should then square the amplitude and sum over phase space and all possible helicity combinations. For this purpose, you can use your favorite Monte Carlo integration program such as VEGAS [1] or BASES [2].

To summarize, you can calculate cross sections of an arbitrary process with the help of HELAS as follows;

- Draw the Feynman diagrams contributing to the process.
- Write the amplitude program as a sequence of `CALL HELAS-Subroutine` lines.
- Prepare the four-momenta and helicities of the external particles in a given reference frame.
- Calculate the amplitude, square it, and sum over phase space and helicities using a standard integration program.

That's all you have to do. The following characteristics of the HELAS system are worth noting.

1. By using the standard HELAS subroutines only, you can calculate arbitrary tree-level helicity amplitudes of the standard model.

   In fact, you can calculate arbitrary tree-level of helicity amplitudes of arbitrary renormalizable quantum field theory by using HELAS*. The SM (Standard Model) and the MSSM (Minimal Supersymmetric Standard Model) are the two most commonly used examples.*

2. In order to minimize the number of Feynman diagrams, the weak boson propagators are given in the unitary gauge form. Numerical accuracy of the program is found to be good up to a few TeV in the weak boson pair invariant mass (see Sect. 2.5).

   In fact, no improvement in the numerical accuracy is found by using the renormalizable t'Hooft-Feynman gauge. The gauge invariance of the HELAS *amplitudes should be tested by using the BRS-invariance, which is the global invariance of the quantum gauge field theory after gauge-fixing (see Sec.x.x). The BRS-invariance test of the amplitudes are found far superior (non-trivial) than the test due to the gauge-fixing parameter independence.*

3. The original HELAS subroutines used single precision manipulation in order to make the program run fast.

   HELAS *ver. 3.0 uses double precision numbers.*

The double precision version `DHELAS`(written by J. Kanzaki) was used for testing the numerical accuracy.

> `DHELAS` *is the bases of* `HELAS` *ver. 3.0. The numerical accuracy of the program has been tested for the weak boson scattering process by comparing with the analytic result.*

4. No effort has been made to develop efficient subroutines for purely gluonic vertices. 3-and 4-gluon vertices are handled merely as a special case of the 3- and 4-weak boson vertices. The QCD gauge coupling constant and the color factors should be supplied by the user (see Sect. 2.9).

> `HELAS` *ver. 3.0 has subroutines for* `GGG` *and* `GGGG` *vertices.*

5. It is relatively easy to add a user-made subroutine for the non-standard vertices expected in many extensions of the standard model and also in the loop-level corrections (see Sect. 2.9).

> `HELAS` *ver. 3.0 contains three examples of* `HELAS` *subroutines for non-renormalizable vertices. Magnetic and electric dipole moments of massive fermions (Sec.2.5.1), Goldstino, or very light Gravitino couplings to supersymmetric pairs of fermions and vector bosons (Sec.2.5.2) and the masive tensor boson couplings to the SM particles (Sec.2.5.3).*

If you want to learn how to use `HELAS` as quickly as possible, then we advise you to read only Chap. 2, and after that try to use the example programs. This will make you familiar with most of the `HELAS` subroutines. We will present some examples of the typical use of `HELAS` and they will be sufficient for you to start using the `HELAS` subroutines. Example programs and many more programs for the SM and MSSM amplitudes can be generated instantly by using the automatic Feynman amplitude generator MadGraph[x] (http://).

In Chap. 3, each subroutine in `HELAS` are explained separately in much more detail. We believe that you don't have to read the chapter until you wish to modify `HELAS` to include non-standard couplings and new particles.

You may worry whether your program written with `HELAS` is correct or not. The programs of `HELAS_CHECK` have been made to help you find bugs. Its use is just the same as `HELAS`, except that you should *link* `HELAS_CHECK` instead of `HELAS` when you compile the program. Then `HELAS_CHECK` will test the consistency of all the inputs of the subroutines. If there appears to be something wrong with the inputs, `HELAS_CHECK` will tell you either `HELAS-ERROR` or `HELAS-warn`. They will be displayed on your terminal, and you can find their meanings in Chap. 4.

Most importantly, by using the `HELAS_CHECK` programs you can perform the BRS-invariance test of the helicity amplitudes, because the vector boson subroutines in `HELAS_CHECK` accept the polarization

```
NHEL = 4
```

for which the corresponding wave function is proportional to the vector-boson four momentum (see Secẋ.x).

We have summarized our conventions for spinors, polarization vectors and coupling constants in Appendix A. Various example programs are presented in Appendix B, in which brief comments are given on what you should remember when programming amplitudes with `HELAS`. Appendix C contains a list of `HELAS`subroutines, `HELAS.LIST1` and `HELAS.LIST2`. Once you have read through Chap. 2, then all you need is `HELAS.LIST1`, which contains the list, as well as brief descriptions, of the inputs and outputs of each `HELAS`subroutine. When you become accustomed to `HELAS`, a much shorter list, `HELAS.LIST2`, will suffice when coding programs.

The subroutine packages `HELAS.FOR` and `HELAS_CHECK.FOR`, together with `HELAS.LIST1`, `HELAS.LIST2` and the example programs that appear in Appendix B, are available on request from the authors (Bitnet address: murayama@jpnkekvx).

Acknowledgments for the original HELAS Manual (KEK Report 91-11):

We would like to thank B.K. Bullock, H. Iwasaki, J. Kanzaki, A. Miyamoto, D. Zeppenfeld, and the members of the JLC (Japan Linear Collider) physics working group for collaborations that contributed to the present form of `HELAS`. We also thank Ben Bullock and Neil McDougall for many useful comments on the manuscript.

Chapter 2

How to use HELAS

In this chapter, we will describe the use of the HELAS package, using the process

$$W^+W^- \to t\bar{t} \tag{2.1}$$

as an example. We believe that reading this chapter is enough for learning the basic use of HELAS.

2.1 Basic Idea

Let us first discuss the general characteristics of tree-level diagrams. As can be seen from the word 'tree', they have a common structure. However there may be many external lines. As they approach the centre of Feynman diagrams, the external lines meet to give an off-shell internal line, and then meet again to make another internal line, until all the lines meet at a single point.

The basic idea of HELAS is to begin with the external lines by creating the wavefunctions explicitly using a fixed notation, and to give rules to join the lines. You may suppose that there are too many possibilities for a complete set of joining rules but in fact they can all be classified into a finite set in renormalizable theories. We will show all the possible rules of renormalizable theories in Table 2.1.

Thus the package basically consists of two parts: wavefunctions and vertices. Then the amplitudes can be computed as follows; first, the external wave functions are evaluated as functions of the particle momenta and helicities. Second, off-shell scalar/spinor/vector (S/F/V) lines obtained from the external lines via renormalizable vertices are evaluated as functions of the external wave functions. This second step can be repeated, giving internal off-shell lines as functions of external off-shell lines, until all the off-shell lines meet. We will first present the list of necessary subroutines, and then discuss their application.

2.2 List of Subroutines

In this section, we will explain what kind of subroutines are provided in the HELAS package. The whole package can be divided into three areas;

- External lines
- Vertices
- Utilities

Table 2.1: List of the possible vertices in renormalizable theories. All these vertices are incorporated in the HELAS system.

Vertex	interaction
FFV	vector or axial vector couplings
FFS	Yukawa couplings
VVV	Yang-Mills couplings
VVS	Higgs interaction
SSV	scalar gauge couplings
SSS	scalar self-couplings
VVVV	Yang-Mills couplings
VVSS	scalar gauge couplings (seagull)
SSSS	scalar self-couplings

Table 2.2: List of 'External Lines' Subroutines

External line	Subroutine
Flowing-In Fermion	IXXXXX
Flowing-Out Fermion	OXXXXX
Vector Boson	VXXXXX
Scalar Boson	SXXXXX

The external lines are computed with the subroutines IXXXXX, OXXXXX, VXXXXX and SXXXXX from input four-momenta and helicities.

As discussed in the first section, the number of possible renormalizable vertices in quantum field theory is finite. For each type of vertex, we can obtain either one off-shell internal line, or an amplitude. Detailed description of each subroutine is found in Chapter 3.

The special subroutines provided to compute the singularities from electron-photon coupling are EAIXXX, EAOXXX, JEEXXX.

There are also some utility subroutines. Most of them are for dealing with phase space variables and kinematics. These can compute four-momenta from angles, and can also rotate or boost four-momenta. In Appendix B.5, we demonstrate their use in the calculation of $e^-e^+ \to e^-\bar{\nu}_e W^+$.

The rest of the utilities are coupling subroutines for the Standard Model. The outputs of these subroutines should be regarded as templates for defining the couplings appropriate for the HELAS subroutines. COUP1X computes couplings among gauge bosons, COUP2X gauge couplings of fermions, COUP3X the Higgs and gauge boson couplings, and COUP4X computes Higgs couplings to fermions. These subroutines provide all possible coupling constants in the Standard Model except for modification by Kobayashi-Maskawa matrix elements, which should be multiplied with the amplitudes 'by hand' outside the subroutines.

2.3 Example: $W^+W^- \to t\bar{t}$

To show how HELAS subroutines are used, we will present part of an example program in this section. The complete example program will be presented later.

The Feynman diagrams of the process $W^+W^- \to t\bar{t}$ are presented in Fig. 1. There are four Feynman diagrams. The first diagram, with t-channel bottom quark exchange, will be referred to as the t-channel diagram. The next two diagrams with s-channel photon and Z exchange,

Table 2.3: List of the vertex subroutines in HELAS system.

Vertex	Inputs	Output	Subroutine
FFV	FFV	Amplitude	IOVXXX
	FF	V	JIOXXX
	FV	F	FVIXXX, FVOXXX
FFS	FFS	Amplitude	IOSXXX
	FF	S	HIOXXX
	FS	F	FSIXXX, FSOXXX
VVV	VVV	Amplitude	VVVXXX
	VV	V	JVVXXX
	GGG	Amplitude	GGGXXX
	GG	G	JGGXXX
VVS	VVS	Amplitude	VVSXXX
	VS	V	JVSXXX
	VV	S	HVVXXX
VSS	VSS	Amplitude	VSSXXX
	SS	V	JSSXXX
	VS	S	HVSXXX
SSS	SSS	Amplitude	SSSXXX
	SS	S	HSSXXX
VVVV	VVVV	Amplitude	WWWWXX, W3W3XX
	VVV	V	JWWWXX, JW3WXX
	GGGG	Amplitude	GGGGXX
	GGG	G	JGGGXX
VVSS	VVSS	Amplitude	VVSSXX
	VSS	V	JVSSXX
	VVS	S	HVVSXX
SSSS	SSSS	Amplitude	SSSSXX
	SSS	S	HSSSXX

Table 2.4: List of 'Utility' Subroutines

Utilities for Momentum Manipulations:

p^μ(energy,mass,costh,phi)	: set up momentum	MOMNTX
p_1^μ & p_2^μ	: set up two momenta in c.m. frame	MOM2CX
p_1^μ & p_2^μ	: set up two momenta in lab. frame	MOM2LX
p_{boosted}	: Lorentz boost of momentum	BOOSTX
p_{rotated}	: rotation of momentum	ROTXXX

Standard Model Coupling Constants:

for VVV,VVVV vertices	COUP1X
for FFV vertices	COUP2X
for VVS,SSS,VVSS,SSSS vertices	COUP3X
for FFS vertices	COUP3X
for all the SM vertices	COUPSM

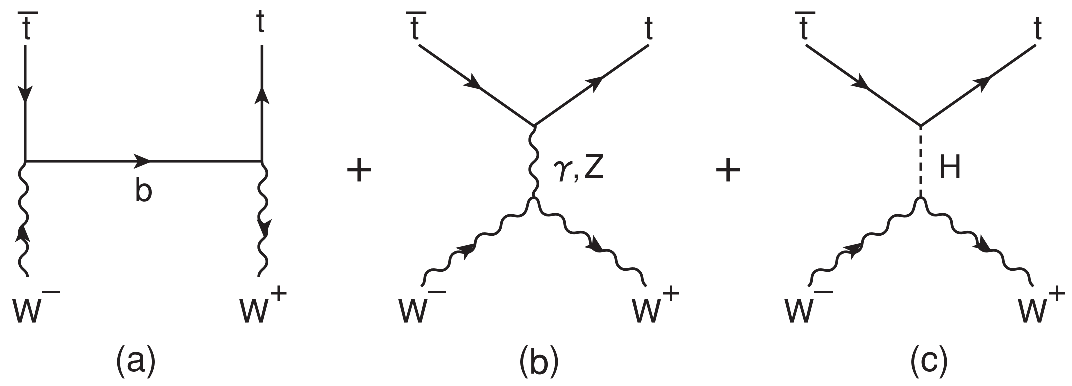


Figure 1: Feynman diagrams for the process $W^+W^- \to t\bar{t}$.

will be called s-channel diagrams. We will call the last diagram with s-channel Higgs boson exchange the Higgs diagram. All these diagrams can be computed with just the following `CALL` sequences.

As mentioned in the previous section, we start from the *external lines*. We call the following subroutines to compute the external wavefunctions of the initial W^-, W^+, and final t, $\bar{t}$.

```
CALL VXXXXX(PWM,WMASS,NHWM,-1 , WM)
CALL VXXXXX(PWP,WMASS,NHWP,-1 , WP)
```

and

```
CALL OXXXXX(PT ,TMASS,NHT ,+1 , FO)
CALL IXXXXX(PTB,TMASS,NHTB,-1 , FI)
```

The subroutine `VXXXXX` computes the wavefunction of a vector boson (polarization vector), `IXXXXX` the flowing-`In` spinor (u- or v-spinor), and `OXXXXX` the flowing-`Out` spinor ($\bar{u}$- or $\bar{v}$-spinor). Here, the inputs are the four-momenta of the external particles `PWM(0:3)`, `PWP(0:3)`, `PT(0:3)`, `PTB(0:3)` of W^- (`WM`inus), W^+ (`WP`lus), t (`T`op) and $\bar{t}$ (`TB`ar), respectively. `WMASS` and `TMASS` are the masses of W and t. `NHWM, NHWP, NHT, NHTB` are their helicities. Note that helicities $\pm 1/2$ of fermions will be referred to as ± 1. The final input requires some explanation. For vector bosons and scalar bosons, `+1` means that they are *flowing out* from the Feynman diagram (hence final state particles), and `-1` the contrary.[1] For fermions, `+1` means that it is a *particle* (u- or $\bar{u}$-spinor) while `-1` means an *anti-particle* (v- or $\bar{v}$-spinor). The wavefunctions will be contained in the outputs `WM(6)`,`WP(6)`, `FO(6)`, `FI(6)`. Actually, the four-momenta of the external lines are also contained in these output arrays.

The t-channel diagram Fig. 1(a) is very simple.

```
CALL FVIXXX(FI,WM,GWF,0.,0. , FVI)
CALL IOVXXX(FVI,FO,WP,GWF , AMPT)
```

The subroutine `FVIXXX` computes the internal fermion lines from the flowing-`In` fermion `FI` (final $\bar{t}$) and a `V`ector boson `WM` (initial W^-). `GWF` is the charged current coupling universally to all fermions. The next two inputs are the mass and width of the off-shell bottom quark, which we are neglecting. Then the output off-shell spinor `FVI` is combined with the flowing-`Out` spinor `FO` (final t) and vector boson `WP` (initial W^+) to obtain the T-matrix element `AMPT`, by using the 'amplitude' subroutine `IOVXXX`. You can find exactly the same amplitude by first combining W^+ with t;

[1]We use the terms *flowing-in* and *flowing-out* as seen from the vertex. On the other hand, *incoming* particles are the initial states while *outgoing* particles are the final states.

Table 2.5: An example HELAS program to compute the amplitude $W^-W^+ \to t\bar{t}$.

```
C
C The initial state wavefunction of the W's:
C
      CALL VXXXXX(PWM,WMASS,NHWM,-1 , WM)
      CALL VXXXXX(PWP,WMASS,NHWP,-1 , WP)
C
C The final state wavefunction of top and tbar.
C
      CALL OXXXXX(PT ,TMASS,NHT ,+1 , FO)
      CALL IXXXXX(PTB,TMASS,NHTB,-1 , FI)
C
C First, we compute the t-channel bottom exchange diagram.
C
      CALL FVIXXX(FI,WM,GWF,0.,0. , FVI)
      CALL IOVXXX(FVI,FO,WP,GWF , AMPT)
C
C Next we compute the s-channel Z, photon exchange diagram.
C
      CALL JXXXXX(FI,FO,GAU,0.,0. , JA)
      CALL VVVXXX(WP,WM,JA,GWWA , AMPA)
      CALL JXXXXX(FI,FO,GZU,ZMASS,ZWIDTH , JZ)
      CALL VVVXXX(WP,WM,JZ,GWWZ , AMPZ)
C
C Finally we compute the s-channel Higgs exchange diagram.
C
      CALL HIOXXX(FI,FO,GCHT,HMASS,HWIDTH , HTT)
      CALL VVSXXX(WM,WP,HTT,GWWH , AMPH)
C
```

```
CALL FVOXXX(FO,WP,GWF,0.,0. , FVO)
CALL IOVXXX(FI,FVO,WM,GWF , AMPT)
```

which gives the same answer.

The s-channel diagram Fig. 1(b) is computed as follows

```
CALL J3XXXX(FI,FO,GAU,GZU,ZMASS,ZWIDTH , J3)
CALL VVVXXX(WP,WM,J3,GW , AMPS)
```

The subroutine `J3XXXX` computes the weighted sum of the photon and Z currents emerging from fermion lines. The subroutine `VVVXXX` computes the Feynman amplitudes from three vector bosons. `FI, FO, WM, WP` are the external wavefunctions obtained above. You can already see here the general rules of using `HELAS`. You combine several external lines (here, `FI, FO`) to obtain internal lines (here, `J3`, off-shell photon and Z current) including their propagators, and the computed internal lines can be used again to compute further internal lines (not necessary here). Finally, you take several external or internal lines to compute the T-matrix elements (here, `AMPS`).

The use of `J3XXXX` requires more explanation. The current from the top quark line here will couple to W-bosons with the gauge boson non-abelian vertex, hence only the combination

$$W^3_\mu = A_\mu \sin\theta_W + Z_\mu \cos\theta_W \tag{2.2}$$

is relevant. `J3XXXX` automatically takes this combination. If you want to do this in a different way,

```
CALL JIOXXX(FI,FO,GZU,ZMASS,ZWIDTH , JZ)
CALL JIOXXX(FI,FO,GAU,0.   ,0.     , JA)
CALL VVVXXX(WP,WM,JZ,GWWZ , AMPSZ)
CALL VVVXXX(WP,WM,JA,GWWA , AMPSA)
AMPS = AMPSZ + AMPSA
```

will do exactly the same job. The subroutine `JIOXXX` computes any `J` (vector current) from flowing-`In` and flowing-`Out` fermions. However, if you don't need to compute the Z- and photon-exchange amplitudes separately, it is always better to use `J3XXXX` since the cancellation between Z- and photon-exchange is done analytically in `J3XXXX` when possible, which makes `J3XXXX` better numerically than computing the Z- and photon-exchange amplitudes separately and adding them afterwards.

As you may have guessed, `GZU` and `GAU` are coupling constants of `U`p-type quarks to the `Z` (Z-boson) and `A` (photon). `GWWZ` and `GWWA` are the coupling constants of `W` (W-boson) with `Z` and `A`. The ordering of the three input vectors in `VVVXXX` is essential to fix the sign of the amplitude, due to the anti-symmetry of the structure constants of the $SU(2)$ Lie group. It should be W^-, W^+, W^3 for *flowing-out* vectors; see subsection 3.4.1 in the next chapter for details. In this case, `WP` is an incoming W^+ boson, or equivalently, it is a *flowing-out* W^- boson as seen from the vertex. Thus, `WP` takes the first position in the subroutine `VVVXXX`.

The Higgs diagram Fig. 1(c) is also very simple.

```
CALL HIOXXX(FI,FO,GCHT,HMASS,HWIDTH , HTT)
CALL VVSXXX(WP,WM,HTT,GWWH , AMPH)
```

The subroutine `HIOXXX` computes the off-shell scalar wave function (generically denoted `H`), analogous to `JIOXXX` for the vector case. The output off-shell scalar is combined with initial W^-, W^+ using `VVSXXX` to obtain the T-matrix element `AMPH`. Or, we can combine the vector bosons first, as

```
CALL HVVXXX(WP,WM,GWWH,HMASS,HWIDTH , HWW)
CALL IOSXXX(FI,FO,HWW,GCHT , AMPH)
```

The ordering of the two vectors in `VVSXXX` and `HVVXXX` is arbitrary. Now the calculation of all the diagrams is complete, and the helicity amplitude is simply the sum `AMPT + AMPS + AMPH`.

In the above, we showed two different ways to calculate the s-channel amplitude `AMPS`. It is instructive to see that you can easily find yet another expression to obtain the same amplitude.

```
CALL JVVXXX(WP,WM,GWWZ,ZMASS,ZWIDTH , JZWW)
CALL JVVXXX(WP,WM,GWWA,0.    ,0.     , JAWW)
CALL IOVXXX(FI,FO,JZWW,GZU , AMPSZ)
CALL IOVXXX(FI,FO,JAWW,GAU , AMPSA)
AMPS = AMPSZ + AMPSA
```

Here the subroutine `JVVXXX` computes the off-shell vector currents from the two vector bosons.

The coupling constants which appeared in this section are computed by `CALL`ing the `COUP` subroutines.

```
CALL COUP1X(SW2 , GW,GWWA,GWWZ)
CALL COUP2X(SW2 , GAL,GAU,GAD,GWF,GZN,GZL,GZU,GZD,G1)
CALL COUP3X(SW2,ZMASS,HMASS , GWWH,GZZH,GHHH,GWWHH,GZZHH,GHHHH)
CALL COUP4X(SW2,ZMASS,TMASS , GCHT)
```

The inputs required here are `SW2` ($\sin^2\theta_W$), `ZMASS, HMASS, TMASS`. The weak-scale fine-structure constant $\alpha = 1/128$ is built-in. If you need to use $\alpha = 1/137$ instead, you modify this by multiplying the final amplitude with the appropriate power of 128/137.

As we have shown with this simple (but non-trivial) example, there are always many ways to compute one T-matrix element. This provides a useful check of the numerical accuracy of `HELAS`. For most electroweak processes, the subroutines we described in this example should be sufficient.

One point should be noted concerning the `VVVV` vertex. In any process which possesses the four-vector-boson vertex, there also appear two of the t-, s- or u-channel vector boson exchange diagrams as well. Furthermore, it is well known that there is a cancellation between those diagrams. To make the numerical accuracy better even in the presence of such a cancellation, we combined all the $VV \to VV$ diagrams into a single subroutine, which is computed using double precision internally. By combining three diagrams into a single subroutine, the `HELAS` program is made more compact. For example, two-photon production of a W-pair can be computed with a single subroutine `W3W3XX`,

```
CALL VXXXXX(PWM,WMASS,NHWM,+1 , WM)
CALL VXXXXX(PWP,WMASS,NHWP,+1 , WP)
CALL VXXXXX(PA1,0.    ,NHA1,-1 , A1)
CALL VXXXXX(PA2,0.    ,NHA2,-1 , A2)
CALL W3W3XX(WM,A1,WP,A2,GWWA,GWWA,WMASS,WWIDTH , AMP)
```

That's all. Here, first four `VXXXXX`'s compute the wavefunctions of `WMinus`, `WPlus`, photon #1 (`A1`) and photon #2 (`A2`) from their four-momenta (`PWM`, `PWP`, `PA1`, `PA2`) and their helicities (`NHWM`, `NHWP`, `NHA1`, `NHA2`). Then `W3W3XX` computes their four-point scattering amplitude, including t- and u-channel W exchange. Further examples are given in Appendix B.

2.4 Collinear Singularities

Though the subroutines we described in the last section may be sufficient for most purposes, we may also encounter collinear photon singularities. Since such singularities appear mostly for the electron-photon couplings with an electron initial state, it is useful to have a numerically safe expression for the collinear photon or electron emerging from the initial electron beam. Even though the quantitative predictions for such a configuration require careful treatment of the

QED radiative effects, we find it very useful to be able to compute accurately exact tree-level amplitudes with such singularities.

The simplest process with such a singularity is $e^-e^+ \to Z\gamma$, where the t-channel electron propagator produces a singularity. We provide special subroutines EAIXXX, EAOXXX for this purpose. By using the usual HELAS subroutines FVIXXX and FVOXXX described in the previous section, you can evaluate the amplitudes by

```
C External lines
      CALL IXXXXX(PEM,EMASS,NHEM,+1 , FI)
      CALL OXXXXX(PEP,EMASS,NHEP,-1 , FO)
      CALL VXXXXX(PZ ,ZMASS,NHZ ,+1 , Z )
      CALL VXXXXX(PA ,0.   ,NHA ,+1 , A )
C 1st amplitude
      CALL FVIXXX(FI,A,GAL,EMASS,0. , FVI)
      CALL IOVXXX(FVI,FO,Z,GZL , AMP1)
C 2nd amplitude
      CALL FVOXXX(FO,A,GAL,EMASS,0. , FVO)
      CALL IOVXXX(FI,FVO,Z,GAL , AMP2)
```

However, this program will suffer from numerical errors when computing the electron propagators in FVIXXX, FVOXXX. (See Appendix A.4 for a detailed explanation of the numerical problem associated with the collinear configurations.) This problem is resolved by using the special subroutines EAIXXX, EAOXXX for the particular γee vertex. You can now compute the above amplitude as follows:

```
C External  lines
      CALL VXXXXX(PZ,ZMASS,NHZ,+1 , Z)
C 1st amplitude
      CALL EAIXXX(EB,EA,SHLF,CHLF,PHI,NHEM,NHA , EAI)
      CALL IOVXXX(EAI,FO,Z,GZL , AMP1)
C 2nd amplitude
      CALL EAOXXX(EB,EA,SHLF,CHLF,PHI,NHEP,NHA , EAO)
      CALL IOVXXX(FI,EAO,Z,GZL , AMP2)
```

The subroutine EAIXXX computes the off-shell electron line from an initial electron with four-momentum EB$(1, 0, 0, \beta)$ (EB is the Energy of the Beam), by emitting a photon with energy EA at polar angle θ. The inputs SHLF, CHLF are $\sin(\theta/2)$ and $\cos(\theta/2)$ respectively, which are found to be useful in efficiently evaluating the electron propagator factor to the necessary accuracy. EAOXXX does the same job from the initial positron with four-momentum EB$(1, 0, 0, -\beta)$.

A similar singularity occurs for off-shell photons emitted from the initial electron or positron line. If you use the standard HELAS subroutine JIOXXX for such a configuration, then you will encounter a severe numerical problem because of the subtle gauge theory cancellation. In order to compute accurately the amplitude with nearly on-shell photon propagators, we provide you with another special subroutine JEEXXX. In this subroutine, the output current is modified to avoid this subtle gauge cancellation by shifting a term proportional to its four-momentum q^μ :

$$J_A^\mu(\langle e'|, |e\rangle) \to J_A^\mu - \frac{J^0}{q^0} q^\mu. \tag{2.3}$$

We note here that it is nontrivial to prove that the above modification leaves the helicity amplitudes invariant in the electroweak theory, especially when two incoming off-shell photon lines are replaced by the above shifted currents. (We give a proof [3] based on the BRS invariance [4] of the electroweak theory in Appendix A.4.) The subroutine JEEXXX can be used in much the same way as the other special subroutines EAIXXX, EAOXXX. (We give an example of its use in Appendix B.4, where we show an example program for calculating the process $e^-e^+ \to \nu_e e^+ W^-$.)

2.5 Numerical Accuracy

The numerals used in HELAS are INTEGER*4, REAL*8 and COMPLEX*16. The output amplitudes or currents are given in the double precision complex number COMPLEX*16 (or its array).

The HELAS program has been coded with a great deal of care, in order that it does not lose numerical accuracy. For example, in getting SINTH (= $\sin\theta$) from COSTH (= $\cos\theta$), we adopted

```
SINTH = SQRT((1.0-COSTH)*(1.0+COSTH))
```

rather than

```
SINTH = SQRT(1.0-COSTH**2)
```

which is less accurate at |COSTH| $\sim$ 1.0. Furthermore, the gauge cancellations and collinear singularities have been prudently treated in VVVXXX [2] and EAIXXX, EAOXXX, JEEXXX subroutines; see Chapter 3 and Appendix A for the details.

The numerical accuracy of the HELAS outputs is evaluated in the process ${\rm e_L}^-{\rm e_R}^+ \to {\rm W_L}^-{\rm W_L}^+$. There are two diagrams in this process, i.e. t-channel by ν_e exchange and s-channel with $\gamma/{\rm Z}$ current. Each amplitude is given as

$$\mathcal{M}_{\rm t} = -\frac{g^2}{4}\frac{s}{m_{\rm W}{}^2}\sin\theta\frac{2\cos\theta-3\beta+\beta^3}{1+\beta^2-2\beta\cos\theta}, \tag{2.4}$$

$$\mathcal{M}_{\rm s} = -\frac{g^2}{4}\frac{s}{m_{\rm W}{}^2}\sin\theta\beta(3-\beta^2)\left(\sin^2\theta_{\rm W}+\frac{1/2-\sin^2\theta_{\rm W}}{1-m_{\rm Z}{}^2/s}\right), \tag{2.5}$$

where $\sin\theta$ and β are the scattering angle and the velocity of the ${\rm W}^-$. Note here that both amplitudes are proportional to $s/m_{\rm W}{}^2$ in the high energy limit. However, the sum of above two behaves as a constant asymptotically, due to the gauge cancellation between the two diagrams.

$$\begin{aligned}\mathcal{M}_{\rm tot} &= \mathcal{M}_{\rm t}+\mathcal{M}_{\rm s} \\ &= \frac{g^2}{4}\frac{s}{s-m_{\rm Z}{}^2}\frac{\sin\theta}{1+\beta^2-2\beta\cos\theta}\Big[2\beta(3-\beta^2)-4(2-\beta^2)\cos\theta \\ &\quad -\frac{m_{\rm Z}{}^2}{m_{\rm W}{}^2}\Big\{(3\beta-\beta^3-2\cos\theta)-\sin^2\theta_{\rm W}\beta(3-\beta^2)(1+\beta^2-2\beta\cos\theta)\Big\}\Big].\end{aligned} \tag{2.6}$$

Thus, there is a cancellation of order $O(s/m_{\rm W}{}^2)$ which is about 10^2 at $\sqrt{s}=1$ TeV, and 10^6 at $\sqrt{s}=100$ TeV. Since HELAS computes amplitudes diagram by diagram, it is impossible to make it do efficiently the gauge cancellation between diagrams of different structure. Thus, the HELAS output may not reproduce the required cancellation in the above process. We first study the numerical accuracy of the HELAS output by comparing it with the above analytic formula for the process

$${\rm e_L}^- + {\rm e_R}^+ \to {\rm W_L}^- + {\rm W_L}^+. \tag{2.7}$$

The program which produces the momenta and the amplitude can be written as follows.

```
C  momenta of beam  e- and e+
      CALL MOM2CX(ROOTS,EMASS,EMASS, +1.0,0.0 , PEM,PEP)
C
C  momenta of final W- and W+
      CALL MOM2CX(ROOTS,WMASS,WMASS,COSTH,PHI , PWM,PWP)
C
```

[2] We do *not* take special care with the JVVXXX subroutine.

$\sqrt{s} \setminus \cos\theta$	-0.9	-0.5	$0.$	0.5	0.9
0.2 TeV	6.0×10^{-16}	3.5×10^{-16}	4.8×10^{-16}	5.9×10^{-15}	8.0×10^{-16}
1 TeV	2.6×10^{-14}	4.5×10^{-16}	3.4×10^{-14}	1.4×10^{-14}	5.8×10^{-14}
10^2 TeV	3.8×10^{-11}	1.7×10^{-10}	1.4×10^{-10}	4.9×10^{-10}	6.2×10^{-10}
10^4 TeV	4.5×10^{-6}	2.3×10^{-6}	2.9×10^{-6}	2.3×10^{-6}	2.5×10^{-5}
10^6 TeV	4.6×10^{-3}	3.8×10^{-2}	2.9×10^{-3}	5.0×10^{-3}	1.2×10^{-2}

Table 2.6: Relative errors of the total amplitude of the process $\mathrm{e_L}^-\mathrm{e_R}^+ \to \mathrm{W_L}^-\mathrm{W_L}^+$, $\mathcal{M}_{\rm tot}$ calculated by HELAS are shown for some c.m. scattering angles.

```
C  external e- e+ wavefunctions
      CALL IXXXXX(PEM,EMASS,-1,+1 , FI)
      CALL OXXXXX(PEP,EMASS,+1,-1 , FO)
C
C  external W- W+ polarization vectors
      CALL VXXXXX(PWM,WMASS, 0,+1 , WM)
      CALL VXXXXX(PWP,WMASS, 0,+1 , WP)
C
C  t-channel diagram
      CALL FVIXXX(FI,WM,GWF,0.D0,0.D0 , FVI)
      CALL IOVXXX(FVI,FO,WP,GWF , AMPT)
C
C  s-channel diagram
      CALL JIOXXX(FI,FO,GAL,0.D0 ,0.D0    , JA)
      CALL JIOXXX(FI,FO,GAL,ZMASS,ZWIDTH , JZ)
      CALL VVVXXX(WM,WP,JA,GWWA , AMPSA)
      CALL VVVXXX(WM,WP,JZ,GWWZ , AMPSZ)

      AMPS = AMPSA + AMPSZ
C
C  sum of two diagrams
      AMPL = AMPT + AMPS
```

The SM coupling constant subroutines COUPSM must be called before the above program. The output of this program is then compared to the analytic formulae (2.4), (2.5) and (2.6) as functions of $\cos\theta$ at three energies $\sqrt{s} = 0.2$, 1, and 10TeV. The orders of the errors in the amplitudes as compared with the above analytic formulae averaged over many COSTH points are summarized in Table 2.6. You can see from this table that HELAS keeps an accuracy of the order 10^{-4} up to a 10^4 TeV for the gauge cancellation in this process.

The severest gauge theory cancellation in the $2 \to 2$ process is expected to take place in the longitudinally polarized weak boson processes

$$\begin{aligned} \mathrm{W_L}^+\mathrm{W_L}^- &\to \mathrm{W_L}^+\mathrm{W_L}^-, && (2.8)\\ &\to \mathrm{Z_L}\,\mathrm{Z_L}, && (2.9) \end{aligned}$$

at high energies when the Higgs boson is light. If the off-shell weak boson is in the unitary gauge, the helicity amplitude of the above processes receives three types of contributions: s- and t-channel γ/Z exchange amplitudes, $\mathcal{M}_{\rm s}$ and $\mathcal{M}_{\rm t}$,

$$\begin{aligned} \mathcal{M}_{\rm s} &= -g^2\Big[4\gamma^4\cos\theta + \gamma^2\cos\theta \\ &\quad + \frac{(s/4)\cos\theta}{s-m_{\rm Z}{}^2}\left[\sec^2\theta_{\rm W} - (3+x)\left(4 - x\tan^2\theta_{\rm W}\right)\right]\Big] \end{aligned}$$

$$
\begin{aligned}
\mathcal{M}_{\rm t} &= g^2\Bigg[\gamma^4\left(3-2\cos\theta-\cos^2\theta\right)-\frac{3\gamma^2}{2}\left(1-5\cos\theta\right) \\
&\quad +\frac{s/4}{t-m_{\rm Z}{}^2}\Bigg[-\frac{1}{2}\left(3+\cos\theta\right)\sec^2\theta_{\rm W}+\left(1-\cos\theta\right)\left(1+5\cos\theta\right) \\
&\quad +x\left[2\left(1+\cos\theta\right)+\frac{s}{2t}\left(\left(1+\cos\theta\right)\beta^2+2\cos^2\theta\right)\tan^2\theta_{\rm W}\right]\Bigg]\Bigg] \quad \left(x\equiv\gamma^{-2}\right)
\end{aligned}
$$

an amplitude of a contact four weak boson vertex, $\mathcal{M}_{\rm c}$,

$$
\mathcal{M}_{\rm c} = g^2\left[\gamma^4\left(-3+6\cos\theta+\cos^2\theta\right)+2\gamma^2\left(1-3\cos\theta\right)\right]
$$

and the sum of the s- and t-channel Higgs boson exchange amplitudes, $\mathcal{M}_{\rm h}$,

$$
\begin{aligned}
\mathcal{M}_{\rm h} &= g^2\Bigg[-\frac{\gamma^2}{2}\left(1+\cos\theta\right)+1-\frac{x}{4}+\frac{s}{4t}\left(\sin^2\theta-x\right) \\
&\quad -\frac{1}{16m_{\rm W}{}^2}\left(\left(2-x\right)^2\frac{sm_{\rm H}{}^2}{s-m_{\rm H}{}^2}+\left(\beta^2-\cos\theta\right)^2\frac{s^2m_{\rm H}{}^2}{t\left(t-M_H^2\right)}\right)\Bigg].
\end{aligned}
$$

In the high energy limit $E\gg m_W$, both the amplitudes $\mathcal{M}_{\rm s}+\mathcal{M}_{\rm t}+\mathcal{M}_{\rm c}$ and $\mathcal{M}_{\rm h}$ are known to behave as E^2

$$
\begin{aligned}
\mathcal{M}_{\rm s}+\mathcal{M}_{\rm t}+\mathcal{M}_{\rm c} &= g^2\Bigg[\frac{\gamma^2}{2}\left(1+\cos\theta\right)-\frac{(s/4)\cos\theta}{s-m_{\rm Z}{}^2}\left[\sec^2\theta_{\rm W}-\left(3+x\right)\left(4-x\tan^2\theta_{\rm W}\right)\right] \\
&\quad +\frac{s/4}{t-m_{\rm Z}{}^2}\Bigg[-\frac{1}{2}\left(3+\cos\theta\right)\sec^2\theta_{\rm W}+\left(1-\cos\theta\right)\left(1+5\cos\theta\right) \\
&\quad +x\left[2\left(1+\cos\theta\right)+\frac{s}{2t}\left(\left(1+\cos\theta\right)\beta^2+2\cos^2\theta\right)\tan^2\theta_{\rm W}\right]\Bigg]\Bigg]
\end{aligned}
$$

while their sum, $\mathcal{M}_{\rm tot}(\equiv\mathcal{M}_{\rm s}+\mathcal{M}_{\rm t}+\mathcal{M}_{\rm c}+\mathcal{M}_{\rm h})$ remains as a constant at $E\gg m_H$.

$$
\begin{aligned}
\mathcal{M}_{\rm tot} &= g^2\Bigg[1-\frac{x}{4}+\frac{s}{4t}\left(\sin^2\theta-x\right) \\
&\quad -\frac{(s/4)\cos\theta}{s-m_{\rm Z}{}^2}\left[\sec^2\theta_{\rm W}-\left(3+x\right)\left(4-x\tan^2\theta_{\rm W}\right)\right] \\
&\quad +\frac{s/4}{t-m_{\rm Z}{}^2}\Bigg[-\frac{1}{2}\left(3+\cos\theta\right)\sec^2\theta_{\rm W}+\left(1-\cos\theta\right)\left(1+5\cos\theta\right) \\
&\quad +x\left[2\left(1+\cos\theta\right)+\frac{s}{2t}\left(\left(1+\cos\theta\right)\beta^2+2\cos^2\theta\right)\tan^2\theta_{\rm W}\right]\Bigg] \\
&\quad -\frac{1}{16m_{\rm W}{}^2}\left(\left(2-x\right)^2\frac{sm_{\rm H}{}^2}{s-m_{\rm H}{}^2}+\left(\beta^2-\cos\theta\right)^2\frac{s^2m_{\rm H}{}^2}{t\left(t-M_H^2\right)}\right)\Bigg]
\end{aligned}
$$

Even worse, each amplitude ($\mathcal{M}_{\rm s}$, $\mathcal{M}_{\rm t}$, $\mathcal{M}_{\rm c}$) behaves as E^4 initially, and it is only because of the electroweak gauge invariance of all the vector boson couplings that the sum $\mathcal{M}_{\rm s}+\mathcal{M}_{\rm t}+\mathcal{M}_{\rm c}$ behaves as E^2. Therefore, we should expect cancellation between order $(E/m_{\rm W})^4$ terms in these processes.

We show in Tables 2.7, the magnitude of the amplitudes for the process (2.8); $\mathcal{M}_{\rm s}$, $\mathcal{M}_{\rm t}$, $\mathcal{M}_{\rm c}$, $\mathcal{M}_{\rm s}+\mathcal{M}_{\rm t}+\mathcal{M}_{\rm c}$, $\mathcal{M}_{\rm h}$ and $\mathcal{M}_{\rm tot}$. at $\sqrt{s}$'s from 0.2 TeV to 1000 TeV. In these tables, the relative errors of the amplitudes evaluated by HELAS and those obtained by above analytic formulae:

$$
\left|\frac{\mathcal{M}_{\rm HELAS}-\mathcal{M}_{\rm EXACT}}{\mathcal{M}_{\rm EXACT}}\right|, \tag{2.10}
$$

$\sqrt{s}$	$\mathcal{M}_{\rm s}$	$\mathcal{M}_{\rm t}$	$\mathcal{M}_{\rm c}$	$\mathcal{M}_{\rm s}+\mathcal{M}_{\rm t}+\mathcal{M}_{\rm c}$	Error
0.2 TeV	1.6	2.8×10^{-1}	-2.7	9.7×10^{-2}	4.9×10^{-15}
0.5 TeV	8.3×10^{1}	1.3×10^{2}	-2.1×10^{2}	1.0	6.9×10^{-15}
1 TeV	1.3×10^{3}	2.4×10^{3}	-3.7×10^{3}	4.1	4.4×10^{-13}
10 TeV	1.3×10^{7}	2.4×10^{7}	-3.7×10^{7}	4.2×10^{1}	1.9×10^{-11}
100 TeV	1.3×10^{11}	2.4×10^{11}	-3.7×10^{11}	4.2×10^{4}	1.5×10^{-9}
1000 TeV	1.3×10^{15}	2.4×10^{11}	-3.7×10^{15}	4.2×10^{6}	6.3×10^{-8}

$\sqrt{s}$	$\mathcal{M}_{\rm s}+\mathcal{M}_{\rm t}+\mathcal{M}_{\rm c}$	$\mathcal{M}_{\rm h}$	$\mathcal{M}_{\rm tot}$	Error
0.2 TeV	9.7×10^{-2}	-1.7×10^{-1}	-7.7×10^{-2}	7.5×10^{-15}
0.5 TeV	1.0	-1.1	-4.8×10^{-2}	1.4×10^{-13}
1 TeV	4.1	-4.2	-3.7×10^{-2}	5.0×10^{-11}
10 TeV	4.2×10^{1}	-4.2×10^{1}	-3.3×10^{-2}	2.3×10^{-7}
100 TeV	4.2×10^{4}	-4.2×10^{4}	-3.3×10^{-2}	1.9×10^{-3}
1000 TeV	4.2×10^{6}	-4.2×10^{6}	-3.3×10^{-2}	7.9

Table 2.7: The test of the numerical accuracies in the process ${\rm W_L}^-{\rm W_L}^+ \to {\rm W_L}^-{\rm W_L}^+$. Amplitudes and their relative errors calculated at $\cos\theta = -0.5$ are shown. We set $\sin^2\theta_{\rm W} = 0.2311$, $m_{\rm Z} = 91.1867$ GeV, $m_{\rm H} = 100$ GeV and $\Gamma_{\rm Z} = \Gamma_{\rm H} = 0$.

$\sqrt{s} \setminus \cos\theta$	-0.9	-0.5	0.	0.5	0.9
0.2 TeV	2.0×10^{-15}	7.5×10^{-15}	3.2×10^{-15}	1.3×10^{-16}	7.1×10^{-16}
0.5 TeV	1.5×10^{-12}	1.4×10^{-13}	2.1×10^{-13}	4.8×10^{-14}	1.4×10^{-14}
1 TeV	2.7×10^{-11}	5.0×10^{-11}	7.9×10^{-12}	4.0×10^{-13}	3.7×10^{-14}
10 TeV	2.1×10^{-7}	2.3×10^{-7}	1.3×10^{-8}	4.2×10^{-9}	1.0×10^{-9}
100 TeV	2.6×10^{-3}	1.9×10^{-3}	9.5×10^{-4}	1.3×10^{-4}	1.9×10^{-5}
1000 TeV	1.7×10^{1}	7.9	2.9	7.3×10^{-2}	0.37

Table 2.8: Relative errors of the total amplitude of the process ${\rm W_L}^-{\rm W_L}^+ \to {\rm W_L}^-{\rm W_L}^+$, $\mathcal{M}_{\rm tot}$ calculated by HELAS are shown for some c.m. scattering angles.

are also shown for $\mathcal{M}_{\rm s}+\mathcal{M}_{\rm t}+\mathcal{M}_{\rm c}$ and $\mathcal{M}_{\rm tot}$. In Table 2.8, this relative error for $\mathcal{M}_{\rm tot}$ are shown for some c.m. scattering angles at the same $\sqrt{s}$ values.

We can clearly see from Tables 2.7 and 2.8 that the numerical error of the HELAS amplitudes increases rapidly with rising energies E for $\mathcal{M}_{\rm s}+\mathcal{M}_{\rm t}+\mathcal{M}_{\rm c}$, exhibiting the cancellation between order E^4 terms resulting in the E^2 behaviour.

The numerical error in the Higgs boson exchange diagrams in $\mathcal{M}_{\rm h}$ remains small since there is no subtle cancellation between the s- and t-channel exchange diagrams. The terms of order E^2 in $\mathcal{M}_{\rm s}+\mathcal{M}_{\rm t}+\mathcal{M}_{\rm c}$ and $\mathcal{M}_{\rm h}$ then cancel in the sum, and the numerical accuracy of the HELAS program is lost at $\sqrt{s} = 1000$ TeV. If one requests that the numerical error should be less than 1% at every phase space point, then the two weak boson invariant mass should be restricted to be less than 100 TeV when the process contains the VVVV vertices.

It is worth noting here that the loss of numerical accuracy discussed in this section is solely due to the subtle gauge theory cancellation among several Feynman diagrams and that it has nothing to do with our choice of the unitary gauge for the weak boson propagators. In fact, we repeated the above calculation of the ${\rm W_L W_L} \to {\rm W_L W_L}$ amplitude in the t'Hooft-Feynman gauge and found virtually no improvement in the accuracy of the results. This can be easily understood if one recognizes that the strongest cancellation between order E^4 terms inside $\mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_c$ (which may now include the Goldstone boson exchange diagrams) occurs between the contact four weak boson vertex and the weak boson exchange diagrams which are present in an arbitrary gauge. The gauge term $p_V^\mu p_V^\nu / m_V^2$ in the unitary gauge propagator does *not* give rise to a cancellation of order E^4 terms, because inside the `HELAS` subroutine, the scalar quantities like $p_V \cdot J_V(W^+W^-)$ are computed first. Inside of this scalar quantity, there is a cancellation between order E^4 terms to reproduce the order E^3 Goldstone boson di-weak boson vertex of the t'Hooft-Feynman gauge. This cancellation of order E^4 terms, leaving a term of order E^3, leads to numerical errors which are negligibly small as compared to those coming from the cancellation between the order E^4 terms, leaving a term of order E^2, common to both gauges.

In conclusion, the standard `HELAS` subroutines give reliable numerical results (error < 1 %) in the region $m_{VV} \lesssim$??? TeV for processes with three weak boson vertices, and at $m_{VV} \lesssim 100$ TeV in processes with four weak boson vertices. We do not anticipate any other sources of severe numerical inaccuracy in the use of `HELAS` subroutines, as long as one makes proper use of the special subroutines `JEEXXX`, `EAIXXX` and `EAOXXX` when dealing with collinear singularities.

2.6 Naming Schemes and Conventions

You may worry that there are too many subroutines to remember. However, the names of the subroutines and their inputs/outputs are systematically determined, and one can easily guess the name of the desired subroutine, or find out its function from its name.

The necessary ingredients to determine the names for subroutines are listed in Tables 2.8–2.12. The most frequently used codes are `I` or `O` for fermion, `V` for vector boson and `S` for scalar boson. Only if one deals with the output, does one use other codes: `F` for fermion, `J` for vector boson (like currents J^μ), and `H` for scalar.[3] There are only a few exceptions. `E` stands for electron or positron in subroutines which deal with collinear singularities (electron mass singularity). `W` means $W^\pm$ boson, and `3` stands for W^3 boson (including photon or Z). The `X`'s in subroutine names don't mean anything, except to tell the user that they are `HELAS` subroutines.

When you get accustomed to `HELAS`, then you will surely want to modify or extend it, to compute your favourite new physics model. Then the important point is how the coupling constants and amplitudes in `HELAS` are defined. We will explain these points briefly, though the details are postponed to Chapter 3.

The basic rule for fixing the convention of the coupling constants is that the coupling constants are the coefficients in the Lagrangian themselves. For example, if you wish to compute the vertex

$$\mathcal{L} = -\frac{\lambda}{4!}\phi^4, \tag{2.11}$$

[3] We couldn't think of a good letter for an output scalar. Here, the letter `H` refers to a Higgs boson, though the subroutine can be used for any scalar boson such as s-fermions in supersymmetric theories.

Table 2.9: Naming schemes for inputs and outputs. The codes in brackets stand for particular particles.

particle	input	output
F	I,O,(E)	F,(E)
V	V,(A,W,3)	J
S	S	H

Table 2.10: Naming schemes of the 'external' subroutines.

external	subroutine name
F	I,O
V	V
S	S

Table 2.11: Naming schemes of the 'vertex' subroutines.

vertex	inputs	output	subroutine name
FFV	FFV	amplitude	IOV
	FV	F	FVI,FVO
	FF	V	JIO
FFS	FFS	amplitude	IOS
	FS	F	FSI,FSO
	FF	S	HIO
VVV	VVV	amplitude	VVV
	VV	V	JVV
	GGG	amplitude	GGG
	GG	G	JGG
VVS	VVS	amplitude	VVS
	VS	V	JVS
	VV	S	HVV
VSS	VSS	amplitude	VSS
	SS	V	JSS
	VS	S	HVS
SSS	SSS	amplitude	SSS
	SS	S	HSS
VVVV	VVVV	amplitude	WWWW,W3W3
	VVV	V	JWWW,JW3W
	GGGG	amplitude	GGGG
	GGG	G	JGGG
VVSS	VVSS	amplitude	VVSS
	VSS	V	JVSS
	VVS	S	HVVS
SSSS	SSSS	amplitude	SSSS
	SSS	S	HSSS

Table 2.12: Naming schemes for special subroutines, to treat the collinear electron (positron) photon vertex.

vertex	input	output	subroutine name
EEA	EA	E	EAI,EAO
	EE	A	JEE

then you take

$$-\lambda \tag{2.12}$$

as the input of the SSSSXX or HSSSXX. Though you may think λ itself is more appropriate, since it is positive to make the potential stable, we took the coefficient in the Lagrangian itself to avoid confusion between the various vertices.

One confusing situation appears when dealing with derivative interactions, which appear in VSS and VVV vertices. Since derivative interactions are anti-symmetric under the interchange of particle and anti-particle, we used the following convention. We always start from the covariant derivative

$$D_\mu \equiv \partial_\mu + igV_\mu^a T^a \tag{2.13}$$

for all gauge groups, with a postive gauge coupling constant g. Then the self-interactions among the gauge bosons are fixed uniquely using the structure constants of the Lie groups. However, they depend on which gauge bosons are flowing out from the vertex. So, we fixed the ordering and couplings explicitly for the $W^\pm$, Z and A (photon) using COUP1X.

The output of the 'amplitude' subroutines is just a T-matrix element. In other words, the outputs are Feynman amplitudes with one i stripped off. If you want to add effective higher order amplitudes to the HELAS amplitudes, the sign and phase conventions are essential.

The conventions of the external wavefunctions are given in Appendix A. If you want to add an amplitude computed by other means, please be careful about whether the conventions of the external wavefunctions are the same. If they differ, one cannot simply add a new amplitude to the output of HELAS.

The propagators for the gauge bosons are written in unitary gauge if they are massive, and in Feynman gauge if they are massless. The choice of the unitary gauge is partly because this gauge requires the minimum number of Feynman diagrams. However, there is a physics motivation for this gauge which is that you can easily go to the infinitely heavy Higgs boson limit just by dropping the Higgs exchange diagrams.

2.7 How You Actually Work with HELAS

Now we will outline how you actually work with HELAS to compute the cross sections.

The first thing to do is to draw the Feynman diagrams. Though there may be many diagrams to be drawn for multi-body process, you should always include at least a set of diagrams which preserve the gauge invariance of the theory. Note that HELAS is written in the unitary gauge, so you don't have to include the Goldstone boson exchange diagrams.

Then you write down the amplitudes (T-matrix elements) for each Feynman diagram. If you have a four-point vector boson coupling, then three diagrams, such as a four-point contact vertex and t- and u-channel vector boson exchange, will be combined into one HELAS diagram. The diagrams representing each subroutine are given in the next section 2.8. If you wish to include the decay of final state particles, we recommend you to write down a program without decays first. Once a program without decays is ready, then it is always straightforward to include their decays.

The phase space integration should be made by your favourite method. For most purposes, we find that the Monte Carlo method is suitable.

Next you do a test run, linking your program to HELAS_CHECK first, to see whether the inputs and outputs of HELAS subroutines are appropriate. You will sometimes encounter error messages even if your program is correct; this happens at the extreme boundaries of phase space. Apart from this bug, the HELAS-ERROR messages will tell you if they find your program has something

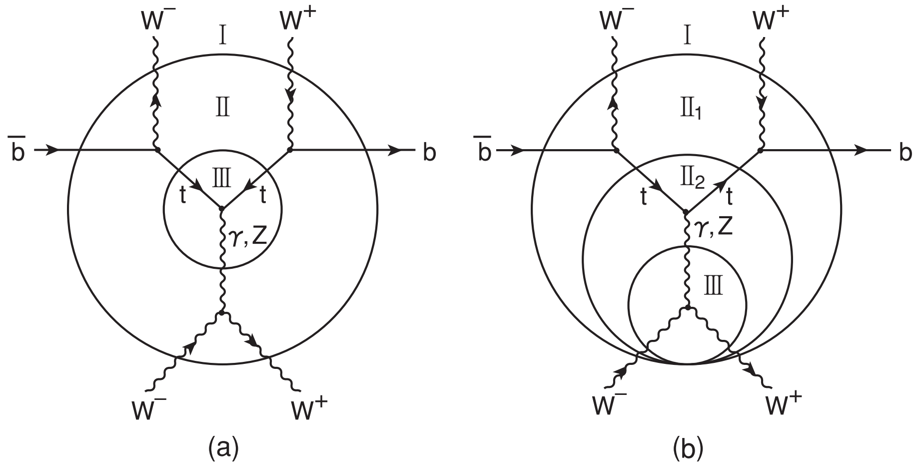


Figure 2: The steps you follow when coding with HELAS. The shown example is the s-channel γ-, Z-exchange amplitudes in the process $W^-W^+ \to t\bar{t}$, with the subsequent decays $t \to bW^+$ and $\bar{t} \to \bar{b}W^-$. The steps are labeled by the roman numbers. See the discussions in the text. The figures (a) and (b) show the different ways to compute the same diagrams.

wrong with its amplitudes. HELAS-warn messages tell you that some part of the calculations may be unnecessary, such as a charged-current interaction of the right-handed electron. If you are worried about the meanings of the error or warning messages, then Chapter 4 will help you.

After you have debugged your program, the final step is to link it to HELAS and run it.

When you are actually writing down the code for the amplitudes using HELAS subroutines, the only information you need is what the inputs and outputs are. Once you get used to HELAS coding, then you will need only HELAS.LIST2 given in Appendix C, which is just a listing of all SUBROUTINE program lines in HELAS. If you cannot figure out what the inputs and outputs stand for, then look at HELAS.LIST1. It is a listing of the comment lines in HELAS, which explains all the inputs and outputs. We believe that our naming scheme is systematic enough that you can easily remember the names of all the subroutines.

We will explain the steps you follow when writing code using HELAS, with the example

$$W^+W^- \to t\bar{t} \to bW^+\bar{b}W^-. \tag{2.14}$$

Part of this diagram was dealt with in section 2.3. We will take the s-channel γ, Z exchange diagram as the example here; see Fig. 4(a).

I. You start from the external lines for the initial W^-, W^+:

```
CALL VXXXXX(PWMI,WMASS,NHWMI,-1 , WMI)
CALL VXXXXX(PWPI,WMASS,NHWPI,-1 , WPI)
```

and final b, W^+, $\bar{b}$, W^-:

```
CALL OXXXXX(PB,    0.,   -1,   +1 , B   )
CALL IXXXXX(PBBAR,0.,    +1,   -1 , BBAR)
CALL VXXXXX(PWMF, WMASS,NHWMF,+1 , WMF )
CALL VXXXXX(PWPF, WMASS,NHWPF,+1 , WPF )
```

This completes step I.

II. You start combining the external lines computed above to approach the centre of the diagram, in step II.

```
CALL FVOXXX(B,   WPF,GWF,TMASS,TWIDTH , FTO )
CALL FVIXXX(BBAR,WMF,GWF,TMASS,TWIDTH , FTI )
CALL JVVXXX(WPI,WMI,GWWA,0.,    0.     , JWWA)
CALL JVVXXX(WPI,WMI,GWWZ,ZMASS,ZWIDTH , JWWZ)
```

Here, we compute the off-shell t-quark wave function `FTO(6)` and the $\bar{t}$-quark wave function `FTI(6)`, and the off-shell s-channel photon current `JWWA(6)` and the Z-current `JWWZ(6)`.

III. Finally, in step III, we combine the s-channel photon or Z current, off-shell t-quark and $\bar{t}$-quark spinors to obtain the amplitudes,

```
CALL IOVXXX(FTI,FTO,JWWA,GAU , AMPA)
CALL IOVXXX(FTI,FTO,JWWZ,GZU , AMPZ)
AMP = AMPA + AMPZ
```

In this way, you can easily write down the code for using `HELAS` subroutines.

The way to combine various lines depends on your taste. For example, the same diagram above can be computed in an alternative way. Here we follow the Fig. 4(b).

I. The step I for computing the external lines is the same as above.

II. Now you can combine b and W^+, $\bar{b}$ and W^- to obtain the internal t-quark line `FTO(6)` and the $\bar{t}$-quark line `FTI(6)` just like the first two lines in the second step above (step II_1 in Fig. 4(b)). After that, you combine these off-shell top quark lines to obtain the s-channel W^3 current `JTT3(6)`:

```
CALL J3XXXX(FTI,FTO,GAU,GZU,ZMASS,ZWIDTH , JTT3)
```

This is step II_2 in Fig. 4(b).

III. Finally we combine the initial W^-, W^+ and the above s-channel W^3 current

```
CALL VVVXXX(WPI,WMI,JTT3,GW , AMP)
```

in step III.

Of course, the amplitude `AMP` obtained here should be the same as that obtained in the previous method to within a numerical accuracy of $O(10^{-6})$.

Summing up, the `HELAS` amplitudes are calculated by the following three steps:

HELAS:	step I:	setting up external particle wave functions
	step II:	computing off-shell `S/F/V` lines
	step III:	computing the helicity amplitude

The subroutines that appear in each of the above steps are clearly distinguished by our naming convention:

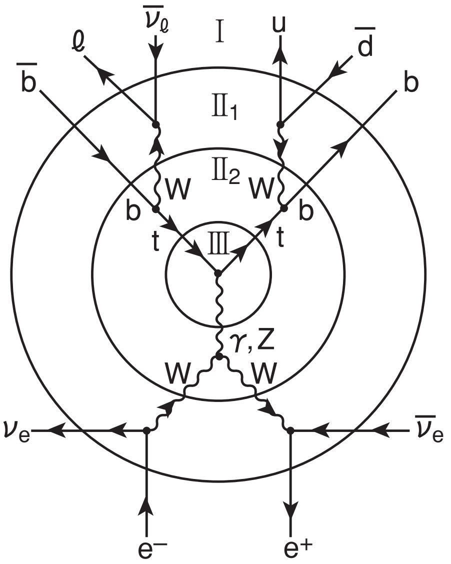


Figure 3: The steps you follow when coding with `HELAS`. The shown example is an extended version of Fig. 4(a), where the final W^+ decays into $u\bar{d}$, and W^- into $l\bar{\nu}_l$.

```
step I:         step II:                step III:

S:  SXXXXX      S:  HIOXXX              FFV:  IOVXXX
                    HVVXXX              FFS:  IOSXXX
F:  IXXXXX          HVSXXX
    OXXXXX          HSSXXX              VVV:  VVVXXX
                                        VVS:  VVVXXX
V:  VXXXXX      F:  FVIXXX  EAIXXX      VSS:  VSSXXX
                    FVOXXX  EAOXXX      SSS:  SSSXXX
                    FSIXXX
                    FSOXXX              VVVV: WWWWXX
                                              W3W3XX
                V:  JIOXXX  JEEXXX      VVSS: VVSSXX
                    J3XXXX              SSSS: SSSSXX
                    JVVXXX
                    JVSXXX
                    JSSXXX
                    JWWWXX
                    JW3WXX
                    JVSSXX
```

In step I, the 4 external wave function subroutines are called the same number of times as the number of external particles. In step II, we calculate off-shell `S/F/V` lines from the external wave functions by the above 19 subroutines. The off-shell lines as calculated in this step can appear repeatedly as inputs of other off-shell line subroutine. Examples of such cases are shown in Fig. 4(b) and in Fig. 5. Finally, in step III, the evaluation of one Feynman diagram terminates with just one of the above 10 amplitude subroutines.

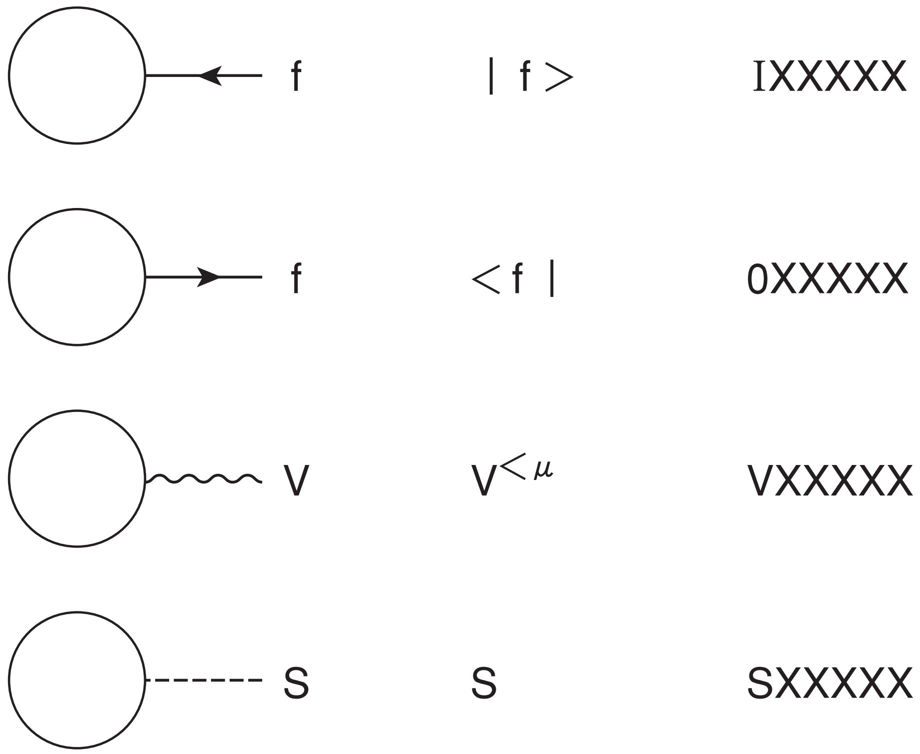


Figure 4: The HELAS subroutines for the external wave functions. IXXXXX for the flowing-in fermion, OXXXXX for the flowing-out fermion, VXXXXX for the vector boson, and SXXXXX for the scalar boson.

2.8 HELAS Feynman Rules

In this subsection, we list all the HELAS subroutines in Figs. 6−12. In each figure, we give a Feynman diagram representation on the left, the HELAS symbol in the middle, and the corresponding HELAS subroutine name on the right. Using these rules, you can write down the amplitude and the FORTRAN code without refering to the conventional Feynman rules. Details of each HELAS subroutine convention are found in Chapter 3. Here, we give the correspondences between HELAS Feynman rules and the conventional one for the most frequently used subroutines.

Shown in Fig. 6 are the four external wave functions; flowing-in and flowing-out spinors, vector and scalar. For fermions, the correspondences are

$$|f(p_f,\lambda_f,S_f)\rangle = \begin{cases} u(p_f,\lambda_f) & \text{for } S_f=+1, \\ v(p_f,\lambda_f) & \text{for } S_f=-1, \end{cases} \tag{2.15}$$

$$\langle f(p_f,\lambda_f,S_f)| = \begin{cases} \bar{u}(p_f,\lambda_f) & \text{for } S_f=+1, \\ \bar{v}(p_f,\lambda_f) & \text{for } S_f=-1, \end{cases} \tag{2.16}$$

where the sign factor S_f distinguishing the particle from the antiparticle appears as an integer variable NSF in the subroutines IXXXXX and OXXXXX. We note that the input four-momentum and the helicity of external particles are physical ones in the HELAS convention. The sign factor S_f takes care of the appropriate crossing relations between an amplitude with an out-going particle and that with an in-coming antiparticle for a flowing-out fermion wave function, and similarly for the flowing-in fermion case.

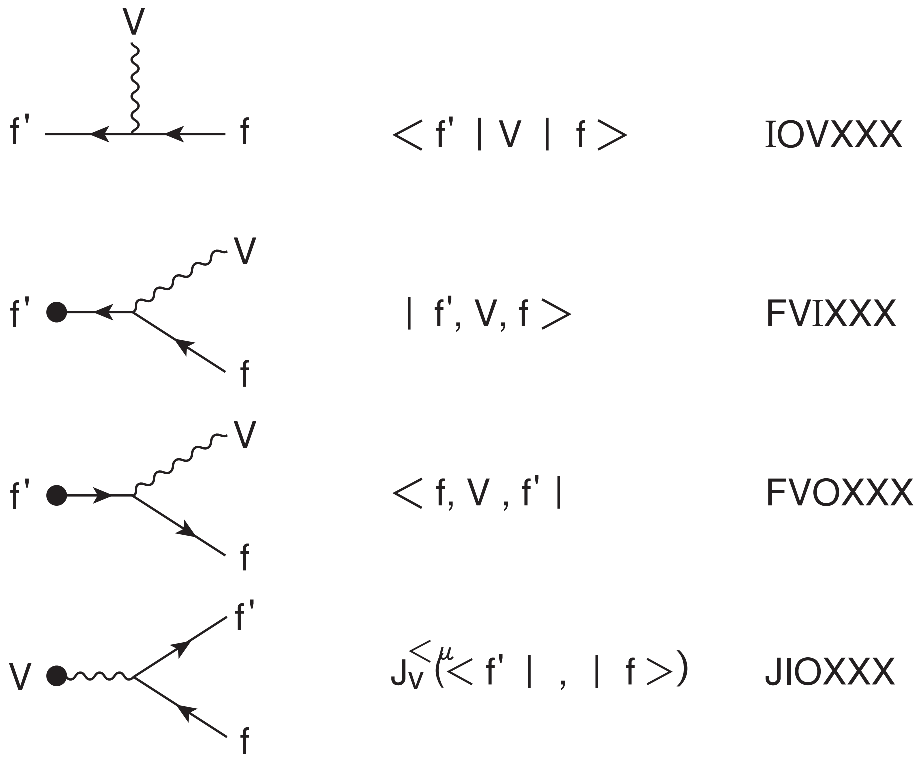


Figure 5: The HELAS subroutines for the FFV vertex. IOVXXX gives the T-matrix element, while FVIXXX and FVOXXX give off-shell fermion lines. Both JIOXXX and J3XXXX give off-shell vector current, however J3XXXX computes a weighted sum of γ- and Z-current appropriate for the inputs of VVVXXX etc.

For vector bosons, we have

$$V^\mu(p_V,\lambda_V,S_V) = \begin{cases} \epsilon^\mu(p_V,\lambda_V)^* & \text{for } S_V=+1, \\ \epsilon^\mu(p_V,\lambda_V) & \text{for } S_V=-1. \end{cases} \tag{2.17}$$

Here we introduced another sign factor S_V, which is denoted by NSV in the subroutine VXXXXX, in order to distinguish an out-going vector boson ($S_V=+1$) and an in-coming one ($S_V=-1$).

The scalar wave function is simply normalized to unity

$$S(p_S,S_S)=1. \tag{2.18}$$

We still assign the sign factor S_S (the integer variable NSS in the subroutine SXXXXX), which distinguishes an out-going scalar ($S_S=+1$) from an in-coming one ($S_S=-1$). The role of the sign factor here is solely to fix the flow of the four-momentum in the Feynman amplitude.

In Figs. 7−12, we list all the vertices that appear in renormalizable local field theory. Each vertex can either appear at the end of an amplitude program to represent the helicity amplitude of a given Feynman diagram, or at an intermediate step to represent an off-shell line (denoted

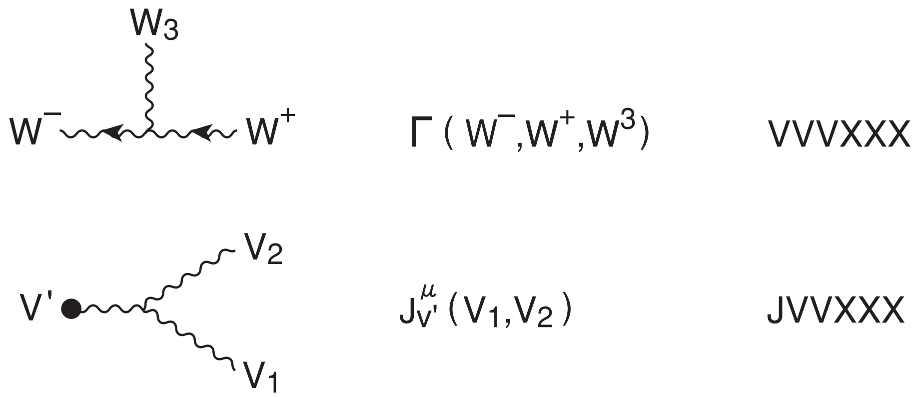


Figure 6: The HELAS subroutines for the VVV vertex. VVVXXX gives the T-matrix element, while JVVXXX gives off-shell vector current.

by a blob in the left-hand-side of diagrams in the figures) made by the vertex. For instance, in Fig. 7, the interaction Lagrangian

$$\mathcal{L}_{int} = \bar{\psi}_{f'} \gamma^{\mu} \left[g_L \frac{1-\gamma_5}{2} + g_R \frac{1+\gamma_5}{2} \right] \psi_f V_{\mu} \tag{2.19}$$

can lead to an amplitude (IOVXXX)

$$\langle f'|V|f\rangle = \bar{u}_{f'} \not{V} \left[g_L \frac{1-\gamma_5}{2} + g_R \frac{1+\gamma_5}{2} \right] u_f \;, \tag{2.20}$$

or to an off-shell f' wave function (FVIXXX)

$$|f', V, f\rangle = \frac{i(\not{p}_{f'} + m_{f'})}{p_{f'}^2 - m_{f'}^2 + i m_{f'} \Gamma_{f'}} \not{V} \left[i g_L \frac{1-\gamma_5}{2} + i g_R \frac{1+\gamma_5}{2} \right] u \;, \tag{2.21}$$

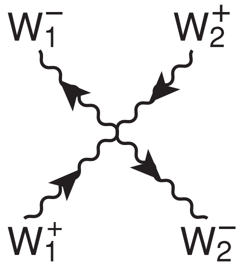
an off-shell f wave function (FVOXXX)

$$\langle f', V, f| = \bar{u}_{f'} \not{V} \left[i g_L \frac{1-\gamma_5}{2} + i g_R \frac{1+\gamma_5}{2} \right] \frac{i(\not{p}_f + m_f)}{p_f^2 - m_f^2 + i m_f \Gamma_f} \;, \tag{2.22}$$

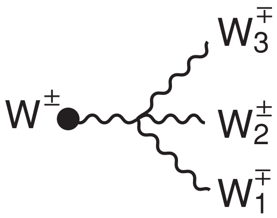
or an off-shell V wave function (JIOXXX)

$$J_V^{\mu}(\langle f'|, |f\rangle) = \bar{u}_{f'} \gamma_{\nu} \left[i g_L \frac{1-\gamma_5}{2} + i g_R \frac{1+\gamma_5}{2} \right] u_f \frac{i(-g^{\mu\mu} + \frac{p_V^{\mu} p_V^{\nu}}{m_V^2})}{p_V^2 - m_V^2 + i m_V \Gamma_V} . \tag{2.23}$$

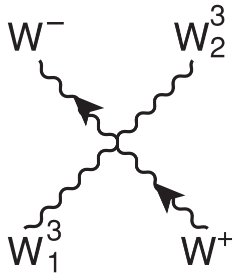
The fermion wave function $u(\bar{u})$ above can also be $v(\bar{v})$ for antiparticles. The real coupling constants g_L and g_R are denoted by G(1) and G(2) in these subroutines. We note that all off-shell wave functions include the propagator of the off-shell particle. We employ the unitary gauge propagator for massive vector bosons and the Feynman gauge one for massless vector bosons ($m_V = 0$). Details of our convention for each subroutine are found in the corresponding section of Chapter 3.



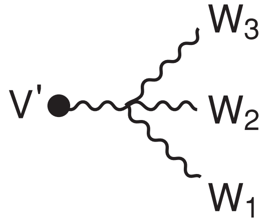
$\Gamma\,(W_1^-, W_1^+, W_2^-, W_2^+)$ WWWWXX



$J^\mu_{W^\pm}(W_1^\mp, W_2^\pm, W_3^\mp)$ JWWWXX



$\Gamma\,(W^-, W_1^3, W^+, W_2^3)$ W3W3XX



$J^\mu_{V'}\,(W_1, W_2, W_3)$ JW3WXX

$V' = W^\pm, \gamma$ or Z

Figure 7: The HELAS subroutines for the $VVVV$ vertex. WWWWXX and W3W3XX give the T-matrix elements, while JWWWXX and JW3WXX give off-shell vector currents. Note that four-point contact term as well as vector boson exchange diagrams are included. These subroutines are written in double precision.

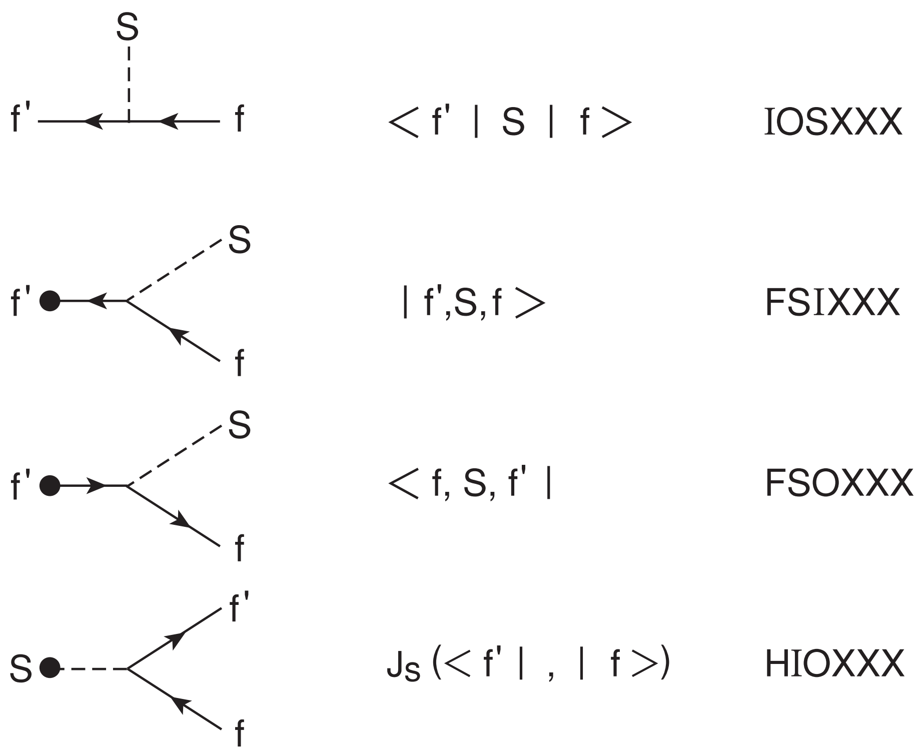


Figure 8: The `HELAS` subroutines for the FFS vertex. `IOSXXX` gives the T-matrix element, while `FSIXXX` and `FSOXXX` give off-shell fermion lines. `HIOXXX` gives the off-shell scalar current.

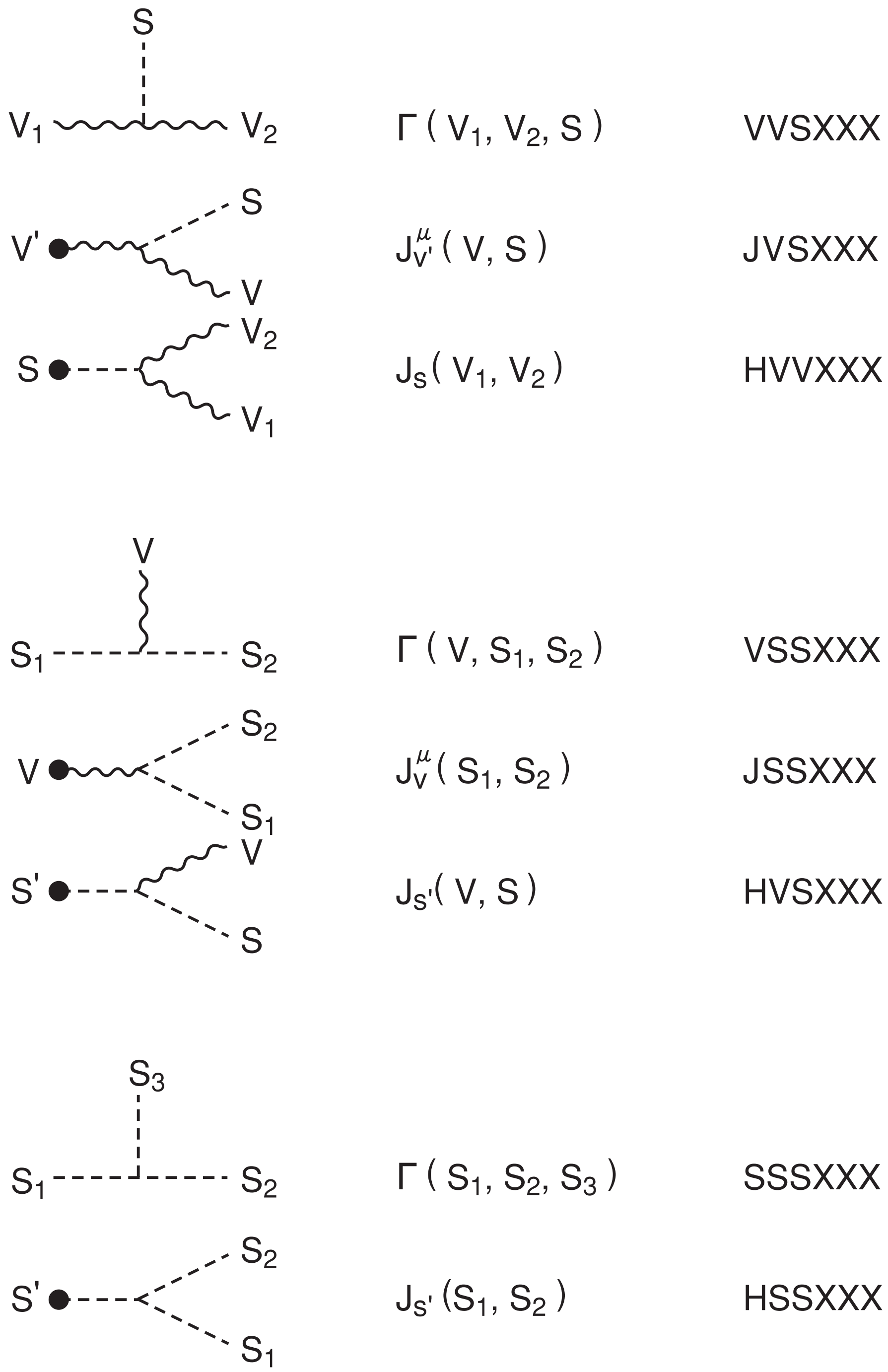


Figure 9: The HELAS subroutines for the VVS, VSS and SSS vertices. VVSXXX, VSSXXX and SSSXXX give the T-matrix elements, JVVXXX, JSSXXX give off-shell vector currents, and HVVXXX, HVSXXX and HSSXXX give off-shell scalar currents.

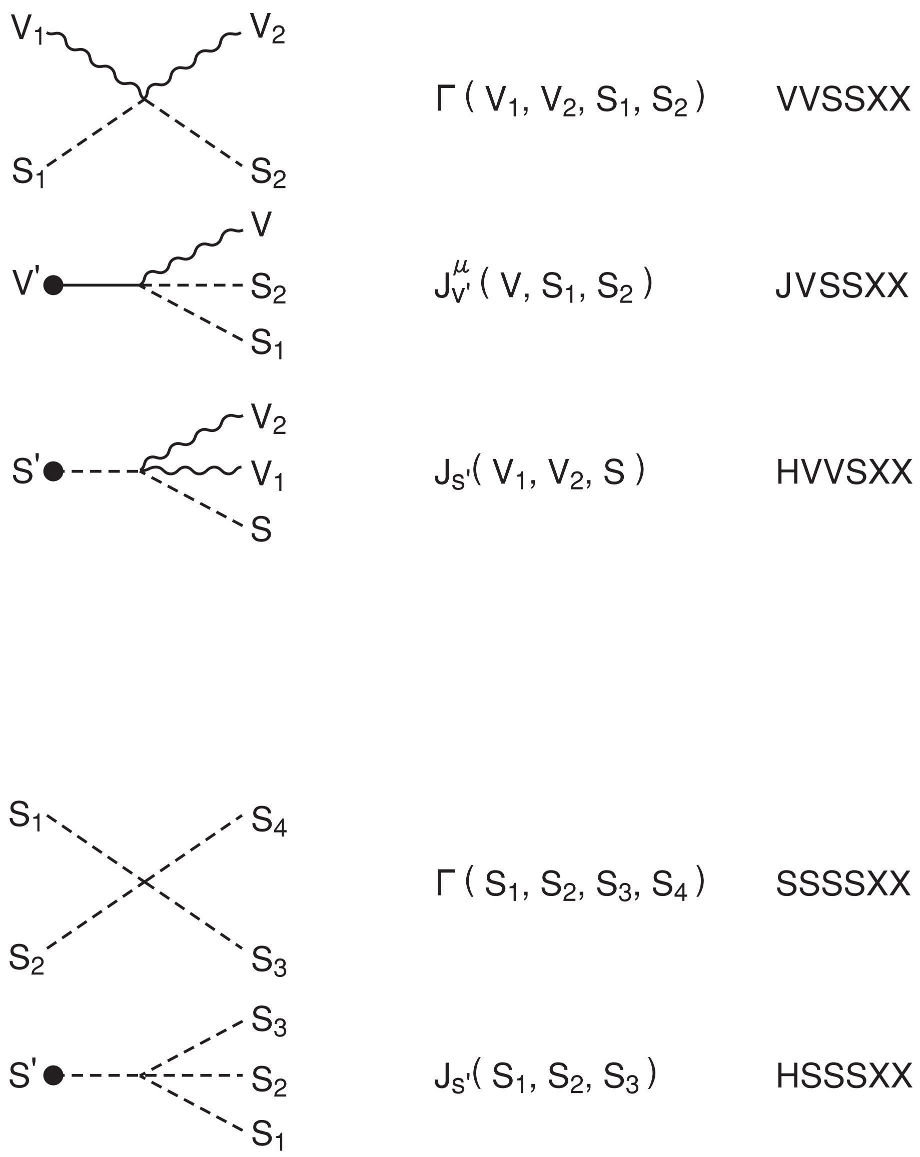


Figure 10: The HELAS subroutines for the $VVSS$ and $SSSS$ vertices. VVSSXX and SSSSXX

These examples show our convention rather clearly. All external particle wave functions are assigned a sign factor (S_f for a fermion, S_V for a vector, and S_S for a scalar) which controls the fermion number and the four-momentum flow in the diagram. Because of this convention, the crossing relation among amplitudes is made trivial in the HELAS system.

The conventional factors of i in the vertices and those in the propagators are both included in the off-shell wave functions, such as eqs. (2.21), (2.22), or (2.23) above. This convention allows us to use the off-shell wave functions as inputs of all HELAS subroutines, in place of external wave functions. For instance, the off-shell f' wave function (2.21) can appear as an input of another off-shell fermion wave function made by a $V'f''f'$ vertex

$$|f'', V', (|f', V, f\rangle)\rangle \equiv |f\text{''}, V', f', V, f\rangle, \tag{2.24}$$

or as an input of an off-shell vector wave function

$$J^{\mu}_{V'}(\langle f''|, |f', V, f\rangle), \tag{2.25}$$

from the same vertex.

Likewise, the off-shell vector wave function (2.23) can appear as an input of an off-shell fermion wave function

$$\langle f''', J_V(\langle f'|, |f\rangle\rangle), f''|, \tag{2.26}$$

in a $Vf'''f''$ vertex.

Finally, the HELAS amplitude, obtained by one of the 10 vertices, gives the contribution to the T matrix element without the factor of i. For example, the amplitude (2.20) does **not** have the usual factor of i for the coupling factor g_L and g_R. These points should be kept in mind when you introduce new interactions and new particles to the HELAS system.

2.9 How to add New Interactions and New Particles

In this section, we briefly give comments that may help HELAS users to evaluate helicity amplitudes with new interactions and new particles. It is on such occasions that the detailed description of each subroutine in Chapter 3 will be useful. Here, we give only general remarks.

As advertised in the introduction, the standard HELAS subroutines handle all possible renormalizable interactions of scalar, fermion and vector bosons. Therefore, you do not need to introduce a new subroutine as long as your new vertices are renormalizable. Only when you introduce non-renormalizable interactions or when you evaluate radiatively corrected amplitudes, will you need to add new subroutines to the HELAS system.

Let us start with a few remarks within the standard mode, where some care is needed in treating QCD interactions (2.9.1), the KM matrix elements (2.9.2), and the Goldstone boson interactions in the renormalizable gauges of the electroweak theory (2.9.3). As examples of new renormalizable interactions, we give brief comments on supersymmetric theories (2.9.4) and theories with additional heavy weak bosons (2.9.5). Possible advantages of using HELAS in studying the consequences of non-renormalizable vertices (2.9.6) and electroweak loop corrections (2.9.7) are also addressed.

2.9.1 QCD

Since there are several excellent programs [5] to evaluate matrix elements of massless quarks and gluons, we made no effort to make efficient HELAS subroutines for the gluon vertices.

The qqg vertex can be evaluated by the subroutines IOVXXX, FVIXXX, FVOXXX and JIOXXX with the real couplings

$$\mathtt{G(1)} = \mathtt{G(2)} = -\sqrt{4\pi\alpha_s}, \tag{2.27}$$

and also

$$\mathtt{VMASS} = \mathtt{VWIDTH} = 0, \tag{2.28}$$

in JIOXXX for an off-shell gluon. The colour factor $(T^a)_{ij}$ should be handled separately. The sign of the coupling (2.27) is fixed by our conventions (2.13) and (3.16).

The ggg vertex is computed by the subroutines VVVXXX and JVVXXX, with the real coupling

$$\mathtt{GS} \equiv \sqrt{4\pi\alpha_s} \tag{2.29}$$

and a vanishing gluon mass (2.28). The colour factor f^{abc} should be multiplied by hand, where the ordering of the three gluons is chosen as

$$\mathtt{CALL\ \ VVVXXX(VA,VB,VC,GS\ ,\ VERTEX)}, \tag{2.30}$$

$$\mathtt{CALL\ \ JVVXXX(VA,VB,GS,0.,0.\ ,\ JVV)}. \tag{2.31}$$

Finally, when the $gggg$ vertex appears, you should employ the special '$\mathtt{WMASS} = \mathtt{0.0}$' option of the HELAS subroutines W3W3XX and JW3WXX. This option was introduced for this purpose, because in the HELAS subroutines for the four vector boson vertices, contributions from the contact interaction and those of the two vector boson exchange diagrams (see Fig. 9) are added inside the subroutines in order to minimize the number of CALL lines in the program. This is, however, not possible in general non-Abelian gauge theories, including QCD. By setting '$\mathtt{WMASS} = \mathtt{0.0}$' in the subroutines W3W3XX and JW3WXX, only half of the contact term and just one massless vector boson exchange diagram contribute to the output. More specifically, a 4 gluon vertex (GGGG) and an off-shell gluon made from three gluons (JGGG) are computed as follows;

$$\mathtt{CALL\ \ W3W3XX(VA,VB,VC,VD,GS,GS,0.0,0.0,GGGG)}, \tag{2.32}$$

$$\mathtt{CALL\ \ JW3WXX(VA,VB,VC,GS,GS,0.0,0.0,0.0,0.0,JGGG)}. \tag{2.33}$$

Here the gluons VA, VB, VC, and VD have the colour a, b, c, and d, respectively, and the off-shell gluon JGGG has the colour d. The associated colour factor is then $f^{abe}f^{cde}$.

By introducing the new subroutines GGGGXX and JGGGXX in place of the above options of the present HELAS subroutines, you can improve the efficiency of QCD manipulations. Summation over the colour degrees of freedom should be done algebraically to get good numerical efficiency.

2.9.2 KM matrix elements

The coupling constants G(1) and G(2) in the FFV vertex subroutines, IOVXXX, FVIXXX and FVOXXX are assumed to be real numbers. This is to ensure fast numerical manipulation since these vertices appear many times in the standard model amplitudes. Therefore, when the complex phase of the KM matrix elements needs to be kept in the amplitude, you should multiply the phase factor to the relevant output of the HELAS subroutine. The magnitude of the KM matrix elements can be included in the input G(1), e.g. as

$$\mathtt{G(1)} = -\frac{g_W}{\sqrt{2}}|V_{ij}| \tag{2.34}$$

with $g_W = e/\sin\theta_W$, for the charged current transition via the Wq_iq_j vertex.

2.9.3 Goldstone bosons and the BRS invariance tests

There are occasions when the calculation of the amplitudes in the renormalizable R_ξ gauge, the t'Hooft-Feynman gauge in particular, of the electroweak theory can be useful. We find that the use of the covariant R_ξ gauge for the weak boson propagators inside tree-level helicity amplitude does not lead to a non-trivial test of the amplitude, nor does it lead to a superior numerical accuracy at high energies as compared to the unitary gauge manipulation (see sect. 2.5). We therefore choose all the massive vector boson propagators to take the unitary gauge form in the HELAS subroutines.

What we find most efficient in testing the helicity amplitudes with one or more external vector bosons is the BRS identity [4]

$$\langle phys;\ \text{out}|\ (\partial^\mu V_\mu - \xi_V m_V \chi_V)\ |phys;\ \text{in}\rangle = 0, \tag{2.35}$$

where ξ_V is the covariant R_ξ gauge parameter and χ_V the Goldstone mode associated with the vector boson V. The states $\langle phys;\ \text{out}|$ and $|phys;\ \text{in}\rangle$ are arbitrary physical states of on-shell external particles. By using the reduction formula, the identity (2.35) leads to an exact relationship between the S-matrix elements of the four-divergence of the vector boson and those of the associated Goldstone boson

$$\langle phys,\ V_S;\ \text{out}|phys;\ \text{in}\rangle \ = \ -\langle phys,\ \chi_V;\ \text{out},|phys;\ \text{in}\rangle, \tag{2.36}$$

where V_S denotes the 'scalar' component of the vector boson. Eq. (2.36) relates an amplitude with a V_S emission, which is obtained from the vector boson emission amplitude with the replacement

$$\epsilon_V^\mu(p_V, \lambda_V, S_V) \rightarrow \frac{p_V^\mu}{m_V}, \tag{2.37}$$

to that with the associated Goldstone boson χ_V emission. The amplitudes with Goldstone boson emission are often very simple and can easily be evaluated numerically with HELAS. It is worth noting that the identity (2.36) between the matrix elements does not depend explicitly on the gauge parameter ξ_V [3] and that it is valid even in the unitary gauge limit $\xi_V \rightarrow \infty$. The standard HELAS subroutines can hence be used in the test. The identity turns out to be very efficient in testing the amplitudes as well as the numerical accuracy of the program. See discussions in refs. [3, 6, 7, 8] and examples in Appendices B.4 and B.6.

In order to perform the tests conveniently, the vector boson wave function subroutine `VXXXXX` has an option tt `NHEL` = 4 in the checking program `HELAS_CHECK.FOR`, for which the polarization vector is simply,

$$\epsilon_V^\mu(p_V, \lambda_V = 4) = \epsilon_V^\mu(p_V, \lambda_V = 4)^* = \begin{cases} p_V^\mu / m_V & \text{if } m_V \neq 0, \\ p_V^\mu / p_V^0 & \text{if } m_V = 0. \end{cases} \tag{2.38}$$

Simply by setting the helicity of an external vector boson to be '4', you can calculate the amplitude for the scalar vector boson emission multiplied by the vector boson mass. The HELAS subroutines for the `VVS, VSS` and `VVSS` vertices are found to be useful in calculating the associated Goldstone boson emission amplitudes: see e.g. Figs. 14 and 16 in Appendices B.4 and B.6, respectively. By writing down the relevant Lagrangian term with the Goldstone boson in the R_ξ gauge, and by comparing the couplings with those appearing in the defining Lagrangian of each vertex in Chapter 3, you can easily determine the input coupling constants for these subroutines. A simple example of the BRS invariance test is worked out in Appendix B.4.

2.9.4 Supersymmetric particles

Superpartners of the standard model particles are among the most popular of the new particles. Since all the interactions in the softly broken supersymmetric standard model are renormalizable, arbitrary tree-level amplitudes in the model can be evaluated by the HELAS subroutines. In fact many subroutines with scalar particles are prepared such that they are convenient in calculating processes with supersymmetric particles. For instance, the couplings in the FFS vertices are chosen chiral and complex for this purpose. Various phases appearing in the mixing phenomenon can therefore be included as complex couplings in these subroutines. This was possible because the vertices with scalar particles rarely appear in the standard model amplitudes, and we did not need to make these subroutines very efficient.

In theories with several vertices with a universal coupling, the relative phases of these vertices should be evaluated correctly. The supersymmetric standard model is a good example of such a theory, where many particles interact with a universal gauge coupling. Our convention of using the coefficients of the interaction Lagrangian as inputs of the HELAS subroutines was introduced for this reason, since the term 'charge' of a particle can be ambiguous.

Finally, there is one technical remark on the HELAS spinor convention which may be helpful in dealing with the interactions with Majorana particles or charge-conjugated fermion operators. We employ the chiral representation of the γ matrices (see Appendix A for details), where the charge conjugation matrix C as defined by

$$C\gamma^\mu C^{-1} = -(\gamma^\mu)^T, \tag{2.39}$$
$$C\gamma_5 C^{-1} = (\gamma_5)^T, \tag{2.40}$$

takes the form

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}, \tag{2.41}$$

which satisfies

$$C = C^* = -C^{-1} = -C^\dagger = -C^T. \tag{2.42}$$

The relative phases of u and v spinors in the HELAS spinor convention (Appendix A) are so chosen that the following identities

$$C\bar{u}(p,\lambda)^T = v(p,\lambda), \tag{2.43}$$
$$C\bar{v}(p,\lambda)^T = u(p,\lambda), \tag{2.44}$$

hold. Therefore, no special care is needed in handling the charge conjugated fermionic operators. Wave functions of Majorana fermions can take either of the u or v spinor form according to the above identity, without introducing an extra phase factor. See ref. [9] for some worked out examples in our convention [10].

2.9.5 Additional weak bosons

New gauge boson interactions with fermions and scalars can be treated exactly the same as the standard weak boson interactions: the couplings in the FFV and VVS subroutines are real and possible complex phase factors from the weak boson mixing should be multiplied explicitly to the output of the HELAS subroutines just as in the case of the KM phase (see sect. 2.9.2). The HELAS subroutines for the VVV couplings can also be used to evaluate all new vertices by choosing the coupling and the vector boson mass inputs appropriately.

The HELAS subroutines for the four vector boson vertices can be used without modification if the new gauge boson belongs to an additional $U(1)$ or $SU(2)$ gauge group. In the former case, you merely change the input WWZ coupling according to the mixing with the new Z boson. In the latter case, you can calculate the effects of the new four weak boson interactions by choosing appropriate coupling and mass inputs. In models with the gauge group $SU(3)$ or a larger group, a modification similar to the 'WMASS = 0.0' option of the subroutines W3W3XX and JW3WXX as explained in subsection 2.9.1 will become necessary.

2.9.6 Non-renormalizable interactions

When studying the consequences of composite models of elementary particles or low energy effective theories of a renormalizable model with heavy particles, you may encounter non-renormalizable effective vertices among the light particles. Since the HELAS subroutines contain only renormalizable vertices, you may need to write a new subroutine for such cases. It may be worth noting, however, that this is not always the case.

When a non-renormalizable vertex is obtained as the limit of a tree-level exchange of a heavy particle with renormalizable interactions, then the standard HELAS subroutines can be used to evaluate the vertex easily. For instance, an arbitrary contact four fermion interaction is obtained by using the FFV and FFS subroutines as the limit of heavy vector or scalar boson exchange amplitudes.

Only when the new interaction cannot be obtained easily in this way, should you add a new subroutine to the HELAS system. You will find that the detailed explanations of each subroutine in the next chapter and those of our conventions in Appendix A are useful in such occasions.

2.9.7 Loop corrected amplitudes

Use of HELAS in radiative correction studies can be advantageous since it allows you to evaluate radiatively corrected helicity amplitudes rather than just the cross sections. We give a few brief remarks which may be helpful when you make such applications.

- When the loop integral contains a dimensionally regularized divergence, then a proper renormalization should be made prior to the use of HELAS, since all the HELAS subroutines assume four dimensional spinors and vector bosons.

- When the loop corrected amplitude is expressed as a sum of terms which are products of a scalar function and renormalizable vertices, then the standard HELAS subroutines are sufficient to generate the radiatively corrected amplitudes.

- When non-renormalizable type vertices are necessary to express the radiatively corrected amplitudes, then the comments in the previous subsection apply. You may or may not need to introduce a new subroutine.

Chapter 3

HELAS Subroutines

In this chapter, we will explain the contents of each subroutines and their use. The `HELAS` subroutines can be grouped in the following three subsets:

I. Wave-Functions

II. Vertices

   I. FFV
   II. FFS
   III. VVV
   IV. VVS
   V. VSS
   VI. SSS
   VII. $VVVV$
   VIII. $VVSS$
   IX. $SSSS$

III. Tools and Standard Model Coupling Constants

The vertex subroutines are further divided into “amplitude” subroutines and “off-shell” subroutines.

We explain the contents of the subroutines with the above groupings in the following sections.

3.1 Wave-Functions

We have four subroutines in this group:

```
IXXXXX  ,
OXXXXX  ,
VXXXXX  ,
SXXXXX  .
```

3.1.1 IXXXXX

This subroutine computes the flowing-In spinor wavefunction of fermion; namely, $u(p)$ and $v(p)$'s combined with its four-momentum.

This subroutine will be called as

```
CALL IXXXXX(P,FMASS,NHEL,NSF , FI)  .
```

We have four inputs P,FMASS,NHEL,NSF and one output FI.

THE INPUTS

I. `real P(0:3)`
This is a real four-dimensional array which contains the four-momentum p^μ of the fermion. The four-momentum may be either time-like ($p^2 > 0$) for a massive fermion or light-like ($p^2 = 0$) for a massless fermion.[1] The energy P(0) must be always positive.

II. `real FMASS`
This is a real variable which contains the mass of the Fermion.

III. `integer NHEL`
This variable specifies the helicity of the fermion. If NHEL = 1 the helicity is $+1/2$, and if NHEL = -1 the helicity is $-1/2$.

IV. `integer NSF`
This variable specifies whether the fermion is particle or anti-particle. If NSF = 1 the fermion is particle, and the subroutine computes the $u(p)$-spinor. If NSF = -1 the fermion is anti-particle, and the subroutine computes the $v(p)$-spinor.

THE OUTPUT

I. `complex FI(6)`
This is a complex six-dimensional array which contains the Fermion's flowing-In spinor combined with its four-momentum. The first two components FI(1) and FI(2) contain the chirality left part of the spinor,

$$\begin{pmatrix} \mathtt{FI(1)} \\ \mathtt{FI(2)} \end{pmatrix} = \frac{1-\gamma_5}{2} u(p) \qquad \text{or} \qquad \frac{1-\gamma_5}{2} v(p). \tag{3.1}$$

The second two components FI(3) and FI(4) contain chirality right part of the spinor,

$$\begin{pmatrix} \mathtt{FI(3)} \\ \mathtt{FI(4)} \end{pmatrix} = \frac{1+\gamma_5}{2} u(p) \qquad \text{or} \qquad \frac{1+\gamma_5}{2} v(p). \tag{3.2}$$

The last two components FI(5) and FI(6) contain the four-momentum along the fermion number flow,

$$\begin{pmatrix} \mathtt{FI(5)} \\ \mathtt{FI(6)} \end{pmatrix} = \mathtt{NSF} \begin{pmatrix} \mathtt{P(0)} + i\mathtt{P(3)} \\ \mathtt{P(1)} + i\mathtt{P(2)} \end{pmatrix}. \tag{3.3}$$

We denote the outputs of the subroutine symbolically as

$$|f\rangle. \tag{3.4}$$

For more about the conventions of the spinors, see Appendix A.

[1] The space-like momentum ($p^2 < 0$) is not allowed. We believe that there is no tachyonic fermion so this is not a restriction in computing helicity amplitudes. However, who knows?

3.1.2 OXXXXX

This subroutine computes the flowing-Out spinor wavefunction of fermion; namely, $\bar{u}(p)$ and $\bar{v}(p)$'s combined with its four-momentum.

This subroutine will be called as

```
CALL OXXXXX(P,FMASS,NHEL,NSF , FO)   .
```

We have four inputs `P,FMASS,NHEL,NSF` and one output `FO`.

THE INPUTS

I. `real P(0:3)`
This is a real four-dimensional array which contains the four-momentum p^μ of the fermion. The four-momentum may be either time-like ($p^2 > 0$) for a massive fermion or light-like ($p^2 = 0$) for a massless fermion. The energy `P(0)` must be always positive.

II. `real FMASS`
This is a real variable which contains the mass of the Fermion.

III. `integer NHEL`
This variable specifies the helicity of the fermion. If `NHEL = 1` the helicity is $+1/2$, and if `NHEL = -1` the helicity is $-1/2$.

IV. `integer NSF`
This variable specifies whether the fermion is particle or anti-particle. If `NSF = 1` the fermion is particle, and the subroutine computes the $u(p)$-spinor. If `NSF = -1` the fermion is anti-particle, and the subroutine computes the $v(p)$-spinor.

THE OUTPUT

I. `complex FO(6)`
This is a complex six-dimensional array which contains the Fermion's flowing-Out spinor combined with its four-momentum. The first two components `FO(1)` and `FO(2)` contain the chirality right part of the spinor,

$$(\mathtt{FO}(1),\, \mathtt{FO}(2)) = \bar{u}(p)\frac{1-\gamma_5}{2} \qquad \text{or} \qquad \bar{v}(p)\frac{1-\gamma_5}{2}. \tag{3.5}$$

The second two components `FI(3)` and `FI(4)` contain chirality left part of the spinor,

$$(\mathtt{FO}(3),\, \mathtt{FO}(4)) = \bar{u}(p)\frac{1+\gamma_5}{2} \qquad \text{or} \qquad \bar{v}(p)\frac{1+\gamma_5}{2}. \tag{3.6}$$

The last two components `FO(5)` and `FO(6)` contain the four-momentum along the fermion number flow,

$$(\mathtt{FO(5)},\mathtt{FO(6)}) = \mathtt{NSF}(\mathtt{P(0)} + i\mathtt{P(3)}, \mathtt{P(1)} + i\mathtt{P(2)}). \tag{3.7}$$

We denote the outputs of the subroutine symbolically as

$$\langle f|. \tag{3.8}$$

For more about the conventions of the spinors, see Appendix A.

3.1.3 VXXXXX

This subroutine computes the Vector particle wavefunction (polarization vector); namely, $\epsilon(p)$ and $\epsilon^*(p)$'s combined with its four-momentum.

This subroutine will be called as

```
CALL VXXXXX(P,VMASS,NHEL,NSV , VC)   .
```

We have four inputs `P,VMASS,NHEL,NSF` and one output `VC`.
THE INPUTS

I. `real P(0:3)`
This is a real four-dimensional array which contains the four-momentum p^μ of the vector boson. The four-momentum may be either time-like ($p^2 > 0$) for a massive vector or light-like ($p^2 = 0$) for a massless vector. The energy `P(0)` must be always positive.

II. `real VMASS`
This is a real variable which contains the mass of the `V`ector boson.

III. `integer NHEL`
This variable specifies the helicity of the vector boson. If `NHEL = 1` the helicity is $+1$, and if `NHEL = -1` the helicity is -1. The longitudinal polarization `NHEL = 0` is allowed only for the massive vector boson. There is a special option `NHEL=4` in `HELAS_CHECK.FOR`, which gives the "scalar polarization" as the output. This output is useful if one uses the BRS-invariance of the amplitude to check the program. However, this option is *not* supported in `HELAS.FOR`.

IV. `integer NSV`
This variable specifies whether the vector boson is final state or initial state particle. If `NSV = 1` the vector boson is in final state, and the subroutine computes the polarization vector $\epsilon(p)^*$. If `NSV = -1` the vector boson is in initial state, and the subroutine computes the polarization vector $\epsilon(p)$.

THE OUTPUT

I. `complex VC(6)`
This is a complex six-dimensional array which contains the `V`e`C`tor boson's polarization vector combined with its four-momentum. The first four components `VC(1)`, `VC(2)`, `VC(3)` and `VC(4)` contain the polarization vector,

$$(\texttt{VC(1)},\ \texttt{VC(2)},\ \texttt{VC(3)},\ \texttt{VC(4)}) = \epsilon^*(p), \tag{3.9}$$

for `NSV = 1`, and

$$(\texttt{VC(1)},\ \texttt{VC(2)},\ \texttt{VC(3)},\ \texttt{VC(4)}) = \epsilon(p), \tag{3.10}$$

for `NSV = -1`. The last two components `VC(5)` and `VC(6)` contain the flowing-out four-momentum,

$$(\texttt{VC(5)}, \texttt{VC(6)}) = \texttt{NSV}(\texttt{P(0)} + i\texttt{P(3)}, \texttt{P(1)} + i\texttt{P(2)}). \tag{3.11}$$

We denote the outputs of the subroutine symbolically as

$$V^\mu. \tag{3.12}$$

For more about the conventions of the polarization vectors, see Appendix A.

3.1.4 SXXXXX

This subroutine computes the '`S`'calar wavefunction of scalar boson, combined with its four-momentum. However, since scalar boson do not have any Lorentz structures, its wavefunction is simply unity.

This subroutine will be called as

```
CALL SXXXXX(P,NSS , SC)   .
```

We have two inputs `P,NSS` and one output `SC`.
THE INPUTS

I. `real P(0:3)`
This is a real four-dimensional array which contains the four-momentum p^μ of the scalar boson. The four-momentum may be either time-like ($p^2 > 0$) for a massive scalar or light-like ($p^2 = 0$) for a massless scalar. The energy `P(0)` must be always positive.

II. `integer NSS`
This variable specifies whether the scalar boson is final state or initial state particle. If `NSS = 1` the scalar boson is in final state, and if `NSS = -1` the scalar boson is in initial state. However, the wavefunction of an external scalar boson is simply unity, and the only difference between initial and final state particles is the sign of their four-momenta contained in the output.

THE OUTPUT

I. `complex SC(3)`
This is a complex three-dimensional array which contains the `SC`alar boson's wavefunction combined with its four-momentum. The first component `SC(1)` contain the wavefunction,

$$\mathtt{SC}(1) \equiv 1. \tag{3.13}$$

The last two components `SC(2)` and `SC(3)` contain the flowing-out four-momentum

$$\begin{pmatrix} \mathtt{SC}(2) \\ \mathtt{SC}(3) \end{pmatrix} = \mathtt{NSS} \begin{pmatrix} \mathtt{P}(0) + i\mathtt{P}(3) \\ \mathtt{P}(1) + i\mathtt{P}(2) \end{pmatrix}. \tag{3.14}$$

We denote the outputs of the subroutine symbolically as

$$S. \tag{3.15}$$

3.2 Vertices I: FFV vertex

This subgroup contains five subroutines.

```
IOVXXX  ,
FVIXXX  ,
FVOXXX  ,
JIOXXX  ,
J3XXXX  .
```

The `FFV` vertex computed by these subroutines are defined by

$$\mathcal{L}_{\mathtt{FFV}} = \bar{f}'\gamma^\mu \left(\mathtt{G}(1)\frac{1-\gamma_5}{2} + \mathtt{G}(2)\frac{1+\gamma_5}{2} \right) f V_\mu^*. \tag{3.16}$$

3.2.1 IOVXXX

The subroutine computes the amplitude of the `FFV` vertex from the flowing-`I`n fermion spinor, flowing-`O`ut fermion spinor and the `V`ector boson polarization vector.

This subroutine will be called as

```
CALL IOVXXX(FI,FO,VC,G , VERTEX)
```

We have four inputs `FI,FO,VC,G` and one output `VERTEX`.
THE INPUTS

I. `complex FI(6)`
This is a complex six-dimensional array which contains the wavefunction of the flowing-`I`n `F`ermion, and its four-momentum. The outputs of the subroutines `IXXXXX`, `FVIXXX`, `FSIXXX` or `EAIXXX` are suitable here.

II. `complex FO(6)`
This is a complex six-dimensional array which contains the wavefunction of the flowing-Out Fermion, and its four-momentum. The outputs of the subroutines OXXXXX, FVOXXX, FSOXXX or EAOXXX are suitable here.

III. `complex VC(6)`
This is a complex six-dimensional array which contains the wavefunction of the VeCtor boson, and its four-momentum. The output of the subroutines VXXXXX, JIOXXX, J3XXXX, JVVXXX, JVSXXX, JSSXXX, JWWWXX, JW3WXX, JVSSXX, or JEEXXX are suitable here.

IV. `real G(2)`
This is a real two-dimensional array which contains the coupling constant of the fermion with the vector boson. G(1) is the coupling of the chirality left fermion, and G(2) is the coupling of the chirality right fermion.

THE OUTPUT

I. `complex VERTEX`
This is a complex number which is the amplitude of the FFV vertex including the coupling constant.

What we compute here is the following T-matrix:

$$\mathtt{VERTEX} = (\mathtt{FO})\left[\not{V}\left(\mathtt{G}(1)\frac{1-\gamma_5}{2}+\mathtt{G}(2)\frac{1+\gamma_5}{2}\right)\right](\mathtt{FI}). \tag{3.17}$$

Here we used the notation

$$\begin{aligned}(\mathtt{FI}) &= \begin{pmatrix}\mathtt{FI}(1)\\ \mathtt{FI}(2)\\ \mathtt{FI}(3)\\ \mathtt{FI}(4)\end{pmatrix},\\ (\mathtt{FO}) &= (\mathtt{FO}(1),\, \mathtt{FO}(2),\, \mathtt{FO}(3),\, \mathtt{FO}(4)),\\ V^\mu &= \mathtt{V}(\mu+1).\end{aligned}$$

We denote the output of the subroutine symbolically as

$$\mathtt{VERTEX} = \langle f'|V|f\rangle. \tag{3.18}$$

3.2.2 FVIXXX

This subroutine computes an off-shell fermion wavefunction attached with the Fermion propagator from the interaction of a Vector boson, or a Vector current with the flowing-In fermion.

This subroutine will be called as

```
CALL FVIXXX(FI,VC,G,FMASS,FWIDTH , FVI)   .
```

We have five inputs FI,VC,G,FMASS,FWIDTH and one output FVI.

THE INPUTS

I. `complex FI(6)`
This is a complex six-dimensional array which contains the wavefunction of the flowing-In Fermion, and its four-momentum. The outputs of the subroutines IXXXXX, FVIXXX, FSIXXX or EAIXXX are suitable here.

II. `complex VC(6)`
This is a complex six-dimensional array which contains the wavefunction and the flowing-out four-momentum of the VeCtor boson, or VeCtor current. The output of the subroutines VXXXXX, JIOXXX, J3XXXX, JVVXXX, JVSXXX, JSSXXX, JWWWXX, JW3WXX, JVSSXX, or JEEXXX are suitable here.

III. `real G(2)`
This is a real two-dimensional array which contains the coupling constant of the fermion with the vector boson. `G(1)` is the coupling of the chirality left fermion, and `G(2)` is the coupling of the chirality right fermion.

IV. `real FMASS, FWIDTH`
These are the mass and the width of the output fermion. Note that the `FMASS` may be different from the mass of the input fermion, as in the case of the $W^{\pm}$ vertex. `FMASS` has to be non-negative.

THE OUTPUT

I. `complex FVI(6)`
This is a complex six-dimensional array including the off-shell `F`ermion wave function obtained from the flowing-`I`n fermion and the `V`ector boson. The fermion propagator is also attached.

What we compute here is the following portion of the Feynman amplitude:

$$(\mathtt{FVI}) = \frac{i(\not{p}+m)}{p^2 - m^2 + im\Gamma} \not{V} \left(i\mathtt{G(1)}\frac{1-\gamma_5}{2} + i\mathtt{G(2)}\frac{1+\gamma_5}{2} \right) (\mathtt{FI}), \tag{3.19}$$

and,

$$\mathtt{FVI}(5) = \mathtt{FI}(5) - \mathtt{VC}(5), \tag{3.20}$$
$$\mathtt{FVI}(6) = \mathtt{FI}(6) - \mathtt{VC}(6). \tag{3.21}$$

Here we used the notation

$$\begin{aligned}
(\mathtt{FVI}) &= \begin{pmatrix} \mathtt{FVI}(1) \\ \mathtt{FVI}(2) \\ \mathtt{FVI}(3) \\ \mathtt{FVI}(4) \end{pmatrix}. \\
(\mathtt{FI}) &= \begin{pmatrix} \mathtt{FI}(1) \\ \mathtt{FI}(2) \\ \mathtt{FI}(3) \\ \mathtt{FI}(4) \end{pmatrix}. \\
V^\mu &= \mathtt{VC}(\mu+1), \\
p^\mu &= (\Re e\mathtt{FVI}(5), \Re e\mathtt{FVI}(6), \Im m\mathtt{FVI}(6), \Im m\mathtt{FVI}(5)) \\
m &= \mathtt{FMASS}, \\
\Gamma &= \mathtt{FWIDTH},
\end{aligned}$$

We denote the outputs of the subroutine symbolically as

$$|FVI\rangle = |f', V, f\rangle. \tag{3.22}$$

3.2.3 FVOXXX

This subroutine computes an off-shell fermion wavefunction attached with the `F`ermion propagator from the interaction of a `V`ector boson, or a `V`ector current with the flowing-`O`ut fermion.

This subroutine will be called as

```
CALL FVOXXX(FO,VC,G,FMASS,FWIDTH , FVO)   .
```

We have five inputs `FO,VC,G,FMASS,FWIDTH` and one output `FVO`.
THE INPUTS

I. `complex FO(6)`
This is a complex six-dimensional array which contains the wavefunction of the flowing-`Out` `F`ermion, and its four-momentum. The outputs of the subroutines `OXXXXX`, `FVOXXX`, `FSOXXX` or `EAOXXX` are suitable here.

II. `complex VC(6)`
This is a complex six-dimensional array which contains the wavefunction and the flowing-out four-momentum of the `VeC`tor boson, or `VeC`tor current. The output of the subroutines `VXXXXX`, `JIOXXX`, `J3XXXX`, `JVVXXX`, `JVSXXX`, `JSSXXX`, `JWWWXX`, `JW3WXX`, `JVSSXX`, or `JEEXXX` are suitable here.

III. `real G(2)`
This is a real two-dimensional array which contains the coupling constant of the fermion with the vector boson. `G(1)` is the coupling of the chirality left fermion, and `G(2)` is the coupling of the chirality right fermion.

IV. `real FMASS, FWIDTH`
This is the mass and the width of the output fermion. Note that the `FMASS` may be different from the mass of the input fermion, as in the case of the $W^{\pm}$ vertex. `FMASS` has to be non-negative.

THE OUTPUT

I. `complex FVO(6)`
This is a complex six-dimensional array consists of the off-shell `F`ermion wave function and its four momentum made from a flowing-`Out` fermion and a `V`ector boson.

What we compute here is the following portion of the Feynman amplitude:

$$(\mathtt{FVO}) = (\mathtt{FO})\ \not{V}\left(i\mathtt{G}(1)\frac{1-\gamma_5}{2} + i\mathtt{G}(2)\frac{1+\gamma_5}{2}\right)\frac{i(\not{p}+m)}{p^2-m^2+im\Gamma}, \tag{3.23}$$

and,

$$\begin{aligned} \mathtt{FVO}(5) &= \mathtt{FO}(5) + \mathtt{VC}(5) & (3.24)\\ \mathtt{FVO}(6) &= \mathtt{FO}(6) + \mathtt{VC}(6). & (3.25) \end{aligned}$$

Here we used the notation

$$\begin{aligned} (\mathtt{FVO}) &= (\mathtt{FVO}(1),\ \mathtt{FVO}(2),\ \mathtt{FVO}(3),\ \mathtt{FVO}(4)),\\ (\mathtt{FO}) &= (\mathtt{FO}(1),\ \mathtt{FO}(2),\ \mathtt{FO}(3),\ \mathtt{FO}(4)),\\ m &= \mathtt{FMASS},\\ p^\mu &= (\Re e\mathtt{FVO}(5),\ \Re e\mathtt{FVO}(6),\ \Im m\mathtt{FVO}(6),\ \Im m\mathtt{FVO}(5))\\ \Gamma &= \mathtt{FWIDTH},\\ V^\mu &= \mathtt{VC}(\mu+1). \end{aligned}$$

We denote the outputs of the subroutine symbolically as

$$\langle FVO| = \langle f, V, f'|. \tag{3.26}$$

3.2.4 JIOXXX

This subroutine computes the bi-spinor vector current J attached with the vector boson propagator from the flowing-In and flowing-Out fermions by a FFV vertex. The gauge of the propagator is taken to be the *unitary gauge* for the massive vector boson, and the *Feynman gauge* for the massless gauge boson.

This subroutine will be called as

```
CALL JIOXXX(FI,FO,G,VMASS,VWIDTH , JIO) .
```

We have five inputs FI,FO,G,VMASS,VWIDTH and one output JIO.

THE INPUTS

I. `complex FI(6)`
   This is a complex six-dimensional array which contains the wavefunction of the flowing-In Fermion, and its four-momentum. The outputs of the subroutines IXXXXX, FVIXXX, FSIXXX or EAIXXX are suitable here.

II. `complex FO(6)`
   This is a complex six-dimensional array which contains the wavefunction of the flowing-Out Fermion, and its four-momentum. The outputs of the subroutines OXXXXX, FVOXXX, FSOXXX or EAOXXX are suitable here.

III. `real G(2)`
   This is a real two-dimensional array which contains the coupling constant of the fermion with the vector boson. G(1) is the coupling of the chirality left fermion, and G(2) is the coupling of the chirality right fermion.

IV. `real VMASS, VWIDTH`
   These are real variables which contain the mass and the width of the vector boson, respectively. VMASS has to be positive, and VWIDTH has to be non-negative.

THE OUTPUT

I. `complex JIO(6)`
   This is a complex six-dimensional array which contains the bi-spinor vector current attached with the massive vector boson propagator in the unitary gauge, combined with its four-momentum.

What we compute here is the following portion of the Feynman amplitude:

$$\begin{aligned}\mathtt{JIO}(\mu+1) &= \frac{i}{q^2-m^2+im\Gamma}\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{m^2}\right) \\ &\quad \times(\mathtt{FO})\gamma_\nu\left(i\mathtt{G}(1)\frac{1-\gamma_5}{2}+i\mathtt{G}(2)\frac{1+\gamma_5}{2}\right)(\mathtt{FI}), \end{aligned} \tag{3.27}$$

for the massive gauge boson, or,

$$\mathtt{JIO}(\mu+1)=\frac{-i}{q^2}(\mathtt{FO})\gamma^\mu\left(i\mathtt{G}(1)\frac{1-\gamma_5}{2}+i\mathtt{G}(2)\frac{1+\gamma_5}{2}\right)(\mathtt{FI}), \tag{3.28}$$

for the massless gauge boson, and

$$\mathtt{JIO}(5) = -\mathtt{FI}(5)+\mathtt{FO}(5), \tag{3.29}$$

$$\mathtt{JIO}(6) = -\mathtt{FI}(6)+\mathtt{FO}(6). \tag{3.30}$$

Here we used the notation

$$\begin{aligned}
(\texttt{FO}) &= (\texttt{FO(1)}, \texttt{FO(2)}, \texttt{FO(3)}, \texttt{FO(4)}),\\
(\texttt{FI}) &= \begin{pmatrix} \texttt{FI(1)} \\ \texttt{FI(2)} \\ \texttt{FI(3)} \\ \texttt{FI(4))} \end{pmatrix},\\
q^\mu &= (\Re e\texttt{JIO(5)}, \Re e\texttt{JIO(6)}, \Im m\texttt{JIO(6)}, \Im m\texttt{JIO(5)})\\
m &= \texttt{VMASS},\\
\Gamma &= \texttt{VWIDTH}.
\end{aligned}$$

We denote the output of the subroutine symbolically as

$$J_V^\mu(\langle f'|, |f\rangle). \tag{3.31}$$

3.2.5 J3XXXX

This subroutine computes the weighted sum of the photon current J_A and Z boson current J_Z: $J_3 = \cos\theta_W J_Z + \sin\theta_W J_A$, to be used as the input of the subroutines `VVVXXX` or `W3W3XX`.[2] [3]

This subroutine will be called as

```
CALL J3XXXX(FI,FO,GAF,GZF,ZMASS,ZWIDTH , J3)   .
```

We have six inputs `FI,FO,GAF,GZF,ZMASS,ZWIDTH` and one output `J3`.

THE INPUTS

I. `complex FI(6)`
This is a complex six-dimensional array which contains the wavefunction of the flowing-In Fermion, and its four-momentum. The outputs of the subroutines `IXXXXX`, `FVIXXX`, `FSIXXX` or `EAIXXX` are suitable here.

II. `complex FO(6)`
This is a complex six-dimensional array which contains the wavefunction of the flowing-Out Fermion, and its four-momentum. The outputs of the subroutines `OXXXXX`, `FVOXXX`, `FSOXXX` or `EAOXXX` are suitable here.

III. `real GAF(2),GZF(2)`
These are real two-dimensional arrays which contain the coupling constants of the fermion with the photon and the Z boson, respectively. `GAF(1),GZF(1)` are the couplings of the chirality left fermion, and `GAF(2),GZF(2)` are the couplings of the chirality right fermion.

IV. `real ZMASS, ZWIDTH`
These are real variables which contain the mass and the width of the Z boson. `ZMASS` has to be positive, and `ZWIDTH` has to be non-negative.

THE OUTPUT

[2]To combine the photon and Z current is more than a matter of convenience. In the Standard Model, the gauge theory cancellation between the photon and Z current occurs in the right-handed current, since in the high-energy limit only the hypercharge gauge boson B_μ couples to the right-handed fermions and the W_μ^3 component decouples. This cancellation is treated carefully in this subroutine.

[3]Note that one *cannot* use this subroutine for the neutral fermions like neutrinos. For the Z current of the neutrino, use `JIOXXX` subroutine. Then the output can be used as the input of `VVVXXX` or `W3W3XX` by setting the coupling constant to `GWWZ` rather than `GW`.

I. `complex J3(6)`
This is a complex six-dimensional array which contains the bi-spinor vector current attached with the Z boson propagator in the unitary gauge and the photon propagator in the Feynman gauge.

What we compute here is the following portion of the Feynman amplitude:

$$\begin{aligned}
\mathtt{J3}(\mu) &= \cos\theta_W \frac{i}{q^2 - m^2 + im\Gamma}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{m^2}\right) \\
&\quad\times(\mathtt{FO})\gamma_\nu\left(-i\mathtt{GZF(1)}\frac{1-\gamma_5}{2} - i\mathtt{GZF(2)}\frac{1+\gamma_5}{2}\right)(\mathtt{FI}) \\
&\quad+\sin\theta_W \frac{i}{q^2}\left(-g^{\mu\nu}\right)(\mathtt{FO})(-ieQ\gamma_\nu)(\mathtt{FI}) \\
&= \left(\frac{-1}{q^2 - m^2 + im\Gamma}\mathtt{GZF(1)} + \frac{-1}{q^2}eQ\right)J_L^\mu \\
&\quad-\frac{-1}{q^2 - m^2 + im\Gamma}\frac{q^\mu}{m^2}(\mathtt{GZF(1)}(q\cdot J_L) + \mathtt{GZF(2)}(q\cdot J_R)) \\
&\quad+eQ\sin\theta_W \frac{m^2 - im\Gamma}{q^2(q^2 - m^2 + im\Gamma)}J_R^\mu,
\end{aligned} \tag{3.32}$$

and,

$$\mathtt{J3(5)} = -\mathtt{FI(5)} + \mathtt{FO(5)}, \tag{3.33}$$

$$\mathtt{J3(6)} = -\mathtt{FI(6)} + \mathtt{FO(6)}. \tag{3.34}$$

Here we used the notation

$$\begin{aligned}
(\mathtt{FO}) &= (\mathtt{FO(1)},\, \mathtt{FO(2)},\, \mathtt{FO(3)},\, \mathtt{FO(4)}), \\
(\mathtt{FI}) &= \begin{pmatrix} \mathtt{FI(1)} \\ \mathtt{FI(2)} \\ \mathtt{FI(3)} \\ \mathtt{FI(4)} \end{pmatrix}, \\
q^\mu &= (\Re e\mathtt{J3(5)},\, \Re e\mathtt{J3(6)},\, \Im m\mathtt{J3(6)},\, \Im m\mathtt{J3(5)}), \\
m &= \mathtt{ZMASS}, \\
\Gamma &= \mathtt{ZWIDTH},
\end{aligned}$$

and

$$\begin{aligned}
J_L^\mu &= (\mathtt{FO})\gamma^\mu\frac{1-\Gamma_5}{2}(\mathtt{FI}), \\
J_R^\mu &= (\mathtt{FO})\gamma^\mu\frac{1+\Gamma_5}{2}(\mathtt{FI}),
\end{aligned}$$

We denote the output of the subroutine symbolically as

$$J_{W^3}^\mu(\langle f'|, |f\rangle). \tag{3.35}$$

3.3 Vertices II: `FFS` vertex

This subgroup contains four subroutines.

```
IOSXXX  ,
FSIXXX  ,
FSOXXX  ,
HIOXXX  .
```

The FFS vertex computed by these subroutines are defined by

$$\mathcal{L}_{\texttt{FFS}} = \bar{f}' \left(\texttt{GC}(1) \frac{1-\gamma_5}{2} + \texttt{GC}(2) \frac{1+\gamma_5}{2} \right) f S^*. \tag{3.36}$$

3.3.1 IOSXXX

The subroutine computes the amplitude of the FFS vertex from the flowing-In fermion spinor, flowing-Out fermion spinor and the Scalar boson wavefunction.

This subroutine will be called as

```
CALL IOSXXX(FI,FO,SC,GC , VERTEX)
```

We have four inputs FI,FO,SC,GC and one output VERTEX.

THE INPUTS

I. `complex FI(6)`
This is a complex six-dimensional array which contains the wavefunction of the flowing-In Fermion, and its four-momentum. The outputs of the subroutines IXXXXX, FVIXXX, FSIXXX or EAIXXX are suitable here.

II. `complex FO(6)`
This is a complex six-dimensional array which contains the wavefunction of the flowing-Out Fermion, and its four-momentum. The outputs of the subroutines OXXXXX, FVOXXX, FSOXXX or EAOXXX are suitable here.

III. `complex SC(3)`
This is a complex three-dimensional array which contains the wavefunction of the SCalar boson, and its four-momentum. The outputs of the subroutines SXXXXX, HIOXXX, HVVXXX, HVSXXX, HSSXXX, HVVSXX or HSSSXX are suitable here.

IV. `complex GC(2)`
This is a Complex two-dimensional array which contains the coupling constant of the fermion with the scalar boson. GC(1) is the coupling of the chirality left fermion and the chirality right anti-fermion, and GC(2) is the coupling of the chirality right fermion and the chirality left anti-fermion.

THE OUTPUT

I. `complex VERTEX`
This is a complex number which is the amplitude of the FFS vertex including the coupling constant.

What we compute here is the following T-matrix:

$$\texttt{VERTEX} = (\texttt{FO}) \left[\texttt{SC}(1) \left(\texttt{GC}(1) \frac{1-\gamma_5}{2} + \texttt{GC}(2) \frac{1+\gamma_5}{2} \right) \right] (\texttt{FI}). \tag{3.37}$$

Here we used the notation

$$\begin{aligned} (\texttt{FI}) &= \begin{pmatrix} \texttt{FI}(1) \\ \texttt{FI}(2) \\ \texttt{FI}(3) \\ \texttt{FI}(4) \end{pmatrix}, \\ (\texttt{FO}) &= (\texttt{FO}(1), \texttt{FO}(2), \texttt{FO}(3), \texttt{FO}(4)). \end{aligned}$$

We denote the outputs of the subroutine symbolically as

$$\texttt{VERTEX} = \langle f' | S | f \rangle. \tag{3.38}$$

3.3.2 FSIXXX

This subroutine computes an off-shell fermion wavefunction attached with the Fermion propagator made from the interaction of a Scalar boson, or a Scalar current with the flowing-In fermion.

This subroutine will be called as

```
CALL FSIXXX(FI,SC,GC,FMASS,FWIDTH , FSI)   .
```

We have five inputs FI,SC,GC,FMASS,FWIDTH and one output FSI.

THE INPUTS

I. complex FI(6)
This is a complex six-dimensional array which contains the wavefunction of the flowing-In Fermion, and its four-momentum. The outputs of the subroutines IXXXXX, FVIXXX, FSIXXX or EAIXXX are suitable here.

II. complex SC(3)
This is a complex three-dimensional array which contains the wavefunction of the SCalar boson, and its four-momentum. The outputs of the subroutines SXXXXX, HIOXXX, HVVXXX, HVSXXX, HSSXXX, HVVSXX or HSSSXX are suitable here.

III. complex GC(2)
This is a Complex two-dimensional array which contains the coupling constant of the fermion with the scalar boson. GC(1) is the coupling of the chirality left fermion and the chirality right anti-fermion, and GC(2) is the coupling of the chirality right fermion and the chirality left anti-fermion.

IV. real FMASS, FWIDTH
These are the mass and the width of the output fermion. Note that the FMASS may be different from the mass of the input fermion, as in the case of the charged Higgs boson vertex. FMASS has to be non-negative.

THE OUTPUT

I. complex FSI(6)
This is a complex six-dimensional array which contains the off-shell Fermion wave function and its momentum made from the flowing-In fermion and the Scalar boson. The fermion propagator is also attached.

What we compute here is the following portion of the Feynman amplitude:

$$(\mathtt{FSI}) = \frac{i(\not{p}+m)}{p^2-m^2+im\Gamma}\mathtt{SC}(1)\left(i\mathtt{GC}(1)\frac{1-\gamma_5}{2}+i\mathtt{GC}(2)\frac{1+\gamma_5}{2}\right)(\mathtt{FI}), \tag{3.39}$$

and,

$$\mathtt{FSI}(5) = \mathtt{FI}(5) - \mathtt{SC}(2), \tag{3.40}$$

$$\mathtt{FSI}(6) = \mathtt{FI}(6) - \mathtt{SC}(3). \tag{3.41}$$

Here we used the notation

$$(\mathtt{FSI}) = \begin{pmatrix} \mathtt{FSI}(1) \\ \mathtt{FSI}(2) \\ \mathtt{FSI}(3) \\ \mathtt{FSI}(4) \end{pmatrix}.$$

$$(\mathtt{FI}) = \begin{pmatrix} \mathtt{FI}(1) \\ \mathtt{FI}(2) \\ \mathtt{FI}(3) \\ \mathtt{FI}(4) \end{pmatrix}.$$

$$\begin{aligned} p^\mu &= (\Re e\texttt{FSI(5)},\, \Re e\texttt{FSI(6)},\, \Im m\texttt{FSI(6)},\, \Im m\texttt{FSI(5)}), \\ m &= \texttt{FMASS}, \\ \Gamma &= \texttt{FWIDTH}. \end{aligned}$$

We denote the outputs of the subroutine symbolically as

$$|FSI\rangle = |f', S, f\rangle. \tag{3.42}$$

3.3.3 FSOXXX

This subroutine computes an off-shell fermion wavefunction attached with the Fermion propagator from the interaction of a Scalar boson, or a Scalar current with the flowing-Out fermion.

This subroutine will be called as

```
CALL FSOXXX(FO,SC,GC,FMASS,FWIDTH , FSO)   .
```

We have five inputs `FO,SC,GC,FMASS,FWIDTH` and one output `FSO`.

THE INPUTS

I. `complex FO(6)`
This is a complex six-dimensional array which contains the wavefunction of the flowing-Out Fermion, and its four-momentum. The outputs of the subroutines `OXXXXX`, `FVOXXX`, `FSOXXX` or `EAOXXX` are suitable here.

II. `complex SC(3)`
This is a complex three-dimensional array which contains the wavefunction of the SCalar boson, and its four-momentum. The outputs of the subroutines `SXXXXX`, `HIOXXX`, `HVVXXX`, `HVSXXX`, `HSSXXX`, `HVVSXX` or `HSSSXX` are suitable here.

III. `complex GC(2)`
This is a Complex two-dimensional array which contains the coupling constant of the fermion with the scalar boson. `GC(1)` is the coupling of the chirality left fermion and the chirality right anti-fermion, and `GC(2)` is the coupling of the chirality right fermion and the chirality left anti-fermion.

IV. `real FMASS, FWIDTH`
These are the mass and the width of the output fermion. Note that the `FMASS` may be different from the mass of the input fermion, as in the case of the charged Higgs boson vertex. `FMASS` has to be non-negative.

THE OUTPUT

I. `complex FSO(6)`
This is a complex six-dimensional array which contains the off-shell Fermion wave function and its momentum, made from the flowing-Out fermion and the Scalar boson. The fermion propagator is also attached.

What we compute here is the following portion of the Feynman amplitude:

$$(\texttt{FSO}) = (\texttt{FO})\texttt{SC(1)}\left(i\texttt{G(1)}\frac{1+\gamma_5}{2} + i\texttt{G(2)}\frac{1-\gamma_5}{2}\right)\frac{i(\not{p}+m)}{p^2 - m^2 + im\Gamma}, \tag{3.43}$$

and,

$$\begin{aligned} \texttt{FSO(5)} &= \texttt{FO(5)} + \texttt{SC(2)}, & (3.44) \\ \texttt{FSO(6)} &= \texttt{FO(6)} + \texttt{SC(3)}. & (3.45) \end{aligned}$$

Here we used the notation

$$\begin{aligned}
(\mathtt{FO}) &= (\mathtt{FO}(1), \mathtt{FO}(2), \mathtt{FO}(3), \mathtt{FO}(4)), \\
(\mathtt{FSO}) &= (\mathtt{FSO}(1), \mathtt{FSO}(2), \mathtt{FSO}(3), \mathtt{FSO}(4)), \\
p^\mu &= (\Re e\mathtt{FSO}(5), \Re e\mathtt{FSO}(6), \Im m\mathtt{FSO}(6), \Im m\mathtt{FSO}(5)), \\
m &= \mathtt{FMASS}, \\
\Gamma &= \mathtt{FWIDTH}.
\end{aligned}$$

We denote the outputs of the subroutine symbolically as

$$\langle FSO| = \langle f, S, f'|. \tag{3.46}$$

3.3.4 HIOXXX

This subroutine computes the bi-spinor scalar current `H` attached with the vector boson propagator from the flowing-`In` and flowing-`Out` fermions by a `FFS` vertex.

This subroutine will be called as

```
CALL HIOXXX(FI,FO,GC,SMASS,SWIDTH , HIO) .
```

We have five inputs `FI,FO,GC,SMASS,SWIDTH` and one output `HIO`.

The Inputs

I. `complex FI(6)`
This is a complex six-dimensional array which contains the wavefunction of the flowing-`In` Fermion, and its four-momentum. The outputs of the subroutines `IXXXXX`, `FVIXXX`, `FSIXXX` or `EAIXXX` are suitable here.

II. `complex FO(6)`
This is a complex six-dimensional array which contains the wavefunction of the flowing-`Out` Fermion, and its four-momentum. The outputs of the subroutines `OXXXXX`, `FVOXXX`, `FSOXXX` or `EAOXXX` are suitable here.

III. `complex GC(2)`
This is a `C`omplex two-dimensional array which contains the coupling constant of the fermion with the scalar boson. `GC(1)` is the coupling of the chirality left fermion and the chirality right anti-fermion, and `GC(2)` is the coupling of the chirality right fermion and the chirality left anti-fermion.

IV. `real SMASS, SWIDTH`
These are real variables which contain the mass and the width of the scalar boson, respectively. `SMASS` has to be positive, and `SWIDTH` has to be non-negative.

The Output

I. `complex HIO(3)`
This is a complex three-dimensional array which contains the bi-spinor scalar current attached with the scalar boson propagator, combined with its four-momentum.

What we compute here is the following portion of the Feynman amplitude:

$$\mathtt{HIO}(1) = \frac{i}{q^2 - m^2 + im\Gamma} \times (\mathtt{FO}) \left(i\mathtt{GC}(1)\frac{1-\gamma_5}{2} + i\mathtt{GC}(2)\frac{1+\gamma_5}{2} \right) (\mathtt{FI}), \tag{3.47}$$

and,

$$\begin{aligned}
\mathtt{HIO}(2) &= -\mathtt{FI}(5) + \mathtt{FO}(5), && (3.48)\\
\mathtt{HIO}(3) &= -\mathtt{FI}(6) + \mathtt{FO}(6). && (3.49)
\end{aligned}$$

Here we used the notation

$$
\begin{aligned}
(\texttt{FO}) &= (\texttt{FO}(1),\, \texttt{FO}(2),\, \texttt{FO}(3),\, \texttt{FO}(4)),\\
(\texttt{FI}) &= \begin{pmatrix} \texttt{FI}(1) \\ \texttt{FI}(2) \\ \texttt{FI}(3) \\ \texttt{FI}(4)) \end{pmatrix},\\
q^\mu &= (\Re e\texttt{HIO}(2),\, \Re e\texttt{HIO}(3),\, \Im m\texttt{HIO}(3),\, \Im m\texttt{HIO}(2)),\\
m &= \texttt{SMASS},\\
\Gamma &= \texttt{SWIDTH},
\end{aligned}
$$

We denote the output of the subroutine symbolically as

$$J_S(\langle f'|, |f\rangle). \tag{3.50}$$

3.4 Vertices III: VVV vertex

This subgroup contains four subroutines.

```
VVVXXX  ,
JVVXXX  ,
GGGXXX  ,
JGGXXX  .
```

The VVV vertex computed by these subroutines are defined by

$$
\begin{aligned}
\mathcal{L}_{\texttt{VVV}} = -i\texttt{G} \;\{ & (\partial_\mu V^*_{1\nu})(V_2^{\mu *}V_3^{\nu *} - V_2^{\nu *}V_3^{\mu *}) \\
& +(\partial_\mu V^*_{2\nu})(V_3^{\mu *}V_1^{\nu *} - V_3^{\nu *}V_1^{\mu *}) \\
& +(\partial_\mu V^*_{3\nu})(V_1^{\mu *}V_2^{\nu *} - V_1^{\nu *}V_2^{\mu *})\}\,.
\end{aligned} \tag{3.51}
$$

3.4.1 VVVXXX

This subroutine computes the three Vector-boson vertex from the polarization vectors or the vector currents. Though this subroutine can be used for any type of the non-Abelian gauge three-point vertex, we refer to the electroweak gauge bosons $W^\pm$, Z and γ in the followings. If you wish to use this subroutine for other non-Abelian gauge bosons like gluons, you have to multiply the vertex with an appropriate group theory factor by hand.

This subroutine will be called as

```
CALL VVVXXX(WM,WP,W3,G , VERTEX)   .
```

We have four inputs `WM,WP,W3,G` and one output `VERTEX`.

THE INPUTS

I. `complex WM(6),WP(6)`
These are complex six-dimensional arrays which contain the wavefunctions and the flowing-out four-momenta of the *flowing-out* W^-, W^+ boson (W-Minus, W-Plus), or vector current of the W^-, W^+ boson, respectively. The output of the subroutines VXXXXX, JIOXXX, JVVXXX, JVSXXX, JSSXXX, JWWWXX, JW3WXX, or JVSSXX are suitable here. Note that the order of WM and WP should be preserved to be consistent with the $SU(2)$ group theory factor appearing in the FFV vertices. Since the quantum numbers are defined by `flowing-out` charges, W^+ boson which flows into the diagram is regarded as flowing-out W^-, and vice versa.

II. `complex W3(6)`
This is a complex six-dimensional array which contains the wavefunction and the flowing-out four-momentum of the Z or A boson (photon), or vector current of the Z, A boson. Since only one linear combination of these vector bosons (namely $Z\cos\theta_W + A\sin\theta_W$) interacts with the $W^\pm$, we generically call this input as the third component of the $SU(2)$ gauge boson W^3. If the input is actually the Z boson, then the coupling should be GWWZ$= g_W\cos\theta_W$, while it should be GWWA$= e = g_W\sin\theta_W$ for photon. If the input is already a linear combination $J_{W^3} = \cos\theta_W J_Z + \sin\theta_W J_A$ like the output of the J3XXXX subroutine, then the $SU(2)$ gauge coupling GW$= g_W$ is appropriate. The output of the subroutines VXXXXX, JIOXXX, J3XXXX, JVVXXX, JVSXXX, JSSXXX, JW3WXX, JVSSXX, or JEEXXX are suitable here.

III. `real G`
This is a real variable which contains the coupling constant of the $W^\pm$ and W^3 boson. If the W^3 boson is the output of the subroutine J3XXXX, G must be the weak gauge coupling GW $(= e/\sin\theta_W)$. If the W^3 boson is the external Z boson or the output of the JIOXXX of the neutrino current, then G must be GWWZ=GW$*\cos\theta_W$. If the W^3 boson is a photon, G must be GWWA $(=$ GW $*\sin\theta_W = e)$.

The Output

I. `complex VERTEX`
This is a complex variable which contains the vertex function of the W^-, W^+ and W^3 (or Z or photon).

What we compute here is the following T-matrix:

$$\begin{aligned}\texttt{VERTEX} &= -\texttt{G}\{((p^- - p^+)\cdot V^3)(V^-\cdot V^+) + ((p^+ - p^3)\cdot V^-)(V^+\cdot V^3) \\ &\quad +((p^3 - p^-)\cdot V^+)(V^3\cdot V^-)\},\end{aligned} \tag{3.52}$$

where we used the notation

$$\begin{aligned}V_\mu^- &= \texttt{WM}(\mu+1)\\ V_\mu^+ &= \texttt{WP}(\mu+1)\\ V_\mu^3 &= \texttt{W3}(\mu+1)\\ p_\mu^- &= (\Re e\texttt{WM(5)},\ \Re e\texttt{WM(6)},\ \Im m\texttt{WM(6)},\ \Im m\texttt{WM(5)})\\ p_\mu^+ &= (\Re e\texttt{WP(5)},\ \Re e\texttt{WP(6)},\ \Im m\texttt{WP(6)},\ \Im m\texttt{WP(5)})\\ p_\mu^3 &= (\Re e\texttt{W3(5)},\ \Re e\texttt{W3(6)},\ \Im m\texttt{W3(6)},\ \Im m\texttt{W3(5)}).\end{aligned}$$

However, there is a gauge theory cancellation between the three terms, which can be hardly handled within the single-precision subroutine. Fortunately, there is an identical expression which can avoid such a cancellation at the high-energy:

$$\begin{aligned}\texttt{VERTEX} = -\texttt{G}\ &\{((p^- - \alpha_- V^- - p^+ + \alpha_+ V^+)\cdot V^3)(V^-\cdot V^+)\\ &+((p^+ - \alpha_+ V^+ - p^3 + \alpha_3 V^3)\cdot V^-)(V^+\cdot V^3)\\ &+((p^3 - \alpha_3 V^3 - p^- + \alpha_- V^-)\cdot V^+)(V^3\cdot V^-)\},\end{aligned} \tag{3.53}$$

for any set of the real numbers α_-, α_+, α_3. [4] Our choice for the α's are

$$\alpha_- = p_0^-/V_0^-, \tag{3.54}$$

$$\alpha_+ = p_0^+/V_0^+, \tag{3.55}$$

$$\alpha_3 = p_0^3/V_0^3. \tag{3.56}$$

[4] We thank H. Iwasaki for this idea.

which significantly reduces the numerical values of the scalar and longitudinal component of the vector current.[5]

We denote the output of the subroutine symbolically as

$$\Gamma(V^-, V^+, V^3). \tag{3.57}$$

3.4.2 JVVXXX

This subroutine computes an off-shell vector current from the VVV vertex attached with the vector boson propagator, from the two Vector boson polarization vectors.

This subroutine will be called as

```
CALL JVVXXX(V1,V2,G,VMASS,VWIDTH , JVV)
```

We have five inputs V1,V2,G,VMASS,VWIDTH and one output JVV.

THE INPUTS

I. complex V1(6),V2(6)
These are complex six-dimensional arrays which contain the wavefunctions of the Vector bosons, and their four-momenta. The output of the subroutines VXXXXX, JIOXXX, J3XXXX, JVVXXX, JVSXXX, JSSXXX, JWWWXX, JW3WXX, JVSSXX, or JEEXXX are suitable here.

II. real G
This is a real coupling constant of the VVV vertex. It should be GW for $W^-W^+W^3$ coupling, GWWZ for W^-W^+Z coupling, and GWWA for W^-W^+A coupling.

III. real VMASS,VWIDTH
The mass and the width of the vector boson emitted from the VVV vertex.

THE OUTPUT

I. complex JVV(6)
This is a complex six-dimensional array which contains the off-shell vector current from the VVV vertex attached with the vector boson propagator, combined with its four-momentum.

The possible choice of the inputs and outputs are:

V1	V2	JVV	G	VMASS	VWIDTH
W^-	W^+	A/Z	GWWA/GWWZ	0./ZMASS	0./ZWIDTH
$W^3/A/Z$	W^-	W^+	GW/GWWA/GWWZ	WMASS	WWIDTH
W^+	$W^3/A/Z$	W^-	GW/GWWA/GWWZ	WMASS	WWIDTH

What we compute is the following portion of the Feynman amplitude:

$$J^\mu = i\mathtt{G}\frac{-i}{s-m^2+im\Gamma}(J_{12}^\mu - q^\mu J_S), \tag{3.58}$$

where,

$$J_{12}^\mu = (p_1-p_2)^\mu (V_1\cdot V_2) + ((p_2-q)\cdot V_1)\, V_2^\mu + ((q-p_1)\cdot V_2)\, V_1^\mu, \tag{3.59}$$

$$J_S = \frac{1}{m^2}\Big((-p_1^2+p_2^2)(V_1\cdot V_2) + (p_1\cdot V_1)(p_1\cdot V_2) - (p_2\cdot V_1)(p_2\cdot V_2)\Big), \tag{3.60}$$

[5]We thank D. Zeppenfeld for this idea.

and,

$$\texttt{JVV(5)} = \texttt{V1(5)} + \texttt{V2(5)}, \tag{3.61}$$
$$\texttt{JVV(6)} = \texttt{V1(6)} + \texttt{V2(6)}. \tag{3.62}$$

Here we used the notation

$$J^\mu = \texttt{JVV}(\mu + 1), \tag{3.63}$$
$$V_1^\mu = \texttt{V1}(\mu + 1), \tag{3.64}$$
$$V_2^\mu = \texttt{V2}(\mu + 1), \tag{3.65}$$
$$p_1^\mu = (\Re e\texttt{V1(5)},\, \Re e\texttt{V1(6)},\, \Im m\texttt{V1(6)},\, \Im m\texttt{V1(5)}), \tag{3.66}$$
$$p_2^\mu = (\Re e\texttt{V2(5)},\, \Re e\texttt{V2(6)},\, \Im m\texttt{V2(6)},\, \Im m\texttt{V2(5)}), \tag{3.67}$$
$$q^\mu = (\Re e\texttt{JVV(5)},\, \Re e\texttt{JVV(6)},\, \Im m\texttt{JVV(6)},\, \Im m\texttt{JVV(5)}), \tag{3.68}$$
$$m = \texttt{VMASS}, \tag{3.69}$$
$$\Gamma = \texttt{VWIDTH}. \tag{3.70}$$

We denote the output of the subroutine symbolically as

$$J^\mu_{V'}(V_1,\, V_2). \tag{3.71}$$

3.4.3 GGGXXX

This subroutine computes the three Vector-boson vertex from the polarization vectors or the vector currents. Though this subroutine can be used for any type of the non-Abelian gauge three-point vertex, we refer to the electroweak gauge bosons $W^\pm$, Z and γ in the followings. If you wish to use this subroutine for other non-Abelian gauge bosons like gluons, you have to multiply the vertex with an appropriate group theory factor by hand.

This subroutine will be called as

```
CALL GGGXXX(GA,GB,GC,G , VERTEX)   .
```

We have four inputs `GA,GB,GC,G` and one output `VERTEX`.

THE INPUTS

I. `complex GA(6),GB(6)`
These are complex six-dimensional arrays which contain the wavefunctions and the flowing-out four-momenta of the *flowing-out* W^-, W^+ boson (`W`-Minus, `W`-Plus), or vector current of the W^-, W^+ boson, respectively. The output of the subroutines `VXXXXX`, `JIOXXX`, `JVVXXX`, `JVSXXX`, `JSSXXX`, `JWWWXX`, `JW3WXX`, or `JVSSXX` are suitable here. Note that the order of `WM` and `WP` should be preserved to be consistent with the $SU(2)$ group theory factor appearing in the `FFV` vertices. Since the quantum numbers are defined by `flowing-out` charges, W^+ boson which flows into the diagram is regarded as flowing-out W^-, and vice versa.

II. `complex GC(6)`
This is a complex six-dimensional array which contains the wavefunction and the flowing-out four-momentum of the Z or A boson (photon), or vector current of the Z, A boson. Since only one linear combination of these vector bosons (namely $Z\cos\theta_W + A\sin\theta_W$) interacts with the $W^\pm$, we generically call this input as the third component of the $SU(2)$ gauge boson W^3. If the input is actually the Z boson, then the coupling should be `GWWZ`$= g_W\cos\theta_W$, while it should be `GWWA`$= e = g_W\sin\theta_W$ for photon. If the input is already a linear combination $J_{W^3} = \cos\theta_W J_Z + \sin\theta_W J_A$ like the output of the `J3XXXX` subroutine, then the $SU(2)$ gauge coupling `GW`$= g_W$ is appropriate. The output of the subroutines `VXXXXX`, `JIOXXX`, `J3XXXX`, `JVVXXX`, `JVSXXX`, `JSSXXX`, `JW3WXX`, `JVSSXX`, or `JEEXXX` are suitable here.

III. `real G`
This is a real variable which contains the coupling constant of the $W^\pm$ and W^3 boson. If the W^3 boson is the output of the subroutine `J3XXXX`, `G` must be the weak gauge coupling `GW` $(= e/\sin\theta_W)$. If the W^3 boson is the external Z boson or the output of the `JIOXXX` of the neutrino current, then `G` must be `GWWZ`=`GW`$*\cos\theta_W$. If the W^3 boson is a photon, `G` must be `GWWA` $(=$ `GW` $*\sin\theta_W = e)$.

THE OUTPUT

I. `complex VERTEX`
This is a complex variable which contains the vertex function of the W^-, W^+ and W^3 (or Z or photon).

What we compute here is the following T-matrix:

$$\begin{aligned}\texttt{VERTEX} &= -\texttt{G}\{((p^- - p^+)\cdot V^3)(V^-\cdot V^+) + ((p^+ - p^3)\cdot V^-)(V^+\cdot V^3) \\ &\quad +((p^3 - p^-)\cdot V^+)(V^3\cdot V^-)\},\end{aligned} \tag{3.72}$$

where we used the notation

$$\begin{aligned}V^-_\mu &= \texttt{WM}(\mu+1) \\ V^+_\mu &= \texttt{WP}(\mu+1) \\ V^3_\mu &= \texttt{W3}(\mu+1) \\ p^-_\mu &= (\Re e\texttt{WM(5)},\ \Re e\texttt{WM(6)},\ \Im m\texttt{WM(6)},\ \Im m\texttt{WM(5)}) \\ p^+_\mu &= (\Re e\texttt{WP(5)},\ \Re e\texttt{WP(6)},\ \Im m\texttt{WP(6)},\ \Im m\texttt{WP(5)}) \\ p^3_\mu &= (\Re e\texttt{W3(5)},\ \Re e\texttt{W3(6)},\ \Im m\texttt{W3(6)},\ \Im m\texttt{W3(5)}).\end{aligned}$$

However, there is a gauge theory cancellation between the three terms, which can be hardly handled within the single-precision subroutine. Fortunately, there is an identical expression which can avoid such a cancellation at the high-energy:

$$\begin{aligned}\texttt{VERTEX} = -\texttt{G}\ &\{((p^- - \alpha_- V^- - p^+ + \alpha_+ V^+)\cdot V^3)(V^-\cdot V^+) \\ &+((p^+ - \alpha_+ V^+ - p^3 + \alpha_3 V^3)\cdot V^-)(V^+\cdot V^3) \\ &+((p^3 - \alpha_3 V^3 - p^- + \alpha_- V^-)\cdot V^+)(V^3\cdot V^-)\},\end{aligned} \tag{3.73}$$

for any set of the real numbers α_-, α_+, α_3. [6] Our choice for the α's are

$$\alpha_- = p^-_0/V^-_0, \tag{3.74}$$
$$\alpha_+ = p^+_0/V^+_0, \tag{3.75}$$
$$\alpha_3 = p^3_0/V^3_0. \tag{3.76}$$

which significantly reduces the numerical values of the scalar and longitudinal component of the vector current.[7]

We denote the output of the subroutine symbolically as

$$\Gamma(V^-, V^+, V^3). \tag{3.77}$$

[6] We thank H. Iwasaki for this idea.
[7] We thank D. Zeppenfeld for this idea.

3.4.4 JGGXXX

This subroutine computes an off-shell vector current from the VVV vertex attached with the vector boson propagator, from the two Vector boson polarization vectors.

This subroutine will be called as

```
CALL JGGXXX(G1,G2,G , JGG)
```

We have five inputs G1,G2,G and one output JGG.

THE INPUTS

I. `complex G1(6),G2(6)`
These are complex six-dimensional arrays which contain the wavefunctions of the Vector bosons, and their four-momenta. The output of the subroutines VXXXXX, JIOXXX, J3XXXX, JVVXXX, JVSXXX, JSSXXX, JWWWXX, JW3WXX, JVSSXX, or JEEXXX are suitable here.

II. `real G`
This is a real coupling constant of the VVV vertex. It should be GW for $W^-W^+W^3$ coupling, GWWZ for W^-W^+Z coupling, and GWWA for W^-W^+A coupling.

THE OUTPUT

I. `complex JGG(6)`
This is a complex six-dimensional array which contains the off-shell vector current from the VVV vertex attached with the vector boson propagator, combined with its four-momentum.

The possible choice of the inputs and outputs are:

V1	V2	JVV	G	VMASS	VWIDTH
W^-	W^+	A/Z	GWWA/GWWZ	0./ZMASS	0./ZWIDTH
$W^3/A/Z$	W^-	W^+	GW/GWWA/GWWZ	WMASS	WWIDTH
W^+	$W^3/A/Z$	W^-	GW/GWWA/GWWZ	WMASS	WWIDTH

What we compute is the following portion of the Feynman amplitude:

$$J^\mu = i\mathtt{G}\frac{-i}{s-m^2+im\Gamma}(J_{12}^\mu - q^\mu J_S), \tag{3.78}$$

where,

$$\begin{aligned} J_{12}^\mu &= (p_1-p_2)^\mu\,(V_1\cdot V_2) + ((p_2-q)\cdot V_1)\,V_2^\mu + ((q-p_1)\cdot V_2)\,V_1^\mu, & (3.79)\\ J_S &= \frac{1}{m^2}\Big((-p_1^2+p_2^2)\,(V_1\cdot V_2) + (p_1\cdot V_1)\,(p_1\cdot V_2) - (p_2\cdot V_1)\,(p_2\cdot V_2)\Big), & (3.80)\end{aligned}$$

and,

$$\begin{aligned} \mathtt{JVV(5)} &= \mathtt{V1(5)}+\mathtt{V2(5)}, & (3.81)\\ \mathtt{JVV(6)} &= \mathtt{V1(6)}+\mathtt{V2(6)}. & (3.82)\end{aligned}$$

Here we used the notation

$$\begin{aligned} J^\mu &= \mathtt{JVV}(\mu+1), & (3.83)\\ V_1^\mu &= \mathtt{V1}(\mu+1), & (3.84)\\ V_2^\mu &= \mathtt{V2}(\mu+1), & (3.85)\\ p_1^\mu &= (\Re e\mathtt{V1(5)},\,\Re e\mathtt{V1(6)},\,\Im m\mathtt{V1(6)},\,\Im m\mathtt{V1(5)}), & (3.86)\end{aligned}$$

$$p_2^\mu = (\Re e\mathtt{V2(5)},\ \Re e\mathtt{V2(6)},\ \Im m\mathtt{V2(6)},\ \Im m\mathtt{V2(5)}), \tag{3.87}$$
$$q^\mu = (\Re e\mathtt{JVV(5)},\ \Re e\mathtt{JVV(6)},\ \Im m\mathtt{JVV(6)},\ \Im m\mathtt{JVV(5)}), \tag{3.88}$$
$$m = \mathtt{VMASS}, \tag{3.89}$$
$$\Gamma = \mathtt{VWIDTH}. \tag{3.90}$$

We denote the output of the subroutine symbolically as

$$J^\mu_{V'}(V_1,\ V_2). \tag{3.91}$$

3.5 Vertices IV: VVS vertex

This subgroup contains three subroutines.

```
VVSXXX   ,
JVSXXX   ,
HVVXXX   .
```

The VVS vertex computed by these subroutines is defined by

$$\mathcal{L}_{\mathtt{VVS}} = \mathtt{G} V_1^{\mu *} V_{2\mu}^* S^*. \tag{3.92}$$

3.5.1 VVSXXX

This subroutine computes the amplitude of the VVS vertex from the two Vector boson polarization vectors and the Scalar wavefunction.

This subroutine will be called as

```
CALL VVSXXX(V1,V2,SC,GC , VERTEX)
```

We have four inputs V1,V2,SC,GC and one output VERTEX.

THE INPUTS

I. complex V1(6),V2(6)
These are complex six-dimensional arrays which contain the wavefunctions of the Vector bosons, and their four-momenta. The output of the subroutines VXXXXX, JIOXXX, J3XXXX, JVVXXX, JVSXXX, JSSXXX, JWWWXX, JW3WXX, JVSSXX, or JEEXXX are suitable here.

II. complex SC(3)
This is a complex three-dimensional array which contains the wavefunction of the SCalar boson, and its four-momentum. The outputs of the subroutines SXXXXX, HIOXXX, HVVXXX, HVSXXX, HSSXXX, HVVSXX or HSSSXX are suitable here.

III. complex GC
This is a complex coupling constant of the VVS vertex.

THE OUTPUT

I. complex VERTEX
This is a complex number which is the amplitude of the VVS vertex including the coupling constant.

What we compute here is the following T-matrix:

$$\mathtt{VERTEX} = \mathtt{G} \sum_{\mu=0}^{3} \mathtt{V1}(\mu+1)\mathtt{V2}(\mu+1)\mathtt{SC}(1). \tag{3.93}$$

We denote the output of the subroutine symbolically as

$$\Gamma(V_1,\ V_2,\ S). \tag{3.94}$$

3.5.2 JVSXXX

This subroutine computes an off-shell vector current from the VVS vertex, from a vector boson polarization vector and a scalar boson wavefunction.

This subroutine will be called as

```
CALL JVSXXX(VC,SC,GC,VMASS,VWIDTH , JVS)   .
```

We have five inputs VC,SC,GC,VMASS,VWIDTH and one output FVI.

THE INPUTS

I. `complex VC(6)`
This is a complex six-dimensional array which contains the wavefunction of the VeCtor boson, and its four-momentum. The output of the subroutines VXXXXX, JIOXXX, J3XXXX, JVVXXX, JVSXXX, JSSXXX, JWWWXX, JW3WXX, JVSSXX, or JEEXXX are suitable here.

II. `complex SC(3)`
This is a complex three-dimensional array which contains the wavefunction of the SCalar boson, and its four-momentum. The outputs of the subroutines SXXXXX, HIOXXX, HVVXXX, HVSXXX, HSSXXX, HVVSXX or HSSSXX are suitable here.

III. `complex GC`
This is a complex coupling constant of the VVS vertex.

IV. `real VMASS, VWIDTH`
The mass and the width of the vector boson emitted from the VVS vertex.

THE OUTPUT

I. `complex JVS(6)`
This is a complex six-dimensional array which contains the off-shell vector current from the VVS vertex attached with the vector boson propagator, combined with its four-momentum.

What we compute here is the following portion of the Feynman amplitude:

$$J^\mu = \mathtt{G}\frac{i}{q^2 - m^2 + im\Gamma}\left(-V^\mu + \frac{q^\mu}{m^2}(q\cdot V)\right)\mathtt{SC(1)}, \tag{3.95}$$

and,

$$\begin{aligned} \mathtt{JVS(5)} &= \mathtt{VC(5)} + \mathtt{SC(2)}, & (3.96)\\ \mathtt{JVS(6)} &= \mathtt{VC(6)} + \mathtt{SC(3)}. & (3.97)\end{aligned}$$

Here we used the notation

$$\begin{aligned} J^\mu &= \mathtt{JVS}(\mu+1),\\ V^\mu &= \mathtt{VC}(\mu+1),\\ q^\mu &= (\Re e(\mathtt{JVS(5)}),\, \Re e(\mathtt{JVS(6)}),\, (\Im m(\mathtt{JVS(6)}).\, \Im m(\mathtt{JVS(5)}),\\ m &= \mathtt{VMASS},\\ \Gamma &= \mathtt{VWIDTH}.\end{aligned}$$

We denote the output of the subroutine symbolically as

$$J^\mu_{V'}(V,\, S). \tag{3.98}$$

3.5.3 HVVXXX

This subroutine computes an off-shell scalar current from the VVS vertex, attached with the scalar propagator, combined with its four-momentum.

This subroutine will be called as

```
CALL HVVXXX(V1,V2,GC,SMASS,SWIDTH , HVV)   .
```

We have five inputs V1,V2,GC,SMASS,SWIDTH and one output HVV.

THE INPUTS

I. `complex V1(6),V2(6)`
These are complex six-dimensional arrays which contain the wavefunctions of the Vector bosons, and their four-momenta. The output of the subroutines VXXXXX, JIOXXX, J3XXXX, JVVXXX, JVSXXX, JSSXXX, JWWWXX, JW3WXX, JVSSXX, or JEEXXX are suitable here.

II. `complex GC`
This is a complex coupling constant of the VVS vertex.

III. `real SMASS, SWIDTH`
The mass and the width of the scalar boson emitted from the VVS vertex.

THE OUTPUT

I. `complex HVV(3)`
This is a complex three-dimensional array which contains the off-shell scalar current from the VVS vertex attached with the scalar boson propagator, combined with its four-momentum.

What we compute here is the following portion of the Feynman amplitude:

$$\mathtt{HVV(1)} = \mathtt{G}\frac{i}{q^2 - m^2 + im\Gamma}(V_1 \cdot V_2), \tag{3.99}$$

and,

$$\begin{aligned} \mathtt{HVV(2)} &= \mathtt{V1(5)} + \mathtt{V2(5)}, & (3.100)\\ \mathtt{HVV(3)} &= \mathtt{V1(6)} + \mathtt{V2(6)}. & (3.101) \end{aligned}$$

Here we used the notation

$$\begin{aligned} V_1^\mu &= \mathtt{V1}(\mu + 1), & (3.102)\\ V_2^\mu &= \mathtt{V2}(\mu + 1), & (3.103)\\ q^\mu &= (\Re e(\mathtt{HVV(2)}),\, \Re e(\mathtt{HVV(3)}),\, \Im m(\mathtt{HVV(3)}),\, \Im m(\mathtt{HVV(2)})), & (3.104)\\ m &= \mathtt{SMASS}, & (3.105)\\ \Gamma &= \mathtt{SWIDTH}. & (3.106) \end{aligned}$$

We denote the output of the subroutine symbolically as

$$J_S(V_1,\, V_2). \tag{3.107}$$

3.6 Vertices V: VSS vertex

This subgroup contains three subroutines.

VSSXXX ,
JSSXXX ,
HVSXXX .

The VSS vertex computed by these subroutines are defined by

$$\mathcal{L}_{\texttt{VSS}} = i\texttt{G}V^*_\mu S^*_2 \overset{\leftrightarrow}{\partial}^\mu S^*_1. \tag{3.108}$$

Note that we should define which is *particle* S1 and *anti-particle* S2. If we reverse the ordering of S1 and S2, then the sign of the coupling will be reversed.

3.6.1 VSSXXX

This subroutine computes the amplitude of the VSS vertex.

This subroutine will be called as

```
CALL VSSXXX(VC,S1,S2,GC , VERTEX)
```

We have four inputs VC,S1,S2,GC and one output VERTEX.

The Inputs

I. complex VC(6)
This is a complex six-dimensional array which contains the wavefunction of the VeCtor boson, and its four-momentum. The output of the subroutines VXXXXX, JIOXXX, J3XXXX, JVVXXX, JVSXXX, JSSXXX, JWWWXX, JW3WXX, JVSSXX, or JEEXXX are suitable here.

II. complex S1(3),S2(3)
These are complex three-dimensional arrays which contain the wavefunctions of the Scalar bosons, and their four-momentum. The outputs of the subroutines SXXXXX, HIOXXX, HVVXXX, HVSXXX, HSSXXX, HVVSXX or HSSSXX are suitable here. Note that the array S1 should contain the *flowing-out* scalar particle, and S2 the *flowing-out* scalar anti-particle. The ordering is crucial since the wrong ordering gives the amplitude in the reversed sign.

III. complex GC
This is a complex coupling constant of the VSS vertex.

The Output

I. complex VERTEX
This is a complex variable which contains the amplitude of the VSS vertex.

What we compute here is the following T-matrix element:

$$\texttt{VERTEX} = -\texttt{G}V_\mu(p^\mu_1 - p^\mu_2). \tag{3.109}$$

Here, we used the notation

$$\begin{aligned} V^\mu &= \texttt{VC}(\mu+1), & (3.110)\\ p^\mu_1 &= (\Re e(\texttt{S1}(2)),\, \Re e(\texttt{S1}(3)),\, \Im m(\texttt{S1}(3)),\, \Im m(\texttt{S1}(2))), & (3.111)\\ p^\mu_2 &= (\Re e(\texttt{S2}(2)),\, \Re e(\texttt{S2}(3)),\, \Im m(\texttt{S2}(3)),\, \Im m(\texttt{S2}(2))). & (3.112) \end{aligned}$$

We denote the output of the subroutine symbolically as

$$\Gamma(V,\, S_1,\, S_2). \tag{3.113}$$

3.6.2 JSSXXX

This subroutine computes an off-shell vector current from the VSS vertex.

This subroutine will be called as

```
CALL JSSXXX(S1,S2,GC,VMASS,VWIDTH , JSS)
```

We have five inputs S1,S2,GC,VMASS,VWIDTH and one output JSS.

THE INPUTS

I. `complex S1(3),S2(3)`
These are complex three-dimensional arrays which contain the wavefunctions of the Scalar bosons, and their four-momentum. The outputs of the subroutines SXXXXX, HIOXXX, HVVXXX, HVSXXX, HSSXXX, HVVSXX or HSSSXX are suitable here. Note that the array S1 should contain the *flowing-out* scalar particle, and S2 the *flowing-out* scalar anti-particle. The ordering is crucial since the wrong ordering gives the amplitude in the reversed sign.

II. `complex GC`
This is a complex coupling constant of the VSS vertex.

III. `real VMASS, VWIDTH`
The mass and the width of the vector boson emitted from the VSS vertex.

THE OUTPUT

I. `complex JSS(6)`
This is a complex six-dimensional array which contains the off-shell vector current from the VSS vertex combined with its four-momentum.

What we compute here is the following portion of the Feynman amplitude:

$$J^{\mu} = -i\mathtt{G}\frac{i}{q^2 - m^2 + im\Gamma}\left(-g^{\mu\nu} + \frac{q_\mu q_\nu}{m^2}\right)(p_{1\nu} - p_{2\nu}). \tag{3.114}$$

for massive vector boson, or

$$J^{\mu} = -i\mathtt{G}\frac{-i}{q^2}(p_1^{\mu} - p_2^{\mu}). \tag{3.115}$$

for massless vector boson, and

$$\mathtt{JSS(5)} = \mathtt{S1(2)} + \mathtt{S2(2)}, \tag{3.116}$$
$$\mathtt{JSS(6)} = \mathtt{S1(3)} + \mathtt{S2(3)}. \tag{3.117}$$

Here, we used the notation

$$J^{\mu} = \mathtt{JSS}(\mu + 1), \tag{3.118}$$
$$V^{\mu} = \mathtt{VC}(\mu + 1), \tag{3.119}$$
$$p_1^{\mu} = (\Re e(\mathtt{S1(2)}),\, \Re e(\mathtt{S1(3)}),\, \Im m(\mathtt{S1(3)}),\, \Im m(\mathtt{S1(2)})), \tag{3.120}$$
$$p_2^{\mu} = (\Re e(\mathtt{S2(2)}),\, \Re e(\mathtt{S2(3)}),\, \Im m(\mathtt{S2(3)}),\, \Im m(\mathtt{S2(2)})), \tag{3.121}$$
$$q_V^{\mu} = (\Re e(\mathtt{JSS(5)}),\, \Re e(\mathtt{JSS(6)}),\, \Im m(\mathtt{JSS(6)}),\, \Im m(\mathtt{JSS(5)})), \tag{3.122}$$
$$m = \mathtt{VMASS}, \tag{3.123}$$
$$\Gamma = \mathtt{VWIDTH}. \tag{3.124}$$

We denote the output of the subroutine symbolically as

$$J_V^{\mu}(S_1,\, S_2). \tag{3.125}$$

3.6.3 HVSXXX

This subroutine computes an off-shell scalar current from VSS vertex.

This subroutine will be called as

```
CALL HVSXXX(VC,SC,GC,SMASS,SWIDTH , HVS)
```

We have five inputs VC,SC,GC,SMASS,SWIDTH and one output HVS.

THE INPUTS

I. `complex VC(6)`
This is a complex six-dimensional array which contains the wavefunction of the VeCtor boson, and its four-momentum. The output of the subroutines VXXXXX, JIOXXX, J3XXXX, JVVXXX, JVSXXX, JSSXXX, JWWWXX, JW3WXX, JVSSXX, or JEEXXX are suitable here.

II. `complex SC(3)`
This is a complex three-dimensional array which contains the wavefunction of the SCalar boson, and its four-momentum. The outputs of the subroutines SXXXXX, HIOXXX, HVVXXX, HVSXXX, HSSXXX, HVVSXX or HSSSXX are suitable here.

III. `complex GC`
This is a complex coupling constant of the VSS vertex. It is assumed that the scalar SC is a *flowing-out particle*. If the scalar is a *flowing-in particle*, then the coupling should have the reversed sign.

IV. `real SMASS, SWIDTH`
The mass and the width of the scalar boson emitted from the VSS vertex.

THE OUTPUT

I. `HVS(3)`
This is a complex three-dimensional array which contains the off-shell scalar current from VSS vertex combined with its four-momentum.

What we compute here is the following portion of the Feynman amplitude:

$$\mathtt{HVS(1)} = -i\mathtt{G}\frac{i}{q^2 - m^2 + im\Gamma} V_\mu (p^\mu - q^\mu), \tag{3.126}$$

and,

$$\begin{aligned} \mathtt{HVS(2)} &= \mathtt{VC(5)} + \mathtt{SC(2)}, &(3.127)\\ \mathtt{HVS(3)} &= \mathtt{VC(6)} + \mathtt{SC(3)}. &(3.128)\end{aligned}$$

Here we used the notation

$$\begin{aligned} V^\mu &= \mathtt{VC}(\mu+1), &(3.129)\\ p^\mu &= (\Re e(\mathtt{SC(2)}),\, \Re e(\mathtt{SC(3)}),\, \Im m(\mathtt{SC(3)}),\, \Im m(\mathtt{SC(2)})), &(3.130)\\ q^\mu &= (\Re e(\mathtt{HVS(2)}),\, \Re e(\mathtt{HVS(3)}),\, \Im m(\mathtt{HVS(3)}),\, \Im m(\mathtt{HVS(2)})), &(3.131)\\ m &= \mathtt{SMASS}, &(3.132)\\ \Gamma &= \mathtt{SWIDTH}. &(3.133)\end{aligned}$$

We denote the output of the subroutine symbolically as

$$J_{S'}(V,\, S). \tag{3.134}$$

3.7 Vertices VI: SSS vertex

This subgroup contains two subroutines.

SSSXXX ,
HSSXXX .

The SSS vertex computed by these subroutines are defined by

$$\mathcal{L}_{\mathtt{SSS}} = \mathtt{G}\, S_1^* S_2^* S_3^*. \tag{3.135}$$

If some of the scalars are identical, then the coupling should be divided by an appropriate statistical factor. Anyway, the coupling G has the same normalization as that appears in the Feynman rule.

3.7.1 SSSXXX

This subroutine computes the amplitude of the SSS vertex.
This subroutine will be called as

```
CALL SSSXXX(S1,S2,S3,GC , VERTEX)
```

We have four inputs S1,S2,S3,GC and one output VERTEX.
The Inputs

I. `complex S1(3),S2(3),S3(3)`
These are complex three-dimensional arrays which contain the wavefunctions of the Scalar bosons, and their four-momentum. The outputs of the subroutines SXXXXX, HIOXXX, HVVXXX, HVSXXX, HSSXXX, HVVSXX or HSSSXX are suitable here.

II. `complex GC`
This is a complex coupling constant of SSS vertex. Though the coupling can be complex in general, the input should be real. If you wish to compute the SSS vertex with complex coupling, you should multiply the output VERTEX by the appropriate factor by hand.

The Output

I. VERTEX
This is a complex amplitude of the SSS vertex including the coupling constant.

What we compute here is the following T-matrix element:

$$\mathtt{VERTEX} = \mathtt{G}\ \mathtt{S1(1)}\ \mathtt{S2(1)}\ \mathtt{S3(1)}. \tag{3.136}$$

We denote the output of the subroutine symbolically as

$$\Gamma(S_1,\, S_2,\, S_3). \tag{3.137}$$

3.7.2 HSSXXX

This subroutine computes an off-shell scalar current from the SSS vertex.
This subroutine will be called as

```
CALL HSSXXX(S1,S2,GC,SMASS,SWIDTH , HSS) .
```

We have five inputs S1,S2,GC,SMASS,SWIDTH and one output HSS.
The Inputs

I. `complex S1(3),S2(3)`
These are complex three-dimensional arrays which contain the wavefunctions of the Scalar bosons, and their four-momentum. The outputs of the subroutines SXXXXX, HIOXXX, HVVXXX, HVSXXX, HSSXXX, HVVSXX or HSSSXX are suitable here.

II. `complex GC`
This is a complex coupling constant of SSS vertex.

III. `real SMASS, SWIDTH`
The mass and the width of the scalar boson emitted from the SSS vertex.

THE OUTPUT

I. `complex HSS(3)`
This is a complex three-dimensional array which contains the off-shell scalar current combined with its four-momentum.

What we compute here is the following portion of the Feynman amplitude:

$$\mathtt{HSS(1)} = i\mathtt{G}\frac{i}{q^2 - m^2 + im\Gamma}\mathtt{S1(1)}\ \mathtt{S2(1)}, \tag{3.138}$$

and,

$$\mathtt{HSS(2)} = \mathtt{S1(2)} + \mathtt{S2(2)}, \tag{3.139}$$
$$\mathtt{HSS(3)} = \mathtt{S1(3)} + \mathtt{S2(3)}. \tag{3.140}$$

Here we used the notation

$$q^\mu = (\Re e(\mathtt{HSS(2)}),\ \Re e(\mathtt{HSS(3)}),\ \Im m(\mathtt{HSS(3)}),\ \Im m(\mathtt{HSS(2)})), \tag{3.141}$$
$$m = \mathtt{SMASS}, \tag{3.142}$$
$$\Gamma = \mathtt{SWIDTH}. \tag{3.143}$$

We denote the output of the subroutine symbolically as

$$J_{S'}(S_1,\ S_2). \tag{3.144}$$

3.8 Vertices VII: VVVV vertex

We have six subroutines in this group:

```
WWWWXX ,
JWWWXX ,
W3W3XX ,
JW3WXX ,
GGGGXX ,
JGGGXX ,
```

In all of these subroutines, we take all the momenta *flowing-out*, and name the particles by the *flowing-out* quantum number. For example, the W^- boson in the initial state will be called flowing-out W^+, and has a negative energy in these subroutines.

The most important point in their use is that these subroutines contain four-point contact vertex as well as s-, t-, or u-channel vector boson exchange. This is convenient since any diagram which contains the four-point contact vertex has always vector boson exchange diagrams, and they have numerical cancellation among themselves. WWWWXX, JWWWXX, W3W3XX and JW3WXX subroutines in HELAS are written in double precision to deal with the cancellation.

The $W^-W^+W^3$ vertex computed by these subroutines are defined by

$$\begin{aligned}\mathcal{L}_{\mathtt{WWW}} &= -i\mathtt{GW}\ \Big\{ (\partial_\mu W_\nu^{-*})(W^{+\mu*}W^{3\nu*} - W^{+\nu*}W^{3\mu*}) \\ &\quad +(\partial_\mu W_\nu^{+*})(W^{3\mu*}W^{-\nu*} - W^{3\nu*}W^{-\mu*}) \\ &\quad + (\partial_\mu W^3_{\nu*})(W^{-\mu*}W^{+\nu*} - W^{-\nu*}W^{+\mu*})\Big\}. \end{aligned} \tag{3.145}$$

The contact four-point vertex is defined by

$$\begin{aligned}\mathcal{L}_{WWWW} &= \frac{\mathtt{GW}^2}{4}\Big\{(W_\mu^{-*}W_\nu^{+*} - W_\nu^{-*}W_\mu^{+*})^2 \\ &\quad -2(W_\mu^{-*}W_\nu^{3*} - W_\nu^{-*}W_\mu^{3*})(W^{\mu+*}W^{\nu3*} - W^{\nu+*}W^{\mu3*})\Big\}. \end{aligned} \tag{3.146}$$

3.8.1 WWWWXX

This subroutine computes the four W boson vertex ($\mathtt{W}^-\mathtt{W}^+\mathtt{W}^-\mathtt{W}^+$) from the polarization vectors or the vector currents. This vertex function contains the four-W contact term and s-channel and t-channel W^3 exchange diagrams as well. Note that the s-channel and t-channel Higgs-exchange diagram is *not* included.[8]

This subroutine will be called as

```
 CALL WWWWXX(WM1,WP1,WM2,WP2,GWWA,GWWZ,ZMASS,ZWIDTH ,
&                                             VERTEX)   .
```

We have eight inputs WM1,WP1,WM2,WP2,GWWA,GWWZ,ZMASS,ZWIDTH and one output VERTEX.

THE INPUTS

I. `complex WM1(6), WM2(6)`
These are complex six-dimensional arrays which contain the wavefunctions and the flowing-out four-momenta of the *flowing-out* W^- boson (W-Minus), or vector current of the W^- boson. The output of the subroutines VXXXXX, JIOXXX, JVVXXX, JVSXXX, JSSXXX, JWWWXX, JW3WXX or JVSSXX are suitable here.

II. `complex WP1(6), WP2(6)`
These are complex six-dimensional arrays which contain the wavefunctions and the flowing-out four-momenta of the *flowing-out* W^+ boson (W-Plus), or vector current of the W^+ boson. The output of the subroutines VXXXXX, JIOXXX, JVVXXX, JVSXXX, JSSXXX, JWWWXX, JW3WXX or JVSSXX are suitable here.

III. `real GWWA,GWWZ`
These are real variables which contain that coupling constants to photon $\mathtt{GWWA} = e$ and Z boson $\mathtt{GWWZ} = e\cos\theta_W/\sin\theta_W$.

IV. `real ZMASS, ZWIDTH`
These are real variables which contain the mass and the width of the Z boson, respectively. ZMASS has to be positive, and ZWIDTH has to be non-negative.

THE OUTPUT

[8] It is well-known that the Higgs-exchange diagram cures the bad high-energy behavior of the four W scattering amplitude. Since the gauge theory cancellation occurs twice, once within the W self-coupling diagrams which are included in this subroutine, and again with the Higgs-exchange amplitude, the W self-coupling diagram must be computed sufficiently precise. Our choice is to make this subroutine in double precision.

I. `complex VERTEX`
This is a complex variable which contains the vertex function of the W^-, W^+, W^-, and W^+.

What we compute here is the following T-matrix:

$$
\begin{aligned}
\texttt{VERTEX} = {} & -g^2 \\
& \times \Big\{ (W_1^- \cdot W_1^+)(W_2^- \cdot W_2^+) + (W_1^- \cdot W_2^+)(W_2^- \cdot W_1^+) - 2(W_1^- \cdot W_2^-)(W_1^+ \cdot W_2^+) \\
& \quad + \Big(D_Z(q^2)\cos^2\theta_W + D_A(q^2)\sin^2\theta_W \Big)(J_{12} \cdot J_{34}) \\
& \quad + \Big(D_Z(k^2)\cos^2\theta_W + D_A(k^2)\sin^2\theta_W \Big)(J_{14} \cdot J_{32}) \\
& \quad - D_Z(q^2)\frac{\cos^2\theta_W}{m_Z^2}(q \cdot J_{12})(q \cdot J_{34}) - D_Z(k^2)\frac{\cos^2\theta_W}{m_Z^2}(k \cdot J_{14})(k \cdot J_{32}) \Big\}
\end{aligned}
\tag{3.147}
$$

where,

$$
\begin{aligned}
W_1^{-\mu} &= \texttt{WM1}(\mu+1), \\
W_1^{+\mu} &= \texttt{WP1}(\mu+1), \\
W_2^{-\mu} &= \texttt{WM2}(\mu+1), \\
W_2^{+\mu} &= \texttt{WP2}(\mu+1), \\
p_1^{-\mu} &= (\Re e\texttt{WM1(5)},\, \Re e\texttt{WM1(6)},\, \Im m\texttt{WM1(6)},\, \Im m\texttt{WM1(5)}), \\
p_1^{+\mu} &= (\Re e\texttt{WP1(5)},\, \Re e\texttt{WP1(6)},\, \Im m\texttt{WP1(6)},\, \Im m\texttt{WP1(5)}), \\
p_2^{-\mu} &= (\Re e\texttt{WM2(5)},\, \Re e\texttt{WM2(6)},\, \Im m\texttt{WM2(6)},\, \Im m\texttt{WM2(5)}), \\
p_2^{+\mu} &= (\Re e\texttt{WP2(5)},\, \Re e\texttt{WP2(6)},\, \Im m\texttt{WP2(6)},\, \Im m\texttt{WP2(5)}), \\
q_\mu &= p_1^{-\mu} + p_1^{+\mu}, \\
k_\mu &= p_1^{-\mu} + p_2^{+\mu}, \\
D_Z(a) &= \frac{-1}{a - m_Z^2 + i m_Z \Gamma_Z}, \\
D_A(a) &= \frac{-1}{a}, \\
m_Z &= \texttt{ZMASS}, \\
\Gamma_Z &= \texttt{ZWIDTH}, \\
g^2 &= \texttt{GWWA}^2 + \texttt{GWWZ}^2, \\
\theta_W &= \tan^{-1}(\texttt{GWWA/GWWZ}),
\end{aligned}
$$

and,

$$
\begin{aligned}
J_{12}^\rho &= (W_1^- \cdot W_1^+)(p_1^- - p_1^+)^\rho + (p_1^+ + q) \cdot W_1^-(W_1^+)^\rho + (-q - p_1^-) \cdot W_1^+(W_1^-)^\rho, \\
J_{34}^\rho &= (W_2^- \cdot W_2^+)(p_2^- - p_2^+)^\rho + (p_2^+ - q) \cdot W_2^-(W_2^+)^\rho + (+q - p_2^-) \cdot W_2^+(W_2^-)^\rho, \\
J_{14}^\rho &= (W_1^- \cdot W_2^+)(p_1^- - p_2^+)^\rho + (p_2^+ + k) \cdot W_1^-(W_2^+)^\rho + (-k - p_1^-) \cdot W_2^+(W_1^-)^\rho, \\
J_{32}^\rho &= (W_2^- \cdot W_1^+)(p_2^- - p_1^+)^\rho + (p_1^+ - k) \cdot W_2^-(W_1^+)^\rho + (+k - p_2^-) \cdot W_1^+(W_2^-)^\rho.
\end{aligned}
$$

As is in the `VVVXXX` subroutine, there is a gauge theory cancellation between the three diagrams, which can be hardly handled within the single-precision subroutine. Thus we decided to compute only the four-vertices in the double precision subroutines. [9]

[9] This is a platitude idea by the authors. We wish to express special regret to ourselves.

We denote the output of the subroutine symbolically as

$$\Gamma(W_1^-, W_1^+, W_2^-, W_2^+). \tag{3.148}$$

3.8.2 JWWWXX

This subroutine computes an off-shell W boson current from the WWWW four-point vertex. This subroutine will be called as

```
CALL JWWWXX(W1,W2,W3,GWWA,GWWZ,ZMASS,ZWIDTH,WMASS,WWIDTH , JWWW)
```

We have nine inputs W1,W2,W3,GWWA,GWWZ,ZMASS,ZWIDTH,WMASS,WWIDTH and one output JWWW.
THE INPUTS

I. `complex W1(6), W2(6), W3(6)`
These are complex six-dimensional arrays which contain the wavefunctions and the flowing-out four-momenta of the W^- bosons. Their ordering should be either *flowing-out* W^-, W^+, W^-, or W^+, W^-, W^+. The output of the subroutines VXXXXX, JIOXXX, JVVXXX, JVSXXX, JSSXXX, JWWWXX, JW3WXX or JVSSXX are suitable here.

II. `real GWWA,GWWZ`
These are real variables which contain that coupling constants to photon GWWA $= e$ and Z boson GWWZ $= e\cos\theta_W / \sin\theta_W$.

III. `real ZMASS, ZWIDTH`
These are real variables which contain the mass and the width of the Z boson, respectively. ZMASS has to be positive, and ZWIDTH has to be non-negative.

IV. `real WMASS, WWIDTH`
These are real variables which contain the mass and the width of the $W^\pm$ boson, respectively. WMASS has to be positive, and WWIDTH has to be non-negative.

THE OUTPUT

I. `complex JWWW(6)`
This is a complex six-dimensional array which contains the vector boson four-point couplings attached with the massive vector boson propagator in the unitary gauge, combined with its four-momentum.

What we compute here is the sum of five diagrams, one is the four-point contact vertex among the vector bosons, and the other four are photon and Z boson exchange diagrams. Their exact expressions are too tedious to be displayed here, although they can be obtained in a straight-forward calculation from the vertices given in this section. Note that the W and Z boson propagators are attached in the unitary gauge.

The possible inputs and outputs are:

W1	W2	W3	GWWA	GWWZ	ZMASS	ZWIDTH	WMASS	WWIDTH	JWWW
W^-	W^+	W^-	GWWA	GWWZ	ZMASS	ZWIDTH	WMASS	WWIDTH	W^+
W^+	W^-	W^+	GWWA	GWWZ	ZMASS	ZWIDTH	WMASS	WWIDTH	W^-

We denote the output of the subroutine symbolically as

$$J^\mu_{W^\pm}(W_1^\mp, W_2^\pm, W_3^\mp). \tag{3.149}$$

3.8.3 W3W3XX

This subroutine computes the four W boson vertex ($\mathtt{W}^{-}\mathtt{W}^{3}\mathtt{W}^{+}\mathtt{W}^{3}$) from the polarization vectors or the vector currents. This vertex function contains the four-W contact term and s-channel and t-channel W exchange diagram as well. The u-channel Higgs-exchange diagram is *not* included.

This subroutine will be called as

```
 CALL W3W3XX(WM,W31,WP,W32,GW31,GW32,WMASS,WWIDTH ,
&                                           VERTEX)   .
```

We have eight inputs WM,W31,WP,W32,GW31,GW32,WMASS,WWIDTH and one output VERTEX.

THE INPUTS

I. `complex WM(6), WP(6)`
These are complex six-dimensional arrays which contain the wavefunctions and the flowing-out four-momenta of the *flowing-out* W^{-}, W^{+} boson (W-Minus, -Plus), or vector current of the W^{-}, W^{+} boson, respectively. The output of the subroutines VXXXXX, JIOXXX, JVVXXX, JVSXXX, JSSXXX, JWWWXX, JW3WXX or JVSSXX are suitable here.

II. `complex W31(6), W32(6)`
These are complex six-dimensional arrays which contain the wavefunctions and the flowing-out four-momenta of the *flowing-out* W^{3} bosons, or vector current of the W^{3} boson. They can be either Z, A (photon), or their linear combination W^{3}, and the coupling constants GW31 and GW32 should be chosen from GWWZ,GWWA,GW correspondingly. The output of the subroutines VXXXXX, JIOXXX, J3XXXX, JVVXXX, JVSXXX, JSSXXX, JW3WXX, JVSSXX, or JEEXXX are suitable here.

III. `real GW31, GW32`
These are real variables which contain the coupling constants of the $W^{\pm}$ and first or second W^{3} boson, respectively. If the W^{3} boson is the output of the subroutine J3XXXX, GW3 must be the weak gauge coupling GW. If the W^{3} boson is the external Z boson or the output of the JIOXXX of the neutrino current, G must be GWWZ ($=\mathtt{GW}*\cos\theta_W$). If the W^{3} boson is the external photon, GW3 must be GWWA ($=\mathtt{GW}*\sin\theta_W=e$).

IV. `real WMASS, WWIDTH`
These are real variables which contain the mass and the width of the $W^{\pm}$ boson, respectively. WMASS and WWIDTH have to be non-negative. If WMASS is zero, the vector boson propagator inside this subroutines will be changed to the Feynman gauge form. This option will be useful when dealing with QCD.

THE OUTPUT

I. `complex VERTEX`
This is a complex variable which contains the vertex function of the W^{-}, W^{3}, W^{+}, and W^{3}(or Z or photon).

The possible inputs and outputs are:

WM	W31	WP	W32	G31	G32
W^{-}	W^{3}	W^{+}	W^{3}	GW	GW
W^{-}	W^{3}	W^{+}	Z	GW	GWWZ
W^{-}	W^{3}	W^{+}	A	GW	GWWA
W^{-}	Z	W^{+}	Z	GWWZ	GWWZ
W^{-}	Z	W^{+}	A	GWWZ	GWWA
W^{-}	A	W^{+}	A	GWWA	GWWA

What we compute here is the following T-matrix:[10]

$$\begin{aligned}
\texttt{VERTEX} &= \texttt{G31 G32} \\
&\times \Big\{ (W^- \cdot W_1^3)(W^+ \cdot W_2^3) + (W^- \cdot W_2^3)(W^+ \cdot W_1^3) - 2(W^- \cdot W^+)(W_1^3 \cdot W_2^3) \\
&\quad + D_W(q^2)(J_{12} \cdot J_{34}) - D_W(q^2)\frac{1}{m_W^2}(q \cdot J_{12})(q \cdot J_{34}) \\
&\quad + D_W(k^2)(J_{14} \cdot J_{32}) - D_W(k^2)\frac{1}{m_W^2}(k \cdot J_{14})(k \cdot J_{32}) \Big\}
\end{aligned} \tag{3.150}$$

where,

$$\begin{aligned}
W^{-\mu} &= \texttt{WM}(\mu+1), \\
W_1^{3\mu} &= \texttt{W31}(\mu+1), \\
W^{+\mu} &= \texttt{WP}(\mu+1), \\
W_2^{3\mu} &= \texttt{W32}(\mu+1), \\
p^{-\mu} &= (\Re e\texttt{WM(5)},\, \Re e\texttt{WM(6)},\, \Im m\texttt{WM(6)},\, \Im m\texttt{WM(5)}), \\
p_1^{3\mu} &= (\Re e\texttt{W31(5)},\, \Re e\texttt{W31(6)},\, \Im m\texttt{W31(6)},\, \Im m\texttt{W31(5)}), \\
p^{+\mu} &= (\Re e\texttt{WP(5)},\, \Re e\texttt{WP(6)},\, \Im m\texttt{WP(6)},\, \Im m\texttt{WP(5)}), \\
p_2^{3\mu} &= (\Re e\texttt{W32(5)},\, \Re e\texttt{W32(6)},\, \Im m\texttt{W32(6)},\, \Im m\texttt{W32(5)}), \\
q_\mu &= p^{-\mu} + p_1^{3\mu}, \\
k_\mu &= p^{-\mu} + p_2^{3\mu}, \\
D_W(a) &= \frac{-1}{a - m_W^2 + i m_W \Gamma_W}, \\
m_W &= \texttt{WMASS}, \\
\Gamma_W &= \texttt{WWIDTH},
\end{aligned}$$

and,

$$\begin{aligned}
J_{12}^\rho &= (W^- \cdot W_1^3)(p^- - p_1^3)^\rho + (p_1^3 + q) \cdot W^-(W_1^3)^\rho + (-q - p^-) \cdot W_1^3(W^-)^\rho, \\
J_{34}^\rho &= (W^+ \cdot W_2^3)(p^+ - p_2^3)^\rho + (p_2^3 - q) \cdot W^+(W_2^3)^\rho + (+q - p^+) \cdot W_2^3(W^+)^\rho, \\
J_{14}^\rho &= (W^- \cdot W_2^3)(p^- - p_2^3)^\rho + (p_2^3 + k) \cdot W^-(W_2^3)^\rho + (-k - p^-) \cdot W_2^3(W^-)^\rho, \\
J_{32}^\rho &= (W^+ \cdot W_1^3)(p^+ - p_1^3)^\rho + (p_1^3 - k) \cdot W^+(W_1^3)^\rho + (+k - p^+) \cdot W_1^3(W^+)^\rho.
\end{aligned}$$

As is in the `WWWWXX` subroutine, we compute with the double precision in this subroutine to deal with the gauge theory cancellation between the three diagrams,

We denote the output of the subroutine symbolically as

$$\Gamma(W^-, W_1^3, W^+, W_2^3). \tag{3.151}$$

3.8.4 JW3WXX

This subroutine computes an off-shell $W^\pm$ boson or W^3 current from the `W3W3` four-point vertex.

This subroutine will be called as

[10] If `WMASS` is zero, then the terms which contain $1/m_W^2$ will be dropped, and hence going to the Feynman gauge. Only half of the remaining terms are kept, *i.e.*,

$$\texttt{VERTEX} = \texttt{G31 G32} \times \Big\{ (W^- \cdot W_2^3)(W^+ \cdot W_1^3) - (W^- \cdot W^+)(W_1^3 \cdot W_2^3) - (J_{12} \cdot J_{34})/q^2 \Big\}.$$

```
 CALL JW3WXX(W1,W2,W3,G1,G2,WMASS,WWIDTH,VMASS,VWIDTH ,
&                                                 JW3W)
```

We have nine inputs `W1,W2,W3,G1,G2,WMASS,WWIDTH,VMASS,VWIDTH` and one output `JW3WXX`.
THE INPUTS

I. `complex W1(6), W2(6), W3(6)`
These are complex six-dimensional arrays which contain the wavefunctions and the flowing-out four-momenta of the W^- boson. Since the output can be either $W^\pm$, γ or Z, the possible inputs depend on what the output is. The possible combinations of the inputs, outputs, and the coupling constants will be summarized below. The output of the subroutines `VXXXXX`, `JIOXXX`, `JVVXXX`, `JVSXXX`, `JSSXXX`, `JWWWXX`, `JW3WXX` or `JVSSXX` are suitable here.

II. `real G1,G2`
These are real variables which contain that coupling constants. The possible combinations of the inputs, outputs, and the coupling constants will be summarized below.

III. `real WMASS, WWIDTH`
These are real variables which contain the mass and the width of the $W^\pm$ boson, respectively, which is exchanged in s-, t- or u-channel. `WMASS` and `WWIDTH` have to be non-negative. If `WMASS` is zero, the vector boson propagator inside this subroutines will be changed to the Feynman gauge form. This option will be useful when dealing with QCD.

IV. `real VMASS, VWIDTH`
These are real variables which contain the mass and the width of the output vector boson, which can be either $0., 0.$ for photon, `ZMASS,ZWIDTH` for Z boson, or `WMASS,WWIDTH` for W boson.

THE OUTPUT

I. `complex JW3W(6)`
This is a complex six-dimensional array which contains the vector boson four-point couplings attached with the massive vector boson propagator in the unitary gauge, or massless vector boson propagator in the Feynman gauge, combined with its four-momentum.

What we compute here is the sum of three diagrams, one is the four-point contact vertex among the vector bosons, and the other two are the W boson exchange diagrams. Their exact expressions are too tedious to be displayed here, although they can be obtained in a straightforward calculation from the vertices given in this section. Note that the photon propagator is attached in the Feynman gauge, while Z boson propagator in the unitary gauge. The W-propagator is basicaly in unitary gauge, as far as `WMASS` is non-vanishing.

If `WMASS` is zero, then the Feynman gauge propagator will be employed, and the terms which contribute to the colour factor $f^{abe}f^{cde}$ with the inputs V^a, V^b, V^c, and the output J_{V^d} are kept (see the footnote in page 91 and sect. 2.9.1).

The possible inputs and outputs are:

W1	W2	W3	G1	G2	WMASS	WWIDTH	VMASS	VWIDTH	JW3W
W^-	W^3	W^+	GW	GWWZ	WMASS	WWIDTH	ZMASS	ZWIDTH	Z
W^-	W^3	W^+	GW	GWWA	WMASS	WWIDTH	0.	0.	A
W^-	Z	W^+	GWWZ	GWWZ	WMASS	WWIDTH	ZMASS	ZWIDTH	Z
W^-	Z	W^+	GWWZ	GWWA	WMASS	WWIDTH	0.	0.	A
W^-	A	W^+	GWWA	GWWZ	WMASS	WWIDTH	ZMASS	ZWIDTH	Z
W^-	A	W^+	GWWA	GWWA	WMASS	WWIDTH	0.	0.	A
W^3	W^-	W^3	GW	GW	WMASS	WWIDTH	WMASS	WWIDTH	W^+
W^3	W^+	W^3	GW	GW	WMASS	WWIDTH	WMASS	WWIDTH	W^-
W^3	W^-	Z	GW	GWWZ	WMASS	WWIDTH	WMASS	WWIDTH	W^+
W^3	W^+	Z	GW	GWWZ	WMASS	WWIDTH	WMASS	WWIDTH	W^-
W^3	W^-	A	GW	GWWA	WMASS	WWIDTH	WMASS	WWIDTH	W^+
W^3	W^+	A	GW	GWWA	WMASS	WWIDTH	WMASS	WWIDTH	W^-
Z	W^-	Z	GWWZ	GWWZ	WMASS	WWIDTH	WMASS	WWIDTH	W^+
Z	W^+	Z	GWWZ	GWWZ	WMASS	WWIDTH	WMASS	WWIDTH	W^-
Z	W^-	A	GWWZ	GWWA	WMASS	WWIDTH	WMASS	WWIDTH	W^+
Z	W^+	A	GWWZ	GWWA	WMASS	WWIDTH	WMASS	WWIDTH	W^-
A	W^-	A	GWWA	GWWA	WMASS	WWIDTH	WMASS	WWIDTH	W^+
A	W^+	A	GWWA	GWWA	WMASS	WWIDTH	WMASS	WWIDTH	W^-

We denote the output of the subroutine symbolically as

$$J^{\mu}_{W^\pm}(W_1^3,\, W_2^\mp,\, W_3^3), \tag{3.152}$$

or,

$$J^{\mu}_{W^3}(W_1^\mp,\, W_2^3,\, W_3^\pm). \tag{3.153}$$

3.8.5 `GGGGXX`

This subroutine computes the four W boson vertex ($\mathtt{W}^-\mathtt{W}^+\mathtt{W}^-\mathtt{W}^+$) from the polarization vectors or the vector currents. This vertex function contains the four-W contact term and s-channel and t-channel W^3 exchange diagrams as well. Note that the s-channel and t-channel Higgs-exchange diagram is *not* included.[11]

This subroutine will be called as

```
CALL GGGGXX(GA,GB,GC,GD,G , VERTEX)   .
```

We have five inputs `GA,GB,GC,GD,G` and one output `VERTEX`.

THE INPUTS

I. `complex WM1(6), WM2(6)`
These are complex six-dimensional arrays which contain the wavefunctions and the flowing-out four-momenta of the *flowing-out* W^- boson (`W`-Minus), or vector current of the W^- boson. The output of the subroutines `VXXXXX`, `JIOXXX`, `JVVXXX`, `JVSXXX`, `JSSXXX`, `JWWWXX`, `JW3WXX` or `JVSSXX` are suitable here.

II. `complex WP1(6), WP2(6)`
These are complex six-dimensional arrays which contain the wavefunctions and the flowing-out

[11] It is well-known that the Higgs-exchange diagram cures the bad high-energy behavior of the four W scattering amplitude. Since the gauge theory cancellation occurs twice, once within the W self-coupling diagrams which are included in this subroutine, and again with the Higgs-exchange amplitude, the W self-coupling diagram must be computed sufficiently precise. Our choice is to make this subroutine in double precision.

four-momenta of the *flowing-out* W^+ boson (W-Plus), or vector current of the W^+ boson. The output of the subroutines VXXXXX, JIOXXX, JVVXXX, JVSXXX, JSSXXX, JWWWXX, JW3WXX or JVSSXX are suitable here.

III. real G
These are real variables which contain that coupling constants to photon $\texttt{GWWA} = e$ and Z boson $\texttt{GWWZ} = e\cos\theta_W/\sin\theta_W$.

THE OUTPUT

I. complex VERTEX
This is a complex variable which contains the vertex function of the W^-, W^+, W^-, and W^+.

What we compute here is the following T-matrix:

$$\begin{aligned}
\texttt{VERTEX} &= -g^2 \\
&\times \Big\{ (W_1^- \cdot W_1^+)(W_2^- \cdot W_2^+) + (W_1^- \cdot W_2^+)(W_2^- \cdot W_1^+) - 2(W_1^- \cdot W_2^-)(W_1^+ \cdot W_2^+) \\
&\quad + \Big(D_Z(q^2)\cos^2\theta_W + D_A(q^2)\sin^2\theta_W\Big)(J_{12}\cdot J_{34}) \\
&\quad + \Big(D_Z(k^2)\cos^2\theta_W + D_A(k^2)\sin^2\theta_W\Big)(J_{14}\cdot J_{32}) \\
&\quad - D_Z(q^2)\frac{\cos^2\theta_W}{m_Z^2}(q\cdot J_{12})(q\cdot J_{34}) - D_Z(k^2)\frac{\cos^2\theta_W}{m_Z^2}(k\cdot J_{14})(k\cdot J_{32}) \Big\}
\end{aligned} \tag{3.154}$$

where,

$$\begin{aligned}
W_1^{-\mu} &= \texttt{WM1}(\mu+1), \\
W_1^{+\mu} &= \texttt{WP1}(\mu+1), \\
W_2^{-\mu} &= \texttt{WM2}(\mu+1), \\
W_2^{+\mu} &= \texttt{WP2}(\mu+1), \\
p_1^{-\mu} &= (\Re e\texttt{WM1(5)},\, \Re e\texttt{WM1(6)},\, \Im m\texttt{WM1(6)},\, \Im m\texttt{WM1(5)}), \\
p_1^{+\mu} &= (\Re e\texttt{WP1(5)},\, \Re e\texttt{WP1(6)},\, \Im m\texttt{WP1(6)},\, \Im m\texttt{WP1(5)}), \\
p_2^{-\mu} &= (\Re e\texttt{WM2(5)},\, \Re e\texttt{WM2(6)},\, \Im m\texttt{WM2(6)},\, \Im m\texttt{WM2(5)}), \\
p_2^{+\mu} &= (\Re e\texttt{WP2(5)},\, \Re e\texttt{WP2(6)},\, \Im m\texttt{WP2(6)},\, \Im m\texttt{WP2(5)}), \\
q_\mu &= p_1^{-\mu} + p_1^{+\mu}, \\
k_\mu &= p_1^{-\mu} + p_2^{+\mu}, \\
D_Z(a) &= \frac{-1}{a - m_Z^2 + im_Z\Gamma_Z}, \\
D_A(a) &= \frac{-1}{a}, \\
m_Z &= \texttt{ZMASS}, \\
\Gamma_Z &= \texttt{ZWIDTH}, \\
g^2 &= \texttt{GWWA}^2 + \texttt{GWWZ}^2, \\
\theta_W &= \tan^{-1}(\texttt{GWWA/GWWZ}),
\end{aligned}$$

and,

$$\begin{aligned}
J_{12}^\rho &= (W_1^- \cdot W_1^+)(p_1^- - p_1^+)^\rho + (p_1^+ + q)\cdot W_1^-(W_1^+)^\rho + (-q - p_1^-)\cdot W_1^+(W_1^-)^\rho, \\
J_{34}^\rho &= (W_2^- \cdot W_2^+)(p_2^- - p_2^+)^\rho + (p_2^+ - q)\cdot W_2^-(W_2^+)^\rho + (+q - p_2^-)\cdot W_2^+(W_2^-)^\rho, \\
J_{14}^\rho &= (W_1^- \cdot W_2^+)(p_1^- - p_2^+)^\rho + (p_2^+ + k)\cdot W_1^-(W_2^+)^\rho + (-k - p_1^-)\cdot W_2^+(W_1^-)^\rho, \\
J_{32}^\rho &= (W_2^- \cdot W_1^+)(p_2^- - p_1^+)^\rho + (p_1^+ - k)\cdot W_2^-(W_1^+)^\rho + (+k - p_2^-)\cdot W_1^+(W_2^-)^\rho.
\end{aligned}$$

As is in the VVVXXX subroutine, there is a gauge theory cancellation between the three diagrams, which can be hardly handled within the single-precision subroutine. Thus we decided to compute only the four-vertices in the double precision subroutines. [12]

We denote the output of the subroutine symbolically as

$$\Gamma(W_1^-, W_1^+, W_2^-, W_2^+). \tag{3.155}$$

3.8.6 JGGGXX

This subroutine computes an off-shell W boson current from the WWWW four-point vertex. This subroutine will be called as

```
CALL JGGGXX(G1,G2,G3,G , JGGG)
```

We have four inputs G1,G2,G3,G and one output JGGG.

The Inputs

I. `complex G1(6), G2(6), G3(6)`
These are complex six-dimensional arrays which contain the wavefunctions and the flowing-out four-momenta of the W^- bosons. Their ordering should be either *flowing-out* W^-, W^+, W^-, or W^+, W^-, W^+. The output of the subroutines VXXXXX, JIOXXX, JVVXXX, JVSXXX, JSSXXX, JWWWXX, JW3WXX or JVSSXX are suitable here.

II. `real GWWA,GWWZ`
These are real variables which contain that coupling constants to photon GWWA $= e$ and Z boson GWWZ $= e\cos\theta_W/\sin\theta_W$.

The Output

I. `complex JGGG(6)`
This is a complex six-dimensional array which contains the vector boson four-point couplings attached with the massive vector boson propagator in the unitary gauge, combined with its four-momentum.

What we compute here is the sum of five diagrams, one is the four-point contact vertex among the vector bosons, and the other four are photon and Z boson exchange diagrams. Their exact expressions are too tedious to be displayed here, although they can be obtained in a straight-forward calculation from the vertices given in this section. Note that the W and Z boson propagators are attached in the unitary gauge.

The possible inputs and outputs are:

W1	W2	W3	GWWA	GWWZ	ZMASS	ZWIDTH	WMASS	WWIDTH	JWWW
W^-	W^+	W^-	GWWA	GWWZ	ZMASS	ZWIDTH	WMASS	WWIDTH	W^+
W^+	W^-	W^+	GWWA	GWWZ	ZMASS	ZWIDTH	WMASS	WWIDTH	W^-

We denote the output of the subroutine symbolically as

$$J^\mu_{W^\pm}(W_1^\mp, W_2^\pm, W_3^\mp). \tag{3.156}$$

[12] This is a platitude idea by the authors. We wish to express special regret to ourselves.

3.9 Vertices VIII: VVSS vertex

This subgroup contains three subroutines.

```
VVSSXX   ,
JVSSXX   ,
HVVSXX   .
```

The VVSS vertex computed by these subroutines are defined by

$$\mathcal{L}_{\texttt{VVSS}} = \texttt{G}\,(V_1^* \cdot V_2^*) S_1^* S_2^* \tag{3.157}$$

If some of the scalars or vectors are identical, then the coupling should be divided by an appropriate statistical factor. Anyway, the coupling G has the same normalization as that appears in the Feynman rule.

3.9.1 VVSSXX

This subroutine computes the amplitude of the VVSS vertex.

This subroutine will be called as

```
CALL VVSSXX(V1,V2,S1,S2,GC , VERTEX) .
```

We have five inputs V1,V2,S1,S2,GC and one output VERTEX.

THE INPUTS

I. `complex V1(6),V2(6)`
These are complex six-dimensional arrays which contain the wavefunctions of the Vector bosons, and their four-momenta. The output of the subroutines VXXXXX, JIOXXX, J3XXXX, JVVXXX, JVSXXX, JSSXXX, JWWWXX, JW3WXX, JVSSXX, or JEEXXX are suitable here.

II. `complex S1(3),S2(3)`
These are complex three-dimensional arrays which contain the wavefunctions of the Scalar bosons, and their four-momenta. The outputs of the subroutines SXXXXX, HIOXXX, HVVXXX, HVSXXX, HSSXXX, HVVSXX or HSSSXX are suitable here.

III. `complex GC`
This is a complex coupling constant of VVSS vertex.

THE OUTPUT

I. `complex VERTEX`
This is a complex amplitude of the VVSS vertex.

What we compute here is the following T-matrix element:

$$\texttt{VERTEX} = \texttt{G}\ (V_1 \cdot V_2)\ \texttt{S1}(1)\ \texttt{S2}(1). \tag{3.158}$$

Here we used the notation

$$V_1^\mu \;=\; \texttt{V1}(\mu + 1), \tag{3.159}$$
$$V_2^\mu \;=\; \texttt{V2}(\mu + 1). \tag{3.160}$$

We denote the output of the subroutine symbolically as

$$\Gamma(V_1,\, V_2,\, S_1,\, S_2). \tag{3.161}$$

3.9.2 JVSSXX

This subroutine computes an off-shell vector current from the VVSS vertex.

This subroutine will be called as

```
CALL JVSSXX(VC,S1,S2,GC,VMASS,VWIDTH , JVSS)
```

We have six inputs VC,S1,S2,GC,VMASS,VWIDTH and one output JVSS.

THE INPUTS

I. complex VC(6)
This is a complex six-dimensional array which contains the wavefunction of the VeCtor boson, and its four-momentum. The output of the subroutines VXXXXX, JIOXXX, J3XXXX, JVVXXX, JVSXXX, JSSXXX, JWWWXX, JW3WXX, JVSSXX, or JEEXXX are suitable here.

II. complex S1(3),S2(3)
These are complex three-dimensional arrays which contain the wavefunctions of the Scalar bosons, and their four-momenta. The outputs of the subroutines SXXXXX, HIOXXX, HVVXXX, HVSXXX, HSSXXX, HVVSXX or HSSSXX are suitable here.

III. complex GC
This is a complex coupling constant of VVSS vertex.

IV. real VMASS, VWIDTH
The mass and the width of the vector boson emitted from the VVSS vertex.

THE OUTPUT

I. complex JVSS(6) This is a complex six-dimensional array which contains the off-shell vector current from the VVSS vertex attached with the vector boson propagator, combined with its four-momentum.

What we compute here is the following portion of the Feynman amplitude:

$$J^\mu = i\mathtt{G}\frac{i}{q^2 - m^2 + im\Gamma}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{m^2}\right)V^\nu \mathtt{S1(1)S2(1)}, \qquad (3.162)$$

and,

$$\mathtt{JVSS(5)} = \mathtt{VC(5)} + \mathtt{S1(2)} + \mathtt{S2(2)}, \qquad (3.163)$$

$$\mathtt{JVSS(6)} = \mathtt{VC(6)} + \mathtt{S1(3)} + \mathtt{S2(3)}. \qquad (3.164)$$

Here we used the notation

$$V^\mu = \mathtt{VC}(\mu + 1), \qquad (3.165)$$

$$J^\mu = \mathtt{JVSS}(\mu + 1), \qquad (3.166)$$

$$q^\mu = (\Re e(\mathtt{JVSS(5)}), \Re e(\mathtt{JVSS(6)}), \Im m(\mathtt{JVSS(6)}), \Im m(\mathtt{JVSS(5)})), \qquad (3.167)$$

$$m = \mathtt{VMASS}, \qquad (3.168)$$

$$\Gamma = \mathtt{VWIDTH}. \qquad (3.169)$$

We denote the output of the subroutine symbolically as

$$J^\mu_{V'}(V,\, S_1,\, S_2). \qquad (3.170)$$

3.9.3 HVVSXX

This subroutine computes

This subroutine will be called as

```
CALL HVVSXX(V1,V2,SC,GC,SMASS,SWIDTH , HVVS)
```

We have six inputs `V1,V2,SC,GC,SMASS,SWIDTH` and one output `HVVS`.

THE INPUTS

I. `complex V1(6),V2(6)`
These are complex six-dimensional arrays which contain the wavefunctions of the `V`ector bosons, and their four-momenta. The output of the subroutines `VXXXXX`, `JIOXXX`, `J3XXXX`, `JVVXXX`, `JVSXXX`, `JSSXXX`, `JWWWXX`, `JW3WXX`, `JVSSXX`, or `JEEXXX` are suitable here.

II. `complex SC(3)`
This is a complex three-dimensional array which contains the wavefunction of the `SC`alar boson, and its four-momentum. The outputs of the subroutines `SXXXXX`, `HIOXXX`, `HVVXXX`, `HVSXXX`, `HSSXXX`, `HVVSXX` or `HSSSXX` are suitable here.

III. `complex GC`
This is a complex coupling constant of the `VVSS` vertex.

IV. `real SMASS, SWIDTH`
The mass and the width of the scalar boson emitted from the `VSS` vertex.

THE OUTPUT

I. `complex HVVS(3)`
This is a complex three-dimensional array which contains the off-shell scalar current from the `VVSS` vertex.

What we compute here is the following portion of the Feynman amplitude:

$$\mathtt{HVVS(1)} = i\mathtt{G}\frac{i}{q^2 - m^2 + im\Gamma}(V_1 \cdot V_2)\mathtt{SC(1)}, \tag{3.171}$$

and,

$$\begin{aligned} \mathtt{HVVS(2)} &= \mathtt{V1(5)} + \mathtt{V2(5)} + \mathtt{SC(2)}, & (3.172)\\ \mathtt{HVVS(3)} &= \mathtt{V1(6)} + \mathtt{V2(5)} + \mathtt{SC(3)}. & (3.173) \end{aligned}$$

Here we used the notation

$$\begin{aligned} \mathtt{V1}^\mu &= (\mathtt{V1(1)},\, \mathtt{V1(2)},\, \mathtt{V1(3)},\, \mathtt{V1(4)}), & (3.174)\\ \mathtt{V2}^\mu &= (\mathtt{V2(1)},\, \mathtt{V2(2)},\, \mathtt{V2(3)},\, \mathtt{V2(4)}), & (3.175)\\ q^\mu &= (\Re e(\mathtt{HVVS(2)}),\, \Re e(\mathtt{HVVS(3)}),\, \Im m(\mathtt{HVVS(3)}),\, \Im m(\mathtt{HVVS(2)})), & (3.176)\\ m &= \mathtt{SMASS}, & (3.177)\\ \Gamma &= \mathtt{SWIDTH}. & (3.178) \end{aligned}$$

We denote the output of the subroutine symbolically as

$$J_{S'}(V_1,\, V_2,\, S). \tag{3.179}$$

3.10 Vertices IX: SSSS vertex

This subgroup contains two subroutines.

```
SSSSXX  ,
HSSSXX  .
```

The SSSS vertex computed by these subroutines are defined by

$$\mathcal{L}_{\mathtt{SSSS}} = \mathtt{G} S_1^* S_2^* S_3^* S_4^* \tag{3.180}$$

If some of the scalars are identical, then the coupling should be divided by an appropriate statistical factor. Anyway, the coupling G has the same normalization as that appears in the Feynman rule.

3.10.1 SSSSXX

This subroutine computes the amplitude of SSSS vertex.
This subroutine will be called as

```
CALL SSSSXX(S1,S2,S3,S4,GC , VERTEX)
```

We have five inputs S1,S2,S3,S4,GC and one output VERTEX.
THE INPUTS

I. `complex S1(3),S2(3),S3(3),S4(3)`
These are complex three-dimensional arrays which contain the wavefunctions of the Scalar bosons, and their four-momenta. The outputs of the subroutines SXXXXX, HIOXXX, HVVXXX, HVSXXX, HSSXXX, HVVSXX or HSSSXX are suitable here.

II. `complex GC`
This is a complex coupling constant of the SSSS vertex.

THE OUTPUT

I. VERTEX
This is a complex amplitude of SSSS vertex.

What we compute here is the following T-matrix element:

$$\mathtt{VERTEX} = \mathtt{G}\ \mathtt{S1}(1)\ \mathtt{S2}(1)\ \mathtt{S3}(1)\ \mathtt{S4}(1). \tag{3.181}$$

We denote the output of the subroutine symbolically as

$$\Gamma(S_1,\, S_2,\, S_3,\, S_4). \tag{3.182}$$

3.10.2 HSSSXX

This subroutine computes an off-shell scalar current from the SSSS vertex.
This subroutine will be called as

```
CALL HSSSXX(S1,S2,S3,GC,SMASS,SWIDTH , HSSS)
```

We have six inputs S1,S2,S3,GC,SMASS,SWIDTH and one output HSSS.
THE INPUTS

I. `complex S1(3),S2(3),S3(3)`
These are complex three-dimensional arrays which contain the wavefunctions of the Scalar bosons, and their four-momenta. The outputs of the subroutines SXXXXX, HIOXXX, HVVXXX, HVSXXX, HSSXXX, HVVSXX or HSSSXX are suitable here.

II. `complex GC`
This is a complex coupling constant of the SSSS vertex.

III. `real SMASS, SWIDTH`
The mass and the width of the scalar boson emitted from the SSSS vertex.

The Output

I. `complex HSSS(3)`
This is a complex three-dimensional array which contains the off-shell scalar current from the SSSS vertex.

What we compute here is the following portion of the Feynman amplitude:

$$\mathtt{HSSS(1)} = i\mathtt{G}\frac{i}{q^2 - m^2 + im\Gamma}\mathtt{S1(1)}\ \mathtt{S2(1)}\ \mathtt{S3(1)}, \tag{3.183}$$

and,

$$\mathtt{HSSS(2)} = \mathtt{S1(2)} + \mathtt{S2(2)} + \mathtt{S3(2)}, \tag{3.184}$$
$$\mathtt{HSSS(3)} = \mathtt{S1(3)} + \mathtt{S2(3)} + \mathtt{S3(3)}. \tag{3.185}$$

Here, we used the notation

$$q = (\Re e(\mathtt{HSSS(2)}),\ \Re e(\mathtt{HSSS(3)}),\ \Im m(\mathtt{HSSS(3)}),\ \Im m(\mathtt{HSSS(2)})), \tag{3.186}$$
$$m = \mathtt{SMASS}, \tag{3.187}$$
$$\Gamma = \mathtt{SWIDTH}. \tag{3.188}$$

We denote the output of the subroutine symbolically as

$$J_{S'}(S_1,\, S_2,\, S_3). \tag{3.189}$$

3.11 Singular Vertices: EEA vertex

This subgroup contains three subroutines.

EAIXXX ,
EAOXXX ,
JEEXXX .

All of them deal with the QED electron-photon coupling defined by

$$\mathcal{L}_{\mathtt{EEA}} = \sqrt{4\pi\alpha}A^*_\mu \bar{e}\gamma^\mu e. \tag{3.190}$$

Note that all of these subroutines assume that the initial electron beam is running along the positive z-axis, while positron the negative z-axis.

3.11.1 EAIXXX

This subroutine computes an off-shell electron line from the initial electron beam after emitting a photon. Though this subroutine is particularly useful for a collinear emission of the photon from the electron line, the analytic expression in the subroutine is also valid for large angles. However, an expansion in terms of m_e/EB is made, which makes its use at low-energy not valid. The numerical accuracy is sufficient if EB is greater than 1 GeV.

This subroutine will be called as

```
CALL EAIXXX(EB,EA,SHLF,CHLF,PHI,NHE,NHA , EAI) .
```

We have seven inputs EB,EA,SHLF,CHLF,PHI,NHE,NHA and one output EAI.

THE INPUTS

I. `real EB,EA`
   These are Energies of the Beam electron and final photon (A) in the unit of GeV, respectively.

II. `real SHLF,CHLF,PHI`
   These are the angular variables of the final photon. Let θ be the polar angle. SHLF is defined to be $\sin(\theta/2)$, and CHLF $\cos(\theta/2)$. PHI is simply the azimuthal angle. We take these variables as the input to obtain completely numerically safe expressions for the off-shell electron spinor.

III. `integer NHE,NHA`
   These are helicities of the initial Electron and final photon (A), respectively. Both of them should be ± 1.

THE OUTPUT

I. `complex EAI(6)`
   This is a complex six-dimensional array which contains the off-shell electron spinor, emerging from the initial electron beam after emitting a photon.

The final result should be the same with that of FVIXXX, as far as the kinematical region is far from the singularity. The explicit expressions for the output spinor EAI is shown in Appendix A. Note that the initial electron beam is assumed to be running along the positive z-axis.

Note that the electron mass $m_e = 0.51099906$ MeV and the weak scale fine-structure constant $\alpha = 1/128$ is built-in in this subroutine. Though the use of the Thomson coupling $\alpha = 1/137$ seems to be more appropriate for the collinear processes, we use weak scale coupling for the consistency with other coupling constants.

We denote the output of the subroutine symbolically as

$$|e, A, e\rangle. \tag{3.191}$$

3.11.2 EAOXXX

This subroutine computes an off-shell electron line from the initial positron beam after emitting a photon. Though this subroutine is particularly useful for a collinear emission of the photon from the positron line, the analytic expression in the subroutine is also valid for large angles. However, an expansion in terms of m_e/EB is made, which makes its use at low-energy not valid. The numerical accuracy is sufficient if EB is greater than 1 GeV.

This subroutine will be called as

```
CALL EAOXXX(EB,EA,SHLF,CHLF,PHI,NHE,NHA , EAO) .
```

We have seven inputs EB,EA,SHLF,CHLF,PHI,NHE,NHA and one output EAO.

THE INPUTS

I. `real EB,EA`
These are Energies of the Beam positron and final photon (`A`) in the unit of GeV, respectively.

II. `real SHLF,CHLF,PHI`
These are the angular variables of the final photon. Let θ be the polar angle. `SHLF` is defined to be $\sin(\theta/2)$, and `CHLF` $\cos(\theta/2)$. `PHI` is simply the azimuthal angle. We take these variables as the input to obtain completely numerically safe expressions for the off-shell electron spinor.

III. `integer NHE,NHA`
These are helicities of the initial positron ($\mathtt{E}^+$) and final photon (`A`), respectively. Both of them should be ± 1.

The Output

I. `complex EAO(6)`
This is a complex six-dimensional array which contains the off-shell electron spinor, emerging from the initial positron beam after emitting a photon.

The final result should be the same with that of `FVOXXX`, as far as the kinematical region is far from the singularity. The explicit expressions for the output spinor `EAO` is shown in Appendix. Note that the initial positron beam is assumed to be running along the negative z-axis.

Note that the electron mass $m_e = 0.51099906$ MeV and the weak scale fine-structure constant $\alpha = 1/128$ is built-in in this subroutine. Though the use of the Thomson coupling $\alpha = 1/137$ seems to be more appropriate for the collinear processes, we use weak scale coupling for the consistency with other coupling constants.

We denote the output of the subroutine symbolically as

$$\langle e,\, A,\, e|. \tag{3.192}$$

3.11.3 JEEXXX

This subroutine computes an off-shell photon current from the initial electron or positron beam. Though this subroutine is particularly useful for a collinear emission of the photon from the electron/positron line, the analytic expression in the subroutine is also valid for large angles. However, an expansion in terms of $m_e/$`EB` is made, which makes its use at low-energy not valid. The numerical accuracy is sufficient if `EB` is greater than 1 GeV.

This subroutine will be called as

```
CAL JEEXXX(EB,EF,SHLF,CHLF,PHI,NHB,NHF,NSF , JEE)
```

We have eight inputs `EB,EF,SHLF,CHLF,PHI,NHB,NHF,NSF` and one output `JEE`.
The Inputs

I. `real EB,EF`
These are Energies of the Beam electron/positron and final electron/positron, in the unit of GeV.

II. `real SHLF,CHLF,PHI`
These are the angular variables of the final electron or positron. Let θ be the polar angle. `SHLF` is defined to be $\sin(\theta/2)$, and `CHLF` $\cos(\theta/2)$. `PHI` is simply the azimuthal angle. We take these variables as the input to obtain completely numerically safe expressions for the off-shell photon current.

III. `integer NHB,NHF`
These are helicities of the initial (Beam) electron/positron and Final electron/positron, respectively. Both of them should be ± 1.

IV. `integer NSF`
This integer specifies whether the beam is electron (`NSF`$= +1$) or positron (`NSF`$= -1$).

THE OUTPUT

I. `complex JEE(6)`
This is a complex six-dimensional array which contains the off-shell photon current, emitted from the initial electron or positron beam.

The final result should be the same with that of `JIOXXX`, only up to a gauge transformation of the photon current. Since the off-shell photon current approaches a pure gauge in the collinear limit, choosing a more appropriate gauge is necessary to avoid a severe gauge theory cancellation. Taking a different gauge for a single photon current in the Feynman amplitude will not cause problems as far as all the external lines are physical states. The explicit expressions for the output current `JEE` is shown in Appendix A. Note that the initial electron beam is assumed to be running along the positive z-axis, and initial positron beam along the negative z-axis.

Note that the electron mass $m_e = 0.51099906$ MeV and the weak scale fine-structure constant $\alpha = 1/128$ is built-in in this subroutine. Though the use of the Thomson coupling $\alpha = 1/137$ seems to be more appropriate for the collinear processes, we use weak scale coupling for the consistency with other coupling constants.

We denote the output of the subroutine symbolically as

$$J_A^\mu(\langle e|, |e\rangle). \tag{3.193}$$

3.12 Tools

The subroutines in this group play supplementary roles in computing helicity amplitudes which are not absolutely necessary but very useful.

We have four subroutines:

```
MOMNTX  ,
MOM2CX  ,
MOM2LX  ,
BOOSTX  ,
ROTXXX  .
```

3.12.1 MOMNTX

This subroutine computes the four-`MOMeNT`um from the energy, the mass and the angles, and useful to compute the four-momentum of the particles from the phase space integration variables.

This subroutine will be called as

```
CALL MOMNTX(ENERGY,MASS,COSTH,PHI , P)  .
```

We have four inputs `ENERGY,MASS,COSTH,PHI` and one output `P`.

THE INPUTS

I. `real ENERGY, MASS`
These are real variables which contain the energy and the mass of the particle, respectively. The `MASS` has to be positive, and the `ENERGY` has to be greater than or equal to `MASS`.

II. `real COSTH, PHI`
These are real variables which contain the cosine of the polar angle, and the azimuthal angle of the momentum of the particle, respectively.

THE OUTPUT

I. `real P(0:3)`
This is a real four-dimensional array which contains the four-vector, whose spatial components have the angle specified by the input `COSTH, PHI`, and required `ENERGY` and `MASS`.

Thus we get

$$\left\{ \begin{array}{lcl} \texttt{P(0)} & = & \texttt{ENERGY} \\ \texttt{P(1)} & = & |\vec{p}| \sin\theta \cos \texttt{PHI} \\ \texttt{P(2)} & = & |\vec{p}| \sin\theta \sin \texttt{PHI} \\ \texttt{P(3)} & = & |\vec{p}| \texttt{COSTH} \end{array} \right. \tag{3.194}$$

with

$$\begin{aligned} |\vec{p}| &= \sqrt{(\texttt{ENERGY})^2 - (\texttt{MASS})^2}, \\ \sin\theta &= \sqrt{1 - \texttt{COSTH}^2}. \end{aligned}$$

3.12.2 MOM2CX

This subroutine computes the four-`MOM`enta of two (`2`) particles in their `C`enter-of-momentum frame.

This subroutine will be called as

```
CALL MOM2CX(ESUM,MASS1,MASS2,COSTH1,PHI1 , P1,P2)
```

We have five inputs `ESUM,MASS1,MASS2,COSTH1,PHI1` and two outputs `P1,P2`.

The Inputs

I. `real ESUM`
This is the invariant mass of the two particles, or equivalently, `SUM` of their `E`nergies in their center-of-momentum frame.

II. `real MASS1,MASS2`
These are `MASS`es of the particles.

III. `real COSTH1,PHI1`
These are angular variables of the particle 1.

The Output

I. `real P1(0:3),P2(0:3)`
These are real four-dimensional arrays which contain the four-momenta of the particles 1 and 2.

What we compute here is the following kinematics.

$$\texttt{P1}^\mu = (E_1,\ P\sin\theta_1 \cos\texttt{PHI1},\ P\sin\theta_1 \sin\texttt{PHI1},\ P\texttt{COSTH1}), \tag{3.195}$$

$$\texttt{P2}^\mu = (E_2,\ -P\sin\theta_1 \cos\texttt{PHI1},\ -P\sin\theta_1 \sin\texttt{PHI1},\ -P\texttt{COSTH1}), \tag{3.196}$$

where,

$$E_1 = \frac{1}{2}\sqrt{\hat{s}}\left(1 + \frac{m_1^2 - m_2^2}{\hat{s}}\right), \tag{3.197}$$

$$E_2 = \frac{1}{2}\sqrt{\hat{s}}\left(1 - \frac{m_1^2 + m_2^2}{\hat{s}}\right), \tag{3.198}$$

$$P = \frac{1}{2}\sqrt{\hat{s}}\left(1 - 2\frac{m_1^2 + m_2^2}{\hat{s}} + \frac{(m_1^2 + m_2^2)^2}{\hat{s}^2}\right)^{1/2}, \tag{3.199}$$

$$\sin\theta_1 = \sqrt{1 - \texttt{COSTH1}^2}. \tag{3.200}$$

Here, we used the notation

$$
\begin{aligned}
m_1 &= \texttt{MASS1}, & (3.201)\\
m_2 &= \texttt{MASS2}, & (3.202)\\
\hat{s} &= \texttt{ESUM}^2. & (3.203)
\end{aligned}
$$

3.12.3 MOM2LX

3.12.4 BOOSTX

This subroutine performs the Lorentz 'BOOST' on a real four-vector.

This subroutine will be called as

```
CALL BOOSTX(P,Q , PBOOST)   .
```

We have two inputs P,Q and one output PBOOST.

THE INPUTS

I. `real P(0:3)`
   This is a real four-dimensional array which contain the four-momentum which will be boosted.

II. `real Q(0:3)`
   This is a real four-dimensional array which contain the four-momentum which is the reference of the Lorentz boost.

THE OUTPUT

I. `real PBOOST(0:3)`
   This is a real four-dimensional array which contains the four-vector, which is obtained by the Lorentz 'BOOST' from the input four-vector P. In the original frame, the reference four-momentum Q was at rest (it had only the time-component). The Lorentz boost is performed such that the four-momentum at rest will be boosted to the input reference momentum Q. The same boost is acted on the input four-momentum P and the result is the output PBOOST. One can use the same variable name as the input P, however not Q.

Thus we get

$$
\begin{aligned}
\begin{pmatrix} \mathtt{PBOOST(0)} \\ \mathtt{PBOOST(i)} \end{pmatrix}
&= \begin{pmatrix} \mathtt{Q}(0)/m & \mathtt{Q}(j)/m \\ \mathtt{Q}(i)/m & \dfrac{\mathtt{Q}(0)-m}{m}\dfrac{\mathtt{Q}(i)\mathtt{Q}(j)}{|\vec{\mathtt{Q}}|^2} + \delta_{ij} \end{pmatrix}
\begin{pmatrix} \mathtt{P}(0) \\ \mathtt{P}(j) \end{pmatrix} \\
&= \begin{pmatrix} (\mathtt{Q}(0)\mathtt{P}(0) + \vec{\mathtt{Q}}\cdot\vec{\mathtt{P}})/m \\ \left(\mathtt{P}(0) + (\mathtt{Q}(0)-m)\dfrac{\vec{\mathtt{Q}}\cdot\vec{\mathtt{P}}}{|\vec{\mathtt{Q}}|^2}\right)\dfrac{\vec{\mathtt{Q}}}{m} + \vec{\mathtt{P}} \end{pmatrix}, \qquad (3.204)
\end{aligned}
$$

where

$$
m = [\mathtt{Q}(0)^2 - |\vec{\mathtt{Q}}|^2]^{1/2}.
$$

3.12.5 ROTXXX

This subroutine computes a spatial ROTation of a four-momentum.

This subroutine will be called as

```
CALL ROTXXX(P,Q , PROT)
```

We have two inputs P,Q and one output PROT.

THE INPUTS

I. `real P(0:3)`
This is a real four-dimensional array which contain the four-momentum which will be rotated.

II. `real Q(0:3)`
This is a real four-dimensional array which contain the four-momentum which is the reference of the rotation.

THE OUTPUT

I. `real PROT(0:3)`
This is a real four-dimensional array which contains the four-momentum, which is obtained by the `ROT`ation from the input four-vector `P`. In the original frame, the reference four-momentum `Q` was pointing the positive z-axis. The spacial rotation is performed such that the four-momentum pointing positive z-axis will be rotated to the input reference momentum `Q`. The same rotation is acted on the input four-momentum `P` and the result is the output `PROT`. One can use the same variable name as the input `P`, however not `Q`.

What we compute here is given as follows. We rewrite the four-momentum `Q` as

$$\mathtt{Q}^\mu = (\mathtt{Q}(0),\ |\vec{\mathtt{Q}}|\sin\theta\cos\phi,\ |\vec{\mathtt{Q}}|\sin\theta\sin\phi,\ |\vec{\mathtt{Q}}|\cos\theta). \tag{3.205}$$

Since we perform only a spacial rotation, the zero-th component of `P` will not be modified,

$$\mathtt{PROT}(0) = \mathtt{P}(0). \tag{3.206}$$

The spacial components of `P` will be rotated as

$$\begin{pmatrix} \mathtt{PROT}(1) \\ \mathtt{PROT}(2) \\ \mathtt{PROT}(3) \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \mathtt{P}(1) \\ \mathtt{P}(2) \\ \mathtt{P}(3) \end{pmatrix}. \tag{3.207}$$

However, the above definition becomes ambiguous for the limiting cases $\cos\theta = \pm 1$. In these cases,

$$\begin{pmatrix} \mathtt{PROT}(1) \\ \mathtt{PROT}(2) \\ \mathtt{PROT}(3) \end{pmatrix} = \mathrm{sgn}(\cos\theta) \begin{pmatrix} \mathtt{P}(1) \\ \mathtt{P}(2) \\ \mathtt{P}(3) \end{pmatrix}. \tag{3.208}$$

If the spatial vector $\vec{\mathtt{Q}}$ vanishes, then the vector will not be rotated at all.

3.13 Standard Model Coupling Constants

We have two subroutines in this group:

```
COUPSM   ,
COUPMSSM   .
```

These subroutines compute the Minimal Standard Model coupling constants appropriate for the inputs of the `HELAS` subroutines.

3.13.1 `COUPSM`

3.13.2 `COUPMSSM`

3.13.3 `COUP1X`

This subroutine computes the Minimal Standard Model coupling constants for the gauge boson self-couplings.

This subroutine will be called as

```
CALL COUP1X(SW2 , GW,GWWA,GWWZ) .
```

We have one input SW2 and three outputs GW,GWWA,GWWZ.
THE INPUT

I. real SW2
This is simply $\sin^2\theta_W$.

THE OUTPUTS

I. real GW,GWWA,GWWZ
These are real variables which contain $SU(2)_L$ gauge coupling $g_W = e/\sin\theta_W$, WWA coupling constant $e \equiv \sqrt{4\pi\alpha}$, and WWZ coupling constant $e/\sin\theta_W\cos\theta_W$, respectively. These coupling constants will be appropriate inputs for the subroutines VVVXXX, JVVXXX, WWWWXX, JWWWXX, W3W3XX and JW3WXX.

3.13.4 COUP2X

This subroutine computes the Minimal Standard Model coupling constants for the fermion gauge couplings.

This subroutine will be called as

```
CALL COUP2X(SW2 , GAL,GAU,GAD,GWF,GZN,GZL,GZU,GZD,G1) .
```

We have one input SW2 and nine outputs GAL, GAU, GAD, GWF, GZN, GZL, GZU, GZD, G1.
THE INPUT

I. real SW2
This is simply $\sin^2\theta_W$.

THE OUTPUTS

I. real GAL(2),GAU(2),GAD(2)
These are real two-dimensional arrays which contain the coupling constants of the charged lepton, up-type quark, and down-type quark, respectively, with the photon. The first component and the second component contain the same value $-eQ$. These are suitable inputs for the subroutines IOVXXX, FVIXXX, FVOXXX, JIOXXX and J3XXXX.

II. real GWF(2)
This is a real two-dimensional array which contain the universal coupling constant of the fermions and $W^\pm$. Thus the left-handed coupling is GWF(1) $= -g_W/\sqrt{2}$, and the right-handed coupling is zero GWF(2) $= 0$. These are suitable inputs for the subroutines IOVXXX, FVIXXX, FVOXXX, JIOXXX and J3XXXX.

III. real GZN(2), GZL(2), GZU(2), GZD(2)
These are real two-dimensional arrays which contain the coupling constants of the neutrinos, the charged leptons, up-type quarks, and down-type quarks, respectively, with the Z boson. The first component contain the left-handed coupling constants $-g_Z(T^3 - Q\sin^2\theta_W)$, and the second component contain the right-handed coupling constants $-g_Z(-Q\sin^2\theta_W)$. Here $g_Z = e/(\sin\theta_W\cos\theta_W)$. These are suitable inputs for the subroutines IOVXXX, FVIXXX, FVOXXX, JIOXXX and J3XXXX.

Note that the fine-structure constant at the weak scale,

$$\alpha(M_Z) = 1/128 \tag{3.209}$$

is built-in to compute the coupling constants.

3.13.5 COUP3X

This subroutine computes the Minimal Standard Model coupling constants for the gauge boson and Higgs boson couplings.

This subroutine will be called as

```
   CALL COUP3X(SW2,ZMASS,HMASS ,
  &                              GWWH,GZZH,GHHH,GWWHH,GZZHH,GHHHH)
```

We have three inputs SW2,ZMASS,HMASS and six outputs GWWH, GZZH, GHHH, GWWHH, GZZHH, GHHHH.

THE INPUTS

I. `real SW2`
   This is simply $\sin^2\theta_W$.

II. `real ZMASS,HMASS`
   These are MASSes of Z boson and Higgs boson.

THE OUTPUTS

I. `real GWWH,GZZH,GHHH,GWWHH,GZZHH,GHHHH`
   These are real coupling constants of gauge boson and Higgs bosons which are defined below. These coupling constants will be appropriate inputs for the subroutines VVSXXX, JVSXXX, HVVXXX, SSSXXX, HSSXXX, VVSSXX, JVSSXX, HVVSXX, SSSSXX and HSSSXX.

The definitions of the coupling constants are

$$\begin{aligned} \texttt{GWWH} &= g_W m_W, && (3.210)\\ \texttt{GZZH} &= g_Z m_Z, && (3.211)\\ \texttt{GHHH} &= -3\lambda v, && (3.212)\\ \texttt{GWWHH} &= g_W^2, && (3.213)\\ \texttt{GZZHH} &= g_Z^2, && (3.214)\\ \texttt{GHHHH} &= -3\lambda, && (3.215) \end{aligned}$$

where we used the notation

$$\begin{aligned} g_W &= \frac{\sqrt{4\pi\alpha}}{\sin\theta_W}, && (3.216)\\ g_Z &= \frac{\sqrt{4\pi\alpha}}{\sin\theta_W\cos\theta_W}, && (3.217)\\ \lambda &= m_H^2/v^2, && (3.218)\\ v^2 &= \frac{4m_Z^2}{g_Z^2}, && (3.219)\\ m_W &= m_Z\cos\theta_W. && (3.220) \end{aligned}$$

Note that the fine-structure constant at the weak scale,

$$\alpha(M_Z) = 1/128 \tag{3.221}$$

is built-in to compute the coupling constants.

3.13.6 COUP4X

This subroutine computes the Minimal Standard Model coupling constants for the fermion Higgs Yukawa couplings.

This subroutine will be called as

```
CALL COUP4X(SW2,ZMASS,FMASS , GCHF)
```

We have three inputs SW2,ZMASS,FMASS and one output GCHF.

THE INPUT

I. real SW2
   This is simply $\sin^2\theta_W$.

II. real ZMASS,FMASS
   These are real MASSes of Z boson and Fermion whose Yukawa coupling is to be computed.

THE OUTPUT

I. complex GCHF(2)
   This is a two-dimensional Complex array which contain the Yukawa coupling of the Fermion with the Higgs boson. Since the Higgs boson couples to the fermions with the same coupling constants to left- and right-handed chiralities, both GCHF(1) and GCHF(2) contain the same values, $-4g_Z$FMASS/ZMASS. One may wonder why we put the Yukawa coupling into a complex array. This is because one may encounter complex Yukawa couplings, as in the matter-smatter-gaugino vertices.

Note that the fine-structure constant at the weak scale,

$$\alpha(M_Z) = 1/128 \tag{3.222}$$

is built-in to compute the coupling constants.

Chapter 4

HELAS_CHECK

The HELAS subroutines are designed to run as fast as possible. For this purpose, the HELAS subroutines do not check the appropriateness of the inputs at all. There are cases that the FORTRAN77 does not give any error messages no matter how wrong your result is. In such cases, the run-time error messages from HELAS_CHECK may be helpful in identifying the bugs in your program. In cases where the FORTRAN77 gives error messages, additional information from the HELAS_CHECK messages will make your job to find mistakes easy. In particular, possible typographical errors can be easily detected by HELAS_CHECK.

For example, if you put a wrong helicity variable by mistake as,

```
CALL IXXXXX(P,FMASS,NHEK,NSF , FI)
```

where NHEK is a typo of NHEL and has not been defined in the previous part of the program, then FORTRAN77 may assume NHEK=0. Since this input value must be -1 or +1, HELAS_CHECK will announce a HELAS-ERROR:

```
HELAS-ERROR : NHEL in IXXXXX is not -1,1
              NHEL =            0
```

By this message, you will easily understand what is going wrong in your program.

The HELAS_CHECK subroutines are made to check the inputs as much as they can. There are two levels of the run-time messages which HELAS_CHECK gives.

I. HELAS-ERROR

II. HELAS-warn

The HELAS-ERROR messages appear if the inputs *cannot* be accepted as suitable inputs, such as negative mass, tachyonic momentum *etc.* The HELAS-warn messages appear if the inputs *may* have some mistakes, however not necessarily errors. Additionaly, we supply a "scalar polarization" for the massive vector fields in VXXXXX to check the program by making use of the BRS invariance (see Appendix B.4).

III. "scalar polarization" option

We recommend the users to link the HELAS_CHECK first and perform a test-run. For the main-run, use HELAS which is much faster than HELAS_CHECK.

4.1 HELAS-ERROR

4.1.1 `zero momentum` message

This error messages occurs if all the components of the input four-momentum are zero. This must not happen in the most of the subroutines. There are occasions that the kinematics allow such zero momenta at the boundary of the phase space, however, they must be extremely rare in the Monte-Carlo evaluations. Thus the most probable origin of the error is a typo-graphical mistake.

This error message appears if the following criterion is satisfied:

$$|p^0| + |\vec{p}| = 0. \tag{4.1}$$

4.1.2 `non-positive energy` message

This error message occurs if the input four-momenta of the wave functions have zero or negative energy. Note that our convention of the four-momentum of the anti-fermions is such that it is the *physical* four-momentum of the anti-fermion, and *not* the four-momentum along the fermion number flow.[1]

This error message appears if the following criterion is satisfied:

$$\texttt{P(0)} \leq 0. \tag{4.2}$$

4.1.3 `inappropriate mass` message

This error message occurs if the input four-momenta of the wave functions has different mass squared compared to the input mass. Though the four-momenta may be computed in terms of four-momenta of other particles, it should have a mass squared consistent with its mass within the numerical accuracy. There should be something wrong with your phase space program.

This error message appears if the following criterion is satisfied:

$$|p^2 - m^2| > 2 \times 10^{-5} (p^0)^2 \tag{4.3}$$

4.1.4 `not -1,0,1` and `not -1, 1` message

This error message occurs if the input helicity variable or sign factor takes a not allowed value. For massless vector and fermion, the helicity variable should be ± 1, and for massive vector ± 1 or 0. The sign factor `NSF, NSV, NSS` should take either $+1$ or -1.

4.1.5 `not balanced momenta` message

This error message occurs if the four-momenta of the input wavefunctions do not balance in an amplitude subroutine. Since all the momentum flows terminate at the amplitude subroutine, their momenta should sum up to zero within the numerical accuracy.

This error message appears if the following criterion is satisfied:

$$|\sum_i p_i^\mu| > 4 \times 10^{-5} \max_{i,\mu}(|p_i^\mu|). \tag{4.4}$$

4.1.6 `zero coupling` message

This error message occurs if an input coupling constant in vertex subroutine are exactly zero. For a coupling constant with two components, such as the couplings of `FFV` or `FFS` vertices, the error message occurs if both of the components vanish simultaneously. Mostly likely mistake is a mis-type of the name of the coupling constant.

[1] This is the same convention as that of Bjorken-Drell.

4.1.7 `MASS is negative` or `non-positive` message

This error message occurs if an input mass of a subroutine is either negative or non-positive. All the input masses in `HELAS` subroutines are expected to be (semi-)positive.

4.1.8 `WIDTH is negative` message

This error message occurs if an input width of a subroutine is negative. All the input widths in `HELAS` subroutines are expected to be positive.

4.1.9 `on MASS pole` message

This error message occurs if the invariant mass of the off-shell propagator is exactly on-pole *and* its width is zero. Since the kinematics hits the pole region, the width of that particle should be included in the computation.

4.1.10 `not positive` energy message

This error message occurs if the input energy of some particle is not positive. In addition, all the zero-th components of the four-momenta as the inputs of the `HELAS` subroutines should be positive. Though the four-momenta which are combined with the wave functions may have negative zero-th component, they appear only as *outputs* of `HELAS` subroutines, not as *inputs*.

4.1.11 `EA/EF is greater than EB` message

This error message occurs only in the "collinear" subroutines, when the final particle (photon in `EAIXXX`, `EAOXXX` and electron/positron in `JEEXXX`) has larger energy than the initial electron/positron beam.

4.1.12 `improper` message

This error message occurs if a real variable does not lie in the expected region, such as `CHLF`, `SHLF`, `COSTH`, `SW2` with absolute value larger than one.

4.1.13 `SHLF and CHLF are inconsistent` message

This error message occurs if the input `SHLF` and `CHLF` does not satisfy the trigonometric relation $\sin^2\theta + \cos^2\theta = 1$.

The error message appears if the following criterion is satisfied:

$$|\mathtt{SHLF}^2 + \mathtt{CHLF}^2 - 1| > 10^{-5}. \tag{4.5}$$

4.1.14 energy `is less than MASS` message

This error message appears if the input energy of a four-momentum is smaller than its input mass, which cannot happen.

4.1.15 `is spacelike` message

This error message occurs if a four-momentum is space-like, while it should be either time-like or light-like as the correct input. However, this may indeed happen due to a small numerical fluctuation if you are dealing with massless particles (light-like). In that case, this message does not mean that the codes are incorrect.

4.1.16 `Q(0:3) in BOOSTX is not timelike` message

This error message occurs if the reference momentum in `BOOSTX` is not time-like. Since `Q` is supposed to be given in its *rest frame* where `P` momentum is defined, not time-like four-momentum will lead to an inconsistency.

4.2 `HELAS-warn`

4.2.1 `zero spinor/vector/scalar` message

These warning messages occur if all the components of the input polarization vector are exactly zero. Since it really happens, as in the case of the $W^{\pm}$ current from the `JIOXXX` output in the chirality plus sector, this might not be an error. However, you are recommended to check the input. Furthermore, you can often make the program faster by suitably removing the irrelevant part of the program.

These messages appear if the following criterion is satisfied:

$$|\texttt{JIO(1)}| + |\texttt{JIO(2)}| + |\texttt{JIO(3)}| + |\texttt{JIO(4)}| = 0 \tag{4.6}$$

where we have generically denoted the input wavefunction (either spinor, vector or scalar) by `JIO`.

4.2.2 `non-standard coupling` message

This warning message occurs if the input coupling constants appear to be very different from the Standard Model couplings. Of course, you may use `HELAS` for various exotic models, and this may not be a mistake. The criterion for this message depends on the couplings.

4.2.3 `PHI does not lie on 0.0 thru 2.0*PI` message

This warning message occurs if the input azimuthal angle `PHI` does not lie between 0 and 2π. The `HELAS` is made to handle any value of the azimuthal angle, but this might be happening due to a mistake.

4.2.4 `EB too low` message

This warning message occurs if the input `EB` (Beam Energy) of the initial electron/positron is less than 1 GeV. Since the analytic expressions used in `EAIXXX, EAOXXX, JEEXXX` uses an expansion in terms of m_e/`EB`, the subroutine cannot be trusted for a very low energy. One can well replace the subroutine by the ordinary `FVIXXX, FVOXXX, JIOXXX` since there will not be any collinear singularities at such a low energy.

4.2.5 `BOOSTX` messages

There are some messages special to the subroutine `BOOSTX`. A warning message occurs if the input momentum `P` is either space-like or has a non-positive energy. Though this might not be a mistake, it seems to be unlikely that `BOOSTX` is used for such a four-momentum, as far as the Lorentz boost of phase space variables is concerned. There will also appear a warning if the space components of the reference momentum `Q` is exactly zero. This might be also OK, but unlikely.

4.3 "scalar polarization" option

When dealing with a relatively higher order process like vector boson fusion processes (for example, see Appendix B.6), we often wish to know whether our results are really gauge-invariant. In fact, most of the mistakes in `HELAS` codes will cause violation of the gauge-invariance of the amplitudes. Thus the check of the gauge-invariance of the amplitudes is a non-trivial test of the correctness of your program.

For that purpose, it is useful to make use of the BRS-invariance of the theory, as discussed briefly in section 2.9.3, and in a detail in Appendix B.4. The essential part of the technique is that for any process which has vector bosons in the external lines, a non-trivial identity holds,

$$\mathcal{M}_{W_S} - i\mathcal{M}_{\chi} = 0, \tag{4.7}$$

where the amplitude $\mathcal{M}_{V_S}$ is the amplitude for the "scalar" polarization

$$\epsilon_S^{\mu}(p) = \begin{cases} p^{\mu}/m_V & \text{when } m_V \neq 0, \\ p^{\mu}/p^0 & \text{when } m_V = 0, \end{cases} \tag{4.8}$$

and the amplitude $\mathcal{M}_{\chi}$ is the corresponding Goldstone boson emission amplitude. Though the Goldstone bosons are unphysical states which appear in the renormalizable (R_{ξ}) gauges, this identity, in fact, holds even in the unitary gauge. Of course, there is no Goldstone bosons for massless gauge bosons, hence the "scalar" amplitudes themselves should vanish,

$$\mathcal{M}_{V_S} = 0. \tag{4.9}$$

When you check this identity in the numerical program written in `HELAS`, we need the "scalar' polarization" of the vector bosons. So there is a special option

$$\texttt{NHEL} = 4 \tag{4.10}$$

in the subroutine `VXXXXX`, whose output is the scalar polarization in Eq. (4.8). Note that this option is *not* supported in `HELAS.FOR`.

Bibliography

[1] G.P. Lepage, Cornel Univ. preprint CLNS-80/447, Mar 1980.

[2] S. Kawabata, *Comput. Phys. Commun.* **41** (1986) 127.

[3] H.Murayama, Doctoral Thesis, Univ. of Tokyo preprint UT-580 (1991).

[4] C. Becchi, A. Rouet and R. Stora, *Ann. Phys.* **98** (1976) 287.

[5] J.Gunion and Z.Kunszt, *Phys. Lett.* **159B** (1985) 167;
S.Parke and T.Taylor, *Nucl. Phys.* **B269** (1986) 410;
Z.Xu, D.-H.Zhang and L.Chang, *Nucl. Phys.* **B291** (1987) 392;
F.A.Berends and W.T.Giele, *Nucl. Phys.* **B294** (1987) 700.

[6] K.Hagiwara, H.Murayama and I.Watanabe, *Nucl. Phys.* **B367** (1991) 257.

[7] K.Hagiwara, H.Iwasaki, A.Miyamoto, H.Murayama and D.Zeppenfeld, *Nucl. Phys.* **B365** (1991) 544.

[8] K.Hagiwara, J.Kanzaki and H.Murayama, KEK preprint KEK-TH-282 (1991).

[9] H. Baer, K. Hagiwara, and X. Tata, *Phys. Rev.* **D35** (1987) 1598.

[10] K.Hagiwara and D.Zeppenfeld, *Nucl. Phys.* **B274**, 1 (1986);
K.Hagiwara and D.Zeppenfeld, *Nucl. Phys.* **B313**, 560 (1989).

Appendix A

Conventions

A.1 γ-matrices and Spinors

Throughout this program, we use the Weyl-basis for the spinors and γ-matrices. We follow the conventions of ref. [10].

We put the chirality-left sector in the upper-component:

$$\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A.1}$$

Correspondingly, the γ-matrices are:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma_+^\mu \\ \sigma_-^\mu & 0 \end{pmatrix}, \tag{A.2}$$

where the σ-matrices are defined by

$$\sigma_\pm^\mu \equiv (1, \ \pm\vec{\sigma}). \tag{A.3}$$

We first define the helicity-eigenspinors χ_+ and χ_-.

$$\begin{aligned} \chi_+(\vec{p}) &= \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}|+p_z)}} \begin{pmatrix} |\vec{p}|+p_z \\ p_x+ip_y \end{pmatrix}, \\ \chi_-(\vec{p}) &= \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}|+p_z)}} \begin{pmatrix} -p_x+ip_y \\ |\vec{p}|+p_z \end{pmatrix}. \end{aligned} \tag{A.4}$$

These helicity eigenspinors satisfy

$$\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\chi_\lambda(\vec{p}) = \lambda\,\chi_\lambda(\vec{p}), \tag{A.5}$$

where $\lambda = \pm 1$. Since the above definition is ambiguous for the momentum $p_z = -|\vec{p}|$, we fix the notation by taking the limit where $p_y = 0$ and $p_x \to +0$:

$$\begin{aligned} \chi_+(\vec{p}) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ \chi_-(\vec{p}) &= \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \end{aligned} \tag{A.6}$$

The four-spinors are defined as follows:

$$\begin{aligned} u(p) &= \begin{pmatrix} \omega_{-\lambda}(p)\chi_\lambda(\vec{p}) \\ \omega_\lambda(p)\chi_\lambda(\vec{p}) \end{pmatrix}, \\ v(p) &= \begin{pmatrix} -\lambda\omega_\lambda(p)\chi_{-\lambda}(\vec{p}) \\ \lambda\omega_{-\lambda}(p)\chi_{-\lambda}(\vec{p}) \end{pmatrix}, \end{aligned} \tag{A.7}$$

where

$$\omega_\pm(p) \equiv \sqrt{E \pm |\vec{p}|}. \tag{A.8}$$

A.2 Polarization Vectors

We fix the notation for the helicity eigenvectors of the vector bosons as follows. If the four-momentum of the vector is

$$k^\mu = (E,\ k_x,\ k_y,\ k_z), \tag{A.9}$$

the three-independent polarization vectors which satisfy $k \cdot \epsilon = 0$ are:

$$\begin{aligned}
\epsilon^\mu(k,1) &= (|\vec{k}|k_T)^{-1}(0,\ k_x k_z,\ k_y k_z,\ -k_T^2), && \text{(A.10)}\\
\epsilon^\mu(k,2) &= (k_T)^{-1}(0,\ -k_y,\ k_x, 0), && \text{(A.11)}\\
\epsilon^\mu(k,3) &= (E/m|\vec{k}|)(|\vec{k}|^2/E,\ k_x,\ k_y,\ k_z), && \text{(A.12)}
\end{aligned}$$

where

$$\begin{aligned}
m &= \sqrt{E^2 - |\vec{k}|^2}, && \text{(A.13)}\\
k_T &= \sqrt{(k_x)^2 + (k_y)^2}. && \text{(A.14)}
\end{aligned}$$

Since the definition of the polarization vector $\epsilon(k,1)$ and $\epsilon(k,2)$ is ambiguous in the limit $k_T \to 0$, we fix our notation by taking the limit where $k_y = 0$ and

$$\begin{cases} k_x \to +0 & (k_z > 0) \\ k_x \to -0 & (k_z < 0). \end{cases}$$

The helicity eigenvectors for $\lambda = \pm, 0$ are:

$$\begin{aligned}
\epsilon^\mu(k, \lambda = \pm) &= \frac{1}{\sqrt{2}}(\mp\epsilon^\mu(k,1) - i\epsilon^\mu(k,2)), && \text{(A.15)}\\
\epsilon^\mu(k, \lambda = 0) &= \epsilon^\mu(k,3). && \text{(A.16)}
\end{aligned}$$

Of course, we don't have the $\lambda = 0$ component for the massless vector boson.

A.3 HELAS coupling constants

In general, we start with the following convention for the covariant-derivative:

$$D_\mu \equiv \partial_\mu + igA_\mu, \tag{A.17}$$

where A_μ is matrix-valued if the gauge-group is non-abelian. This convention for the covariant derivative will lead to the following fermion-fermion-gauge FFV vertex:

$$\mathcal{L} = -g\bar{f}'\gamma^\mu f V_\mu^*. \tag{A.18}$$

For non-derivative couplings, we define the coefficients of the Lagrangian as the HELAS coupling constants, use these as the inputs of the HELAS subroutines. For example, the above FFV vertex has the HELAS coupling constant

$$\texttt{G(I)} = -g, \tag{A.19}$$

where I = 1 for a chirality-left fermion and = 2 for a chirality-right fermion. The same holds for the scalar self-couplings. The Lagrangian

$$\mathcal{L} = -\frac{\lambda}{4!}\phi^4 \tag{A.20}$$

for a real scalar field gives the HELAS coupling constant $-\lambda$ for the SSSSXX subroutine. According to this definition, the electron to photon coupling is

$$\texttt{GAL(1)} = \texttt{GAL(2)} = +\sqrt{4\pi\alpha}, \tag{A.21}$$

since the electron has a *negative* charge. You can regard the outputs of `COUP1X` to `COUP4X` as templates to define the coupling constants in your favorite model. If one has a Feynman rule, then the coupling constant with one i stripped off is the `HELAS` coupling constant.

For derivative couplings, the proper definition is less clear. The definition of the coupling constants depends on the choice of 'particle' and 'anti-particle'. For the VVV vertex, we explicitly fixed the ordering of the input vector bosons in the order $W^-W^+W^3$, where the boson quantum numbers are defined by those flowing out from the vertex. The coupling constants are defined such that they will be positive for this order. If one reverses the ordering, the coupling constant should be changed in sign. For the coupling constants in the SSV vertex, `HELAS` assumes that the *first* input of the scalar wavefunction is 'particle'. For example, if the first scalar wavefunction `S1(3)` in `SSVXXX` represents a scalar electron and `VC(6)` a photon, then the coupling constant will be `GAL(1)` (or equivalently, `GAL(2)`). If the first input is a scalar anti-electron, then the coupling constant should be `-GAL(1)`. In any case, the definition of the coupling constants in the SSV vertex follows that of the FFV vertex as one regards the first input scalar as a 'particle'. Here we note again that all the quantum numbers are defined to be flowing-out from the vertex. Thus, for instance, the scalar neutrino in the initial state has the coupling constant `-GZN(1)` with the Z-boson.

The outputs of off-shell wavefunctions include the i's appearing in the Feynman rules and propagators. These points are explicitly described for each subroutines in Chapter 3. Since `HELAS` coupling constants are just the coefficients of the Lagrangian, the Feynman rule is i times the `HELAS` coupling constant. The propagator for the scalar and fermion line also contains another factor of i, while for the vector line it is a factor of $-i$.

The outputs of the amplitude subroutines (`VERTEX`) are always T-matrix elements, rather than the Feynman amplitudes. A Feynman amplitude is actually an S-matrix element which is related to the T-matrix element by

$$S = I + iT. \tag{A.22}$$

Or, in other words, the `HELAS` amplitudes (T-matrix elements) have the same phase as the Lagrangian itself. This point should be known to the users who wish to add a new type of amplitudes to the `HELAS` outputs, as may be needed in radiative corrections.

A.4 Collinear Singularities

For many processes including the t-channel photon exchange or initial state photon emission, naive application of `HELAS` will break down due to severe numerical cancellations. For example in the process $e^+e^- \to e^-\bar{\nu}_e W^+$, the t-channel photon exchange determined the character of the whole distributions as we discussed in Chapter 4. There a wide plateau appeared in the pseudo-rapidity distribution of the final e^-, which goes up to a very forward region $1-\cos\theta \sim 10^{-17}$. To achieve this accuracy, we need the quadruple precision, which is not very economical.

To avoid this problem, we supply special subroutines to deal with the collinear singularities which include the t-channel photon exchange and initial state soft photon radiation.

First let us explain our treatment of the t-channel photon exchange. From the general framework of `HELAS`, all we need is the off-shell (though almost on-shell) wavefunction of the t-channel photon. From the `HELAS` Feynman rules described above, we need

$$J_A^\mu(\langle e'|, |e\rangle) = -D_A(p_A^2)\hat{J}_A^\mu(\langle e'|, |e\rangle), \tag{A.23}$$

to a high degree of numerical accuracy. The problem is two-fold. The denominator of the propagator factor D_A, and the matrix element $\hat{J}_A^\mu$.

The output of the `JEEXXX` is simply

$$\hat{J}_A^\mu(\langle e'|, |e\rangle) = (-e)\bar{u}(p,\sigma')\gamma^\mu u(k,\sigma), \tag{A.24}$$

for the t-channel photon emission from the electron current. Since the problem is limited to the case that the initial state is the electron (or positron, which will be discussed later) beam, we completely fix the frame in the following. We take the four-momenta to be

$$\begin{aligned} k^\mu &= E(1,\ 0,\ 0,\ \beta) \\ p^\mu &= xE(1,\ \beta'\sin\theta\cos\phi,\ \beta'\sin\theta\sin\phi,\ \beta'\cos\theta), \end{aligned} \tag{A.25}$$

where

$$\begin{aligned} \beta &= \sqrt{1-\frac{m_e^2}{E^2}}, \\ \beta' &= \sqrt{1-\frac{m_e^2}{x^2E^2}}. \end{aligned} \tag{A.26}$$

Now we write down the expression explicitly.

For the helicity non-flip case $\sigma = \sigma'$, the truncated current reads

$$\hat{J}_A^\mu = (-e)\sqrt{x}(2E)\left(\cos\frac{\theta}{2},\ \sin\frac{\theta}{2}e^{-i\sigma\phi},\ i\sigma\sin\frac{\theta}{2}e^{-i\sigma\phi},\ \cos\frac{\theta}{2}\right), \tag{A.27}$$

where the terms of order $O(m_e^2/E^2)$ are neglected. One can check the conservation of the current by contracting with the four-momentum of the photon

$$\begin{aligned} p_A^\mu &= k^\mu - p^\mu \\ &= E(1-x,\ -x\beta'\sin\theta\cos\phi,\ -x\beta'\sin\theta\sin\phi,\ \beta - x\beta'\cos\theta) \\ &= E(1-x,\ -x\sin\theta\cos\phi,\ -x\sin\theta\sin\phi,\ 1-x\cos\theta) + O\left(\frac{m_e^2}{E}\right), \end{aligned} \tag{A.28}$$

to the accuracy of order $O(m_e^2/E^2)$.

Since the problem lies in the collinear limit $\cos\theta \to 1$, it is instructive to give the expression in the limit,

$$\begin{aligned} \hat{J}_A^\mu &= \sqrt{x}(2E)(1,\ 0,\ 0,\ 1), \\ p_A^\mu &= E(1-x,\ 0,\ 0,\ 1-x). \end{aligned} \tag{A.29}$$

Here the truncated current is completely proportional to its four-momentum, or in other words, it is *pure gauge*! In analytic evaluation of the amplitudes, the appearance of the pure gauge current does not cause any harm, however it requires cancellation among the diagrams. For the numerical evaluation of the amplitudes, we wish to avoid possible cancellations as much as we can. Since we can change the gauge of the photon freely as discussed in Chapter 2, we subtract a four-vector proportional to the four-momentum p_A to make its largest component (zeroth component) vanish. Thus, the modified truncated current reads

$$\begin{aligned} \hat{J}_A^\mu - \hat{J}_A^0\frac{p_A^\mu}{p_A^0} &= (-e)\sqrt{x}(2E)\sin\frac{\theta}{2}\left(0,\ e^{-i\sigma\phi} + \frac{2x}{1-x}\cos^2\frac{\theta}{2}\cos\phi,\right. \\ &\qquad \left. i\sigma e^{-i\sigma\phi} + \frac{2x}{1-x}\cos^2\frac{\theta}{2}\sin\phi,\ -\frac{2x}{1-x}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right), \end{aligned} \tag{A.30}$$

which has the safe vanishing limit in the $\cos\theta \to 1$ with the expected behaviour $\sim \sin(\theta/2)$. Note that the expression is safe *if* we knew the $\sin(\theta/2)$ to high accuracy, and $\cos\theta$ is not a suitable input here. Recall that we have to know $1-\cos\theta$ up to the accuracy of 10^{-17}! Thus, we always take the set

$$\sin\frac{\theta}{2}, \qquad \cos\frac{\theta}{2} \tag{A.31}$$

as the inputs of the current. This completes the treatment of the matrix element, which is now completely safe numerically.

Now comes the treatment of the photon propagator. If one may naively take the square from the expression of p_A^μ Eq. (A.28), we obtain a vanishing result. This is due the fact that we have neglected the terms of order $O(m_e^2/E)$ in the expression. We have to go back to the original definition $p_A^\mu = k^\mu - p^\mu$ to compute

$$\begin{aligned} D_A(p_A^2)^{-1} &= (k-p)^2 \\ &= 2m_e^2 - 2k\cdot p \\ &= -x(2E^2)\beta\beta'(1-\cos\theta) - t_{\min}(x), \end{aligned} \tag{A.32}$$

with the notation

$$t_{\min}(x) = m_e^2 \frac{(1-x)^2}{x} + O\left(\frac{m_e^4}{E}\right) \tag{A.33}$$

defined in App. B. Note that the relevant combination is again $1-\cos\theta$, which can be rewritten as

$$1-\cos\theta = 2\sin^2\frac{\theta}{2}, \tag{A.34}$$

which will be numerically safe once we adopt $\sin(\theta/2)$ as the input.

Combining the truncated current and the propagator factor, we have now a special current J_A^μ for the t-channel photon exchange. The case for the positron current goes just analogously; using

$$\begin{aligned} k^\mu &= E(1,\ 0,\ 0,\ -\beta) \\ p^\mu &= xE(1,\ \sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta), \end{aligned} \tag{A.35}$$

the helicity non-flip current reads

$$\hat{J}_A^\mu = -(-e)\sqrt{x}(2E)\left(\sin\frac{\theta}{2}e^{-i\sigma\phi},\ \cos\frac{\theta}{2},\ -i\sigma\cos\frac{\theta}{2},\ -i\sigma e^{-i\sigma\phi}\right). \tag{A.36}$$

We obtain similar expressions for the modified current by subtracting a four-vector proportional to the photon four-momentum to make the zero-th component vanish. It can be worked out easily from the above expression.

We have discussed the helicity-conserving case so far. The helicity-flip currents are much more straightforward to obtain, since there is no large pure gauge part. We give the expressions for the truncated currents below:

$$\begin{aligned} \hat{J}_A^\mu = (-e)\frac{m_e}{\sqrt{x}} \quad &\left(-\sigma(1+x)\sin\frac{\theta}{2}e^{i\sigma\phi},\ \sigma(1-x)\cos\frac{\theta}{2},\right. \\ &\qquad \left. i(1-x)\cos\frac{\theta}{2},\ -\sigma(1-x)\sin\frac{\theta}{2}e^{i\sigma\phi}\right) \end{aligned} \tag{A.37}$$

for the electron, and

$$\begin{aligned} \hat{J}_A^\mu = (-e)\frac{m_e}{\sqrt{x}}e^{i\sigma\phi} \quad &\left(-\sigma(1+x)\cos\frac{\theta}{2}e^{-i\sigma\phi},\ \sigma(1-x)\sin\frac{\theta}{2},\right. \\ &\qquad \left. -i(1-x)\sin\frac{\theta}{2},\ \sigma(1-x)\cos\frac{\theta}{2}e^{-i\sigma\phi}\right) \end{aligned} \tag{A.38}$$

for the positron current. Though they are proportional to the electron mass due to the helicity flip, they survive in the total cross section since the small denominator of the t-channel photon propagator will become as small as $m_e^2(1-x)^2/x$.

It is worth noting here that, even though the shift of the electromagnetic current as in eq.(A.30) is known not to affect the matrix elements in QED, it does not immediately follow that the same applies in the electroweak theory. Here we give a simple proof [3] based on the BRS invariance [4] of the electroweak theory. The BRS identity relevant to our problem is

$$\langle phys;\ \mathrm{out}|\ \partial^\mu A_\mu\ |phys;\ \mathrm{in}\rangle = 0, \tag{A.39}$$

where A_μ is the photon field. The identity follows from the anti-commutation relation

$$\{Q_{BRS},\ \bar{c}_A\} = \partial^\mu A_\mu, \tag{A.40}$$

where Q_{BRS} is the BRS charge and $\bar{c}_A$ is the anti-ghost operator associated with the photon, since Q_B annihilates the physical states

$$\langle phys;\ \mathrm{out}|Q_B = Q_B|phys;\ \mathrm{in}\rangle = 0. \tag{A.41}$$

When just one off-shell photon current is replaced by the shifted current (A.30), the matrix element is shifted by a term proportional to

$$\langle phys;\ \mathrm{out}|A_\mu|phys;\ \mathrm{in}\rangle q^\mu, \tag{A.42}$$

which vanishes by the BRS identity as long as both the other initial state and the final states are all on-shell physical states. Hence, the shifted current gives the correct helicity amplitudes in the electroweak theory.

For the processes where a nearly on-shell t-channel photon is exchanged twice (the 'two-photon' processes), we may want to use the shifted current twice in the same amplitude. The above proof no longer applies, because the shifted current is **not** the physical state that appears in eq.(A.42). We need only a little more algebra [3]

$$\begin{aligned}
&\partial^\mu A_\mu(x)\,\partial^\nu A_\nu(y) \\
&\quad= \{Q_{BRS},\ \bar{c}_A(x)\}\{Q_{BRS},\ \bar{c}_A(y)\} \\
&\quad= Q_{BRS}\bar{c}_A(x)Q_{BRS}\bar{c}_A(y) + \bar{c}_A(x)Q_{BRS}\bar{c}_A(y)Q_{BRS} + Q_{BRS}\bar{c}_A(x)\bar{c}_A(y)Q_{BRS},
\end{aligned} \tag{A.43}$$

where we made use of the nilpotency of the BRS charge $Q_{BRS}^2 = 0$. This identity ensures that the double gauge transformation does not cause any harm as long as the remaining external lines are all on-shell, since the BRS charges at the left-most or right-most annihilate the out- or in-state respectively. Thus, the use of the shifted current eq. (A.30) for the off-shell photon wavefunction is completely justified.

The treatment of the t-channel photon in the `JEEXXX` subroutine is completed so far. The treatment of the collinear emission of the photon is much simpler. The subroutines `EAIXXX` and `EAOXXX` will do the job for the emission of the photon from the electron and positron line, respectively.

For the emission from the electron in `EAIXXX`, we need the off-shell wavefunction of the electron after the emission of the photon:

$$|e', A, e\rangle \equiv U = \frac{1}{\not{q} - m_e}(e\gamma^\mu)\epsilon^*_\mu(p,\lambda)u(p,\sigma) \tag{A.44}$$

according to the `HELAS` Feynman rules. This spinor is the output `EAI(6)` of the `EAIXXX`. Again we separate the propagator factor and the matrix element into

$$\begin{aligned}
U &= \frac{1}{q^2 - m_e^2}\hat{U}, \\
\hat{U} &= (\not{q} + m_e)(e\gamma^\mu)\epsilon^*_\mu(p,\lambda)u(p,\sigma).
\end{aligned} \tag{A.45}$$

Here we adopt the notation similar to the photon case

$$\begin{aligned}
k^\mu &= E(1,\ 0,\ 0,\ \beta), \\
p^\mu &= xE(1,\ \sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta), \\
q^\mu &= k^\mu - p^\mu.
\end{aligned} \tag{A.46}$$

We give the relevant formulae for $\sigma = \pm$ separately. For $\sigma = +$,

$$\hat{U} = e\lambda\sqrt{E}\left[\begin{array}{c} m_e\begin{pmatrix} -\sin\theta(1+\frac{1}{2}x(1+\lambda)) \\ \frac{1}{2}xe^{i\phi}(1+\cos\theta)(1+\lambda) \end{pmatrix} \\ E\begin{pmatrix} \sin\theta(-2+x(1-\lambda)) \\ xe^{i\phi}(1-\lambda)(1-\cos\theta) \end{pmatrix} \end{array}\right], \tag{A.47}$$

and for $\sigma = -$,

$$\hat{U} = -e\lambda\sqrt{E}\left[\begin{array}{c} E\begin{pmatrix} xe^{-i\phi}(1+\lambda)(1-\cos\theta) \\ \sin\theta(2-x(1+\lambda)) \end{pmatrix} \\ m_e\begin{pmatrix} \frac{1}{2}xe^{-i\phi}(1-\lambda)(1+\cos\theta) \\ \sin\theta(1+\frac{1}{2}x(1-\lambda) \end{pmatrix} \end{array}\right]. \tag{A.48}$$

The same is done for the positron spinor `EAO(6)` in the subroutine `EAOXXX`. We will fix the kinematics to the frame

$$\begin{aligned} k^\mu &= E(1,\ 0,\ 0,\ -\beta), \\ p^\mu &= xE(1,\ \sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta), \\ q^\mu &= p^\mu - k^\mu. \end{aligned} \tag{A.49}$$

We compute the truncated spinors

$$\hat{V} = \bar{v}(k,\sigma)(e\gamma^\mu)\epsilon^*_\mu(p,\lambda)(\not{q}+m_e), \tag{A.50}$$

in terms of which the off-shell positron wave function is expressed as

$$\langle e, A, e'| \equiv V = \frac{1}{q^2-m_e^2}\hat{V}. \tag{A.51}$$

The formulae read for $\sigma = +$,

$$\begin{aligned} \hat{V} = \quad -e\lambda\sqrt{E} \quad & \left[E\left(\sin\theta(-2+x(1-\lambda)),\ -xe^{-i\phi}(1+\cos\theta)(1-\lambda)\right),\right. \\ & \left. m_e\left(\sin\theta(1+\frac{1}{2}x(1+\lambda)),\ \frac{x}{2}e^{-i\phi}(1-\cos\theta)(1+\lambda)\right)\right]. \end{aligned} \tag{A.52}$$

and for $\sigma = -$,

$$\begin{aligned} \hat{V} = \quad e\lambda\sqrt{E} \quad & \left[m_e\left(-\frac{x}{2}e^{i\phi}(1-\cos\theta)(1-\lambda),\ \sin\theta(1+\frac{x}{2}(1-\lambda)\right),\right. \\ & \left. E\left(x(1+\cos\theta)(1+\lambda)e^{i\phi},\ \sin\theta(-2+x(1+\lambda))\right)\right]. \end{aligned} \tag{A.53}$$

The propagators should be treated separately just as in the case of the currents, and we use

$$q^2 - m_e^2 = -2xE^2\left(1-\cos\theta+\frac{1}{2}\frac{m_e^2}{E^2}\cos\theta\right), \tag{A.54}$$

for the emission from the electron, and

$$q^2 - m_e^2 = -2xE^2\left(1+\cos\theta-\frac{1}{2}\frac{m_e^2}{E^2}\cos\theta\right), \tag{A.55}$$

for the emission from the positron. Both have again safe expressions if we adopt $\cos(\theta/2)$ and $\sin(\theta/2)$ as the inputs.

Appendix B

Sample Programs

In this chapter, we show several sample programs to clarify the use of `HELAS`.

The first example, $W^+W^- \to t\bar{t}$, is the one discussed in Chap. 2 to introduce the use of `HELAS` subroutines. The whole program, including the setting of the kinematics, is presented in Appendix B.1. The readers are recommended to run this program first for amusement.

The second example of vector boson scattering illustrates the use of $VVVV$ vertices, which might appear complicated to beginners. This also shows that vector boson scattering can be treated in a very simple way using `HELAS` system. The process $W^+W^- \to W^+W^-$ also includes Higgs boson exchange, which can demonstrate the gauge theory cancellation between the vector boson scattering amplitude and the Higgs boson exchange amplitude.

The third example, $e^-e^+ \to \gamma Z$ shows the use of the `EAIXXX` and `EAOXXX` subroutines. Since their use requires special kinematical variables to make the collinear region numerically safe, you should get accustomed to the inputs of these subroutines. Here again the whole program is presented, and it can be run immediately.

The fourth example, $e^-e^+ \to e^-\bar{\nu}_e W^+$ is presented to show how to use `JEEXXX`, which also requires special kinematical inputs. Also a discussion on the test of BRS invariance is shown, which is extremely useful in checking the `HELAS` programs if there are vector bosons in external lines.

The fifth example, $e^-e^+ \to W^-W^+$ shows you how the `ROTXXX` and `BOOSTX` can be used to translate between the W rest frame and the laboratory frame. This example has rich physical significance, since the helicity measurement of the W boson is done exactly in the way presented in this program. This example also proves how easy it is to incorporate decay of the final state particles.

The last example shows the power of the `HELAS` system which makes such a complicated higher order process into a very compact program. Also a discussion on crossing is presented.

We believe that these examples are enough to show the use of the `HELAS` system, and convince you how useful it is.

B.1 $W^-W^+ \to t\bar{t}$

This is the example discussed in Chap. 2. The virtue of this example is that one can explicitly see the gauge theory cancellation between various diagrams. The case with longitudinal W's is interesting. The $J = 1$ partial wave cancels between the s-channel and t-channel diagrams, and the $J = 0$ amplitude cancels between the s-channel and Higgs diagrams. If t and $\bar{t}$ have opposite helicities, then the $J = 0$ wave will be absent. The gauge theory cancellation is a non-trivial check of the HELAS amplitudes and its numerical accuracy.

This program produces a data file for TOPDRAWER. The first half of the program does only the initializations, including setup of the momenta and couplings. The subroutine WWTT computes the amplitudes.

```
C ******************************************************************
C *****                     +  -        _                    *****
C *****     TEST PROGRAM :   W  W  => t   t                    *****
C *****                                                      *****
C *****                      by H. Murayama  :  Feb. 15th 1992   *****
C ******************************************************************
C
C INPUTS:
C     helicities of W- and W+
C     helicities of top and tbar
C
C OUTPUT:
C UNIT=1 :
C absolute value of the total amplitude
C UNIT=2 :
C absolute value of the s-channel amplitude
C UNIT=3 :
C absolute value of the t-channel amplitude
C UNIT=4 :
C absolute value of the Higgs amplitude
C
C *****     MAIN PROGRAM : two body to two body                *****
C
      IMPLICIT REAL (B-H,M,O-Z)
      IMPLICIT COMPLEX (A)
      REAL     PWP(0:3),PWM(0:3),PT(0:3),PTB(0:3)
      REAL     GAL(2),GAU(2),GAD(2),GWF(2),GZN(2),GZL(2),GZU(2),GZD(2),
     &         G1(2)
      COMPLEX  GCHT(2)
      COMMON / MASS /WMASS,TMASS,ZMASS,HMASS,ZWIDTH,HWIDTH
      COMMON / COUP /GAU,GZU,GWF,GCHT,GW,GWWH
C
      WRITE(*,*) 'Input sqrt(s) in GeV'
      READ (*,*) ROOTS
      WRITE(*,*) 'Input mt in GeV'
      READ (*,*) TMASS
      WRITE(*,*) 'Input helicities of W- and W+'
      READ (*,*) NHWM,NHWP
      WRITE(*,*) 'Input helicities of t and tbar'
      READ (*,*) NHT,NHTB
C
      WRITE(1,1000)
      WRITE(2,1000)
      WRITE(3,1000)
      WRITE(4,1000)
1000  FORMAT('   SET LIMITS X FROM -1 TO 1')
C
C
      PI     = REAL(3.141592653589793D0)
      PBGEV2 = 0.389E+09
C
      WMASS  =  80.0
      ZWIDTH =  2.5
      SW2    =  0.23
```

```
      ZMASS  =  WMASS/SQRT(1.0-SW2)
      HMASS  =  100.0
C
C We determine the coupling constants which will be contained in the arrays.
C
      CALL COUP1X(SW2 , GW,GWWA,GWWZ)
      CALL COUP2X(SW2 , GAL,GAU,GAD,GWF,GZN,GZL,GZU,GZD,G1)
      CALL COUP3X(SW2,ZMASS,HMASS , GWWH,GZZH,GHHH,GWWHH,GZZHH,GHHHH)
      CALL COUP4X(SW2,ZMASS,TMASS , GCHT)
C
      S=ROOTS**2
      EBEAM=ROOTS*.5
C
C The four-momenta of the initial W- and W+
C
      COSTH = 1.
      PHI   = 0.
      CALL MOM2CX(ROOTS,WMASS,WMASS,COSTH,PHI , PWM,PWP)
C
C The phase space factor:
C
      BETAF=SQRT(1.-4.*TMASS**2/S)
      SBETA=S*SQRT(1.-4.*WMASS**2/S)
      FACTOR=PBGEV2/2.0/(2.*SBETA)*(BETAF/8./PI)/2.
C
      WRITE(*,*) 'Input PHI of W-'
      READ (*,*) PHI
C
      DO 999 IHCOST=-100,100
C
      COSTH=REAL(IHCOST)*.01
C
C The four-momenta of the final t and tbar
C
      CALL MOM2CX(ROOTS,TMASS,TMASS,COSTH,PHI , PT,PTB)
C
C We call the subroutine which computes the amplitudes.
C
      CALL WWTT(PWM,PWP,PT,PTB,NHWM,NHWP,NHT,NHTB , AMPT,AMPS,AMPH)
C
      WRITE(1,*) COSTH, ABS(AMPT+AMPS+AMPH)
      WRITE(2,*) COSTH, ABS(AMPT)
      WRITE(3,*) COSTH, ABS(AMPS)
      WRITE(4,*) COSTH, ABS(AMPH)
C
 999   CONTINUE
C
      WRITE(1,1010)
      WRITE(2,1010)
      WRITE(3,1010)
      WRITE(4,1010)
1010  FORMAT('   JOIN')
C
   1  CONTINUE
      STOP
      END
C
C234567890=========2=========3=========4=========5=========6=========7==
```

```
      SUBROUTINE WWTT(PWM,PWP,PT,PTB,NHWM,NHWP,NHT,NHTB, AMPT,AMPS,AMPH)
      IMPLICIT REAL (B-H,M,O-Z)
      IMPLICIT COMPLEX (A)
      COMPLEX  FI(6),FO(6),FVI(6),WP(6),WM(6),J3(6),HTT(3)
      REAL     PWP(0:3),PWM(0:3),PT(0:3),PTB(0:3)
      REAL     GAU(2),GWF(2),GZU(2)
      COMPLEX  GCHT(2)
      COMMON / MASS /WMASS,TMASS,ZMASS,HMASS,ZWIDTH,HWIDTH
      COMMON / COUP /GAU,GZU,GWF,GCHT,GW,GWWH
C
      PI    =REAL(3.141592653589793D0)
      HWIDTH = 0.
C
C The initial state wavefunction of the W's:
C
      CALL VXXXXX(PWM,WMASS,NHWM,-1 , WM)
      CALL VXXXXX(PWP,WMASS,NHWP,-1 , WP)
C
C The final state wavefunction of top and tbar.
C
      CALL OXXXXX(PT ,TMASS,NHT ,+1 , FO)
      CALL IXXXXX(PTB,TMASS,NHTB,-1 , FI)
C
C First, we compute the s-channel Z, photon exchange diagram.
C
      CALL J3XXXX(FI,FO,GAU,GZU,ZMASS,ZWIDTH , J3)
      CALL VVVXXX(WP,WM,J3,GW , AMPS)
C
C Next we compute the t-channel bottom exchange diagram.
C
      CALL FVIXXX(FI,WM,GWF,0.,0. , FVI)
      CALL IOVXXX(FVI,FO,WP,GWF , AMPT)
C
C Finally we compute the s-channel Higgs exchange diagram.
C
      CALL HIOXXX(FI,FO,GCHT,HMASS,HWIDTH , HTT)
      CALL VVSXXX(WM,WP,HTT,GWWH , AMPH)
C
      RETURN
      END
```

B.2 Vector Boson Scattering

We present sample programs with the vector boson four-point couplings. The subroutines WWWWXX and W3W3XX compute the amplitudes including the four-point contact vertex and two of s-, t- or u-channel vector boson exchanges. The cancellation between the vector boson four-point coupling and the Higgs exchange amplitude in $W^+W^- \to W^+W^-$ is also non-trivial.

The examples do not have the parts which set up momenta and couplings. One can easily modify the first example in B.1 appropriately for the examples in this section.

B.2.1 $W^-W^+ \to W^-W^+$

```
C
C Polarization vectors of initial W^-, W^+
C
```

```
      CALL VXXXXX(PWMI,WMASS,NHWMI,-1 , WMI)
      CALL VXXXXX(PWPI,WMASS,NHWPI,-1 , WPI)
C
C Polarization vectors of final W^-, W^+
C
      CALL VXXXXX(PWMF,WMASS,NHWMF,+1 , WMF)
      CALL VXXXXX(PWPF,WMASS,NHWPF,+1 , WPF)
C
C Vector boson scattering amplitude
C
      CALL WWWWXX(WPI,WMI,WMF,WPF,GWWA,GWWZ,ZMASS,ZWIDTH , AMPWW)
C
C Higgs boson exchange amplitude (s-channel)
C
      CALL HVVXXX(WMI,WPI,GWWH,HMASS,HWIDTH , HWWS)
      CALL VVSXXX(WMF,WPF,HWWS,GWWH , AMPHS)
C
C Higgs boson exchange amplitude (t-channel)
C
      CALL HVVXXX(WMI,WMF,GWWH,HMASS,HWIDTH , HWWT)
      CALL VVSXXX(WPI,WPF,HWWS,GWWH , AMPHT)
```

B.2.2 $\gamma\gamma \to W^-W^+$

```
C
C Polarization vectors of initial photon 1 and 2
C
      CALL VXXXXX(PA1,0.,NHA1,-1 , A1)
      CALL VXXXXX(PA2,0.,NHA2,-1 , A2)
C
C Polarization vectors of final W^-, W^+
C
      CALL VXXXXX(PWM,WMASS,NHWM,+1 , WM)
      CALL VXXXXX(PWP,WMASS,NHWP,+1 , WP)
C
C Vector boson scattering amplitude
C
      CALL W3W3XX(WM,A1,WP,A2,GWWA,GWWA,WMASS,WWIDTH , AMP)
```

B.2.3 $W^-\gamma \to W^-Z$

```
C
C Polarization vectors of initial W- and photon
C
      CALL VXXXXX(PWI,WMASS,NHWMI,-1 , WMI)
      CALL VXXXXX(PA ,0.   ,NHA  ,-1 , A  )
C
C Polarization vectors of final W- and Z
C
      CALL VXXXXX(PWF,WMASS,NHWMF,+1 , WMF)
      CALL VXXXXX(PZ ,ZMASS,NHZ  ,+1 , Z  )
C
C Vector boson scattering amplitude
C
      CALL W3W3XX(WMF,Z,WMI,A,GWWZ,GWWA,WMASS,WWIDTH , AMP)
```

B.3 $e^-e^+ \to \gamma Z$

This example is presented to illustrate the use of the subroutines EAIXXX and EAOXXX. These subroutines are designed to deal with the emission of collinear photons from the beam electron or positron; note, however, they are also valid for large angle photons. In these subroutines, it is always assumed that the beam electron is running along the positive z-axis, and the beam positron is running along the negative z-axis.

The initialization of the kinematics is also presented to show the use of CHLF, SHLF. In this example, we compute these variables from the pseudo-rapidity variable Y.

```
C ********************************************************************
C *****                      +  -     0                         *****
C *****     TEST PROGRAM :   e  e  => Z   GAMMA                 *****
C *****                                                         *****
C *****          by H. Murayama & I. Watanabe  :  14th June 1990   *****
C ********************************************************************
C
C
C INPUTS:
C       two integers (1,0 or -1),(1,0 or -1) : helicities of Z and gamma
C       one real value (-1.0 TO +1.0)        : polarization of e- beam
C       one positive real value              : beam energy (GeV)
C       one real value                       : azimuthal angle of Z
C
C OUTPUT:
C       file='EEZG.DAT'  : differential cross-sections (pb) vs. cos(theta)
C                          in all channels with all chiralities.
C
C *****     MAIN PROGRAM : two body to two body                 *****
C
C This program is a test program computing the e+e- --> ZG amplitude.
C
      IMPLICIT REAL (B-H,M,O-Z)
      IMPLICIT COMPLEX (A)
      REAL PEM(0:3),PEP(0:3),PG(0:3),PZ(0:3)
      OPEN (UNIT=1, STATUS='NEW', FILE='EEZG.DAT')
C
      PI=REAL(3.14159265)
C
C The followings are the initial data for electron and Z.
C
      EMASS = 0.511E-03
      ZMASS = 91.1
      WRITE(*,*) ' Which helicities do you like for Z and gamma?'
      READ(*,*) NHZ, NHG
C
C The phase space factor is
C
      SQRTS = 1000.
      S     = SQRTS**2
      BETAF = 1.-ZMASS**2/S
      SBETA = S*SQRT(1.-4.*EMASS**2/S)
      PBGEV2= 0.389E+09
      FLUX  = 2.*SBETA
      SPNAVG= 1./4.
      STAT  = 1.
      FACTOR= PBGEV2*SPNAVG/FLUX*(BETAF/8./PI)
C
```

```
C The initial electron is running along the positive z-axis.
C
      CTHETA = 1.
      PHI  = 0.
      CALL MOM2CX (SQRTS,EMASS,EMASS,CTHETA,PHI , PEM,PEP)
C
C The momentum of Z boson described by pseudo-rapidity Y
C
      PHI=1.
      DO 10 I=-100,100
      Y=REAL(I)/2.5E0
      COSTH = TANH(Y/2.0)
      CALL MOM2CX (SQRTS,ZMASS,0.,COSTH,PHI , PZ,PG)
C
C We call the subroutine of the invariant amplitude.
C
      CALL EEZG (EMASS,ZMASS,Y,PHI,PEM,PEP,PG,PZ,NHEM,NHEP,NHG,NHZ ,
     &           PROB)
C
      RJAC = 1.D0/(2.0*COSH(Y/2.0)**2)
      WRITE(1,*) Y, PROB*FACTOR*RJAC
C
10    CONTINUE
      END
C
C234567890=========2=========3=========4=========5=========6=========7==
C **********************************************************************
C *****                                                            *****
C *****     SUBROUTINE: e+ e- --> Z gamma                          *****
C *****                                                            *****
C *****                                                            *****
C **********************************************************************
C
C
C This is the subroutine which computes the invariant amplitude of the
C process e+e- --> ZG.
C
      SUBROUTINE EEZG (EMASS,ZMASS,Y,PHI,PEM,PEP,PG,PZ,
     &                 NHEM,NHEP,NHG,NHZ , PROB)
      IMPLICIT REAL (B-H,M,O-Z)
      IMPLICIT COMPLEX (A)
      COMPLEX EM(6), EP(6), Z(6), G(6), EAI(6), EAO(6)
      REAL    PEM(0:3),PEP(0:3),PG(0:3),PZ(0:3),GZ(2)
      REAL    GAL(2),GAU(2),GAD(2),GW(2),GZN(2),GZL(2),GZU(2),GZD(2),
     &        GS(2),G1(2)
      DIMENSION Q(0:3)
C
      PI=REAL(3.14159265)
C
      SW2=0.23
      CALL COUP2X(SW2 , GAL,GAU,GAD,GWF,GZN,GZL,GZU,GZD,G1)
C
C
C The wave functions of the final Z boson and photon.
C
      CALL VXXXXX (PZ,ZMASS,NHZ,+1 , Z)
      CALL VXXXXX (PG,0.   ,NHG,+1 , G)
C
C We can check the gauge-invariance from the following substitution; the
```

```
C t-channel diagram and u-channel diagram cancel with each other.
C
C      Z(1) = CMPLX(PZ(0))/MZ
C      Z(2) = CMPLX(PZ(1))/MZ
C      Z(3) = CMPLX(PZ(2))/MZ
C      Z(4) = CMPLX(PZ(3))/MZ
C
C some kinematics
C
      CHLF = 1./SQRT(1.+EXP(-Y))
      SHLF = 1./SQRT(1.+EXP( Y))
C
      PROB = 0.
C
C We sum over possible helicities of electron and positron.
C
      DO 10 NHEM = -1, 1, 2
      DO 20 NHEP = -1, 1, 2
C
C The wave functions of the initial electron and positron.
C
      CALL IXXXXX (PEM,EMASS,NHEM,+1 , EM)
      CALL OXXXXX (PEP,EMASS,NHEP,-1 , EP)
C
C We compute two diagrams.
C
C One diagram where photon is attached to the initial
C electron and Z to the positron (t-channel).
C
      CALL EAIXXX (PEM(0),PG(0),SHLF,CHLF,PHI,NHEM,NHG , EAI)
      CALL IOVXXX (EAI,EP,Z,GZL , AMPT)
C
C The other diagram where Z is attached to the initial
C positron and photon to the electron (u-channel).
C
      CALL EAOXXX (PEP(0),PG(0),SHLF,CHLF,PHI,NHEP,NHG , EAO)
      CALL IOVXXX (EM,EAO,Z,GZL , AMPU)
C
      PROB = PROB + ABS(AMPT + AMPU)**2
C
 20   CONTINUE
 10   CONTINUE
C
      RETURN
      END
```

B.4 $e^-e^+ \to e^-\bar{\nu}_e W^+$

This example is given to present the use of the subroutine `JEEXXX` and also to demonstrate the BRS invariance test of the amplitude. This code was actually used in [7]. The code is designed to be called from the Monte Carlo integration package `BASES` [2]. To clarify the meaning of the inputs in `JEEXXX`, the part of the program to set up the kinematics is also included. The four-momenta of the final state e^-, $\bar{\nu}_e$ and W^+ are computed in terms of the integration variables `X(1)` to `X(5)`. The last integration variable `X(6)` is devoted to the summation over the helicities.

Since the `HELAS` codes get bigger as you go to the higher order processes (though in a much more mild way compared to other means), it will be useful to have a method which enables you to check the correctness of your HELAS codes. The best method for this purpose is to check the BRS invariance of the amplitude explicitly in the numerical program. Indeed we have done the check on this process in the paper [7].

The method works by making use of the physical state conditions:

$$Q_{BRS}\,|phys;\,\mathrm{in}\rangle \;=\; \langle phys;\,\mathrm{out}|\,Q_{BRS} = 0. \tag{B.1}$$

These conditions mean that, for any fermionic operator O, the expectation value

$$\langle phys;\,\mathrm{out}|\,\{Q_{BRS},\,O\}\,|phys;\,\mathrm{in}\rangle \;= 0, \tag{B.2}$$

where the anti-commutator $\{Q_{BRS},\,O\}$ should be replaced by a commutator if the operator O is bosonic. On the other hand, one has the gauge fixing terms in the Lagrangian

$$\begin{aligned}\mathcal{L}_{gf} \;&=\; \frac{1}{\xi_W^2}(\partial^\mu W_\mu^+ - \xi_W m_W \chi_W^+)(\partial^\mu W_\mu^- - \xi_W m_W \chi_W^-)\\ &+\; \frac{1}{2\xi_Z^2}(\partial^\mu Z_\mu - \xi_Z m_Z \chi_Z)^2\end{aligned} \tag{B.3}$$

which cancel the mixing terms between gauge boson and Goldstone boson fields, and give mass terms to the Goldstone boson fields. Here we adopt the following parameterization of the Higgs doublet,

$$\Phi \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} i\sqrt{2}\chi_W^+ \\ v + H - i\chi_Z \end{pmatrix}. \tag{B.4}$$

The gauge-fixing terms are introduced to the Lagrangian so that the BRS invariance of the theory is maintained; it requires that the gauge fixing terms can be re-written as an anti-commutator of a certain operator with the BRS charge. Actually, the gauge fixing conditions are BRS transforms of the anti-ghost operators,

$$(\partial^\mu W_\mu^+ - \xi_W m_W \chi_W^+) \;=\; \{Q_{BRS},\,\bar{c}_W^+\}, \tag{B.5}$$

$$(\partial^\mu Z_\mu - \xi_Z m_Z \chi_Z^+) \;=\; \{Q_{BRS},\,\bar{c}_Z\}. \tag{B.6}$$

Thus, we obtain the useful identity

$$\langle phys;\,\mathrm{out}|\,(\partial^\mu W_\mu^- - \xi_W m_W \chi_W^-)\,|phys;\,\mathrm{in}\rangle \;= 0. \tag{B.7}$$

As far as the tree-amplitudes are concerned, the above relation can be re-written in terms of the W^+ production amplitude $\mathcal{M}_W^\mu$ and the χ_W^+ emission amplitude $\mathcal{M}_\chi$,

$$\begin{aligned} ip^\nu\left[\frac{-i}{p^2-m_W^2}\left(g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)+\xi_W\frac{-i}{p^2-\xi_W m_W^2}\frac{p_\mu p_\nu}{p^2}\right]\mathcal{M}_W^\mu&\\ -\xi_W m_W\frac{i}{p^2-\xi_W m_W^2}\mathcal{M}_\chi = 0,&\end{aligned} \tag{B.8}$$

where p^ν is the common outgoing four-momentum of W^+ and χ^+. On the other hand, the amplitude for the emission of the scalar component of W is

$$\mathcal{M}_{W_S} = \epsilon_S^{\mu *}(p)\mathcal{M}_W^\mu, \tag{B.9}$$

where the scalar component of the $W^\pm$ boson is defined by the polarization vector

$$\epsilon_S^\mu(p) = \frac{p^\mu}{m_W}. \tag{B.10}$$

Then, one obtains the *exact* relation between the Goldstone boson and scalar component amplitudes,

$$\mathcal{M}_{W_S} - i\mathcal{M}_\chi = 0. \tag{B.11}$$

Exactly the same relation holds for Z_S and χ_Z. Note that the final relation does not depend on the gauge parameter ξ_W or ξ_Z. Thus, we can take the formal limit $\xi_W, \xi_Z \to \infty$ in the above relation, which leads to the unitary gauge employed in the `HELAS` system. Then, the W_S amplitude is simply the amplitude computed in the unitary gauge with the external wavefunction being ϵ_S in Eq. (B.10).

The amplitudes with a Goldstone boson emission are often very simple and can easily be evaluated numerically in the `HELAS` system [6], or in some cases even analytically. The test of the identity (B.11) turns out to be very useful in checking the numerical accuracy of the program for those amplitudes with longitudinally polarized vector boson emission, which contains at very high energies ($\sqrt{s} \gg m_V$) a numerical cancellation associated with the identity.

In the example we discuss here, the process

$$e^-e^+ \to e^-\bar{\nu}_e\chi_W^+ \tag{B.12}$$

should be compared with the amplitude $e^-e^+ \to e^-\bar{\nu}_e W^+$ for the scalar polarization ϵ_S in Eq. (B.10). The χ_W^+ amplitude has actually only two diagrams in Fig. 14, in the vanishing electron mass limit. The `VVS` coupling of the χ_W is

$$\mathcal{L} = \frac{i}{2}\frac{e^2}{\cos\theta_W \sin\theta_W} v(A_\mu \cos\theta_W - Z_\mu \sin\theta_W)(\chi_W^{-*}W_\mu^{+*} - \chi_W^{+*}W_\mu^{-*}). \tag{B.13}$$

Since `HELAS` `VVS` vertices do not accept complex coupling constants, we compute i times the T-matrix element here, which should give the same answer as the W_S^+ emission T-matrix element. Then, the following `HELAS` amplitude

```
CALL SXXXXX(PW,MW,+1 , SCHI)
GWCA = ABS(GWWA)*MW
GWCZ = - ABS(GWWA)*MZ*SQRT(SW2)
CALL VVSXXX(CEEAT,CENUWT,SCHI,GWCA , AMPCA)
CALL VVSXXX(CEEZT,CENUWT,SCHI,GWCZ , AMPCZ)
AMPCHI = AMPCA + AMPCZ
```

should give the same answer as the `AMPT` in the following program when the polarization vector `W` is replaced by the scalar polarization which is computed by setting `NHW=4` in `VXXXXX` of `HELAS_CHECK` (not `HELAS`). Furthermore, the χ_W amplitude is completely safe numerically even for very high energies, since there is not cancellation among the diagrams. This check will be a good exercise for beginners.

The following program contains only the t-channel amplitude. The inclusion of the s-channel amplitude can be done in a very simple way (by crossing). This topic will be explained in the last example in Appendix B.6. The inclusion of the decay $W^+ \to u\bar{d}$ etc. is also straightforward. An example is presented in Appendix B.7.

```
C234567890---------2---------3---------4---------5---------6---------7--
C
C FUNCTION CROSS (e- e+ --> e- nubar W+)
C
C234567890---------2---------3---------4---------5---------6---------7--
C
      REAL FUNCTION CROSS(X)
C
      IMPLICIT REAL (B-H,M,O-Z)
      IMPLICIT COMPLEX (A)
      REAL X(25)
      REAL    PEMI(0:3),PEP(0:3),PEMF(0:3),PNU(0:3),PW(0:3)
      COMPLEX SEMI(6),SEP(6),SEMF(6),SNU(6),W(6),
     1        CEEAT(6),CEEZT(6),CENUWT(6),
     2        SINIT(6),SFEL(6),SFNU(6)
      REAL    LEEL
      REAL GAL(2),GWF(2),GZN(2),GZL(2)
      COMMON /INPUT/ MEL,MZ,ZWIDTH,MW,WWIDTH,EBEAM,FACTOR,DELTA,RMW4
      COMMON /COUPL/ GAL,GZN,GZL,GWF,GW,GWWA,GWWZ
      PI     = 3.14159265
      CROSS = 0.
C
C The input phase space variables are:
C     X(1) = log(1 - cos theta(e-) + DELTA)
C     X(2) = log(1 + cos theta(nu) + RMW4)
C     X(3) = log(1 - 2 E(e-)/sqrt(s))
C     X(4) = phi(e-)
C     X(5) = phi(nu) - phi(e-)
C     X(6) = helicity summation
C with parameters
C           EELMAX= (S+MEL**2-MW**2)/4./EBEAM
C           XELMAX= EELMAX/EBEAM
C           DELXEL= (MEL**2-MW**2)/S
C           DELTA = MEL**2*DELXEL**2/XELMAX/(2.*EBEAM**2)
C           RMW4  = MW**2/S
C
      ZEL  = X(1)
      ZNU  = X(2)
      LEEL = X(3)
      PHIEL= X(4)
      PHINU= PHIEL+X(5)
      NH   = INT(X(6))
C
C We fix the kinematics of the particles
C
      ROOTS = 2.*EBEAM
      CALL MOM2CX(ROOTS,MEL,MEL,1.,0. , PEMI,PEP)
C
      S      = (2.*EBEAM)**2
      CEL    = 1.-(EXP(ZEL)-DELTA)
      CNU    =-1.+(EXP(ZNU)-RMW4)
      SHLF   = SQRT( (EXP(ZEL)-DELTA)/2. )
      CHLFNU = SQRT( (EXP(ZNU)-RMW4)/2. )
      CHLF   = SQRT( (1.+SHLF)*(1.-SHLF) )
      SHLFNU = SQRT( (1.+CHLFNU)*(1.-CHLFNU) )
      SEL    = 2.*CHLF*SHLF
      SINNU  = 2.*CHLFNU*SHLFNU
      C12    = SEL*SINNU*COS(PHIEL-PHINU)+CEL*CNU
      EEL    = EBEAM*(1.-EXP(LEEL))
```

```
      PEL    = EEL*SQRT(1.-MEL**2/EEL**2)
      ENU    = (S+MEL**2-2.*SQRT(S)*EEL-MW**2)
     &              /(2.*(SQRT(S)-EEL)+2.*PEL*C12)
      EW     = 2.*EBEAM-EEL-ENU
      IF (EW.LT.MW) EW=MW
C
      CALL MOMNTX(EEL,MEL,CEL,PHIEL , PEMF)
      CALL MOMNTX(ENU,0. ,CNU,PHINU , PNU )
      PW(0) = EW
      PW(1) = -PEMF(1)-PNU(1)
      PW(2) = -PEMF(2)-PNU(2)
      PW(3) = -PEMF(3)-PNU(3)
C
      PTW = SQRT( PW(1)**2+PW(2)**2 )
C
C We fix the helicities.
C
      NHEMI = MOD(NH,2)*2-1
      NHEP  = MOD(NH/2,2)*2-1
      NHEMF = MOD(NH/4,2)*2-1
      NHW   = MOD(NH/8,3)-1
C
C234567890---------2---------3---------4---------5---------6---------7--
C
C We compute the amplitude here
C
      CALL IXXXXX(PEMI,MEL,NHEMI,+1 , SEMI)
      CALL OXXXXX(PEP ,MEL,NHEP ,-1 , SEP )
      CALL OXXXXX(PEMF,MEL,NHEMF,+1 , SEMF)
      CALL IXXXXX(PNU ,0. ,+1   ,-1 , SNU )
      CALL VXXXXX(PW  ,MW ,NHW  ,+1 , W   )
C
C t-channel currents
C
      CALL JEEXXX(EBEAM,EEL,SHLF,CHLF,PHIEL,NHEMI,NHEMF,+1 , CEEAT)
      CALL JIOXXX(SEMI,SEMF,GZL,MZ,ZWIDTH , CEEZT )
      CALL JIOXXX(SNU ,SEP ,GWF,MW,WWIDTH , CENUWT)
C
C spinors
C
      CALL FVOXXX(SEP ,W,GWF,0. ,0. , SINIT)
      CALL FVIXXX(SNU ,W,GWF,MEL,0. , SFEL )
      CALL FVOXXX(SEMF,W,GWF,0. ,0. , SFNU )
C
C diagram (a)
C
      CALL IOVXXX(SFEL,SEP,CEEAT,GAL , AMPTAA)
      CALL IOVXXX(SFEL,SEP,CEEZT,GZL , AMPTAZ)
C
C diagram (b)
C
      CALL VVVXXX(CENUWT,W,CEEAT,GWWA , AMPTBA)
      CALL VVVXXX(CENUWT,W,CEEZT,GWWZ , AMPTBZ)
C
C diagram (c)
C
      CALL IOVXXX(SNU,SINIT,CEEZT,GZN , AMPTC)
C
C diagram (d)
```

```
C
      CALL IOVXXX(SEMI,SFNU,CENUWT,GWF , AMPTD)
C
C total t-channel amplitude
C
      AMPT = AMPTAA+AMPTAZ+AMPTBA+AMPTBZ+AMPTC+AMPTD
C234567890---------2---------3---------4---------5---------6---------7--
C
      DSIGMA = REAL(AMPT*CONJG(AMPT))
C
      BETAEL = SQRT(1.-MEL**2/EEL**2)
      RJACEL = EXP(ZEL)
      RJACNU = EXP(ZNU)
      RJAC   = BETAEL*EEL*ENU/(SQRT(S)-EEL+EEL*BETAEL*C12)/8./(2.*PI)**5
      RJACL  = EBEAM*EXP(LEEL)
      CORR   = REAL(128.D0/137.0359895D0)
      CROSS  = DSIGMA*FACTOR*RJAC*RJACEL*RJACNU*RJACL*CORR
C
C This is our integrand.
C
      RETURN
      END
```

B.5 Decay Angle Distribution in $e^-e^+ \to W^-W^+$

The following example illustrates the use of ROTXXX and BOOSTX. To measure the helicity of the final W's, we study the decay angle distributions of the final state fermions. The example computes the decay angle distribution of the final state e^- from W^-, where the angle is defined in the W^- rest frame, with the polar angle measured from the W^- momentum direction. To use HELAS to compute the distribution, we define the angle in the W^- rest frame first, then rotate it such that the positive z-axis will be rotated to the W^- momentum direction, and then boost it to the laboratory frame along the W^- momentum. These steps are done by ROTXXX and BOOSTX.

The program basically computes the angular distribution of the W^- for each helicity combination. The lines in *small* letters are the modifications to compute the decay angle distributions for fixed W^- angle. As one can clearly see, HELAS can easily incorporate the decay of final state particles just by replacing the on-shell wavefunctions by outputs of suitable subroutines. In this sample, the on-shell wavefunction computed by VXXXXX is replaced by the output of JIOXXX, which computes the off-shell W^- current from the final state e^- and $\bar{\nu}_e$ spinors.

```
C ********************************************************************
C *****                         +  -      +   -                  *****
C *****     SAMPLE PROGRAM :   e  e  => W   W                    *****
C +++++                                                          +++++
C +++++     studies the decay angle distribution of W-           +++++
C +++++     with fixed W+ helicity                               +++++
C *****                                                          *****
C ********************************************************************
C
      COMPLEX  I(6),O(6),WM(6),WP(6),FVI(6),J3(6),AMP,AMPT,AMPS
      REAL     PI(0:3),PO(0:3),PWM(0:3),PWP(0:3),
     &         COSTHE,PHIE,COSTHW,PHIW,
     &         EMASS,WMASS,ZMASS,ZWIDTH,NUMASS,NWIDTH,
     &         SW2,GW,GWWA,GWWZ,
     &         GAL(2),GAU(2),GAD(2),GWF(2),GZN(2),GZL(2),GZU(2),GZD(2),
```

```
     &         G1(2)
      INTEGER  HELI,HELO,HELWM,HELWP
      complex  emf(6),nuf(6)
      real     pemhat(0:3),pnuhat(0:3),pemf(0:3),pnuf(0:3)
C
      ROOTS  = 500.0
      COSTHE = 1.
      PHIE   = 0.
C
      SW2    =  0.23
      EMASS  =  0.0
      WMASS  = 80.0
      WWIDTH =  2.0
      ZMASS  =  WMASS/SQRT(1.0-SW2)
      ZWIDTH =  2.5
      NUMASS =  0.0
      NWIDTH =  0.0
C
      write(*,*) 'input cos theta(w-), phi(w-)'
      read (*,*) costhw,phiw
      WRITE(*,*) 'Input helicities of initial e-, e+'
      READ (*,*) HELI,HELO
c      WRITE(*,*) 'Input helicity of W-'
c      READ (*,*) HELWM
      WRITE(*,*) 'Input helicity of W+'
      READ (*,*) HELWP
C
      CALL COUP1X(SW2 , GW,GWWA,GWWZ)
      CALL COUP2X(SW2 , GAL,GAU,GAD,GWF,GZN,GZL,GZU,GZD,G1)
C
      CALL MOM2CX(ROOTS,EMASS,EMASS,COSTHE,PHIE , PI,PO)
      call mom2cx(roots,wmass,wmass,costhw,phiw , pwm,pwp)
C
C      DO 10 J = 100, -100
C         COSTHW = REAL(I)/100.
C         PHIW   = 2.
C         CALL MOM2CX(ROOTS,WMASS,WMASS,COSTHW,PHIW , PWM,PWP)
C
      do 10 j = -100, 100
         coshat = real(j)/100.
         phihat = 1.
         call mom2cx(wmass,emass,numass,coshat,phihat , pemhat,pnuhat)
         call rotxxx(pemhat,pwm , pemhat)
         call rotxxx(pnuhat,pwm , pnuhat)
         call boostx(pemhat,pwm , pemf)
         call boostx(pnuhat,pwm , pnuf)
C
C The initial state spinors
C
         CALL IXXXXX(PI,EMASS,HELI,+1 , I)
         CALL OXXXXX(PO,EMASS,HELO,-1 , O)
C
C The W-, W+ polarization vectors
C
c         CALL VXXXXX(PWM,WMASS,HELWM,+1 , WM)
         call oxxxxx(pemf,emass,-1,+1 , emf)
         call ixxxxx(pnuf,0.   ,+1,-1 , nuf)
         call jioxxx(nuf,emf,gwf,wmass,wwidth , wm)
         CALL VXXXXX(PWP,WMASS,HELWP,+1 , WP)
```

```
C
C t-channel neutrino exchange
C
        IF (HELI.EQ.1) THEN
           AMPT = 0.
        ELSE
           CALL FVIXXX(I,WM,GWF,NUMASS,NWIDTH , FVI)
           CALL IOVXXX(FVI,O,WP,GWF , AMPT)
        END IF
C
C s-channel photon/Z exchange
C
        CALL J3XXXX(I,O,GAL,GZL,ZMASS,ZWIDTH , J3)
        CALL VVVXXX(WM,WP,J3,GW , AMPS)
C
        AMP = AMPT + AMPS
        PROB = ABS(AMP)**2
        write(10,*) coshat,prob
c
 10   continue
C
      END
```

B.6 $e^-e^+ \to e^-\bar{\nu}_e W^+ Z$

This sample is given to demonstrate the compactness of the HELAS codes for higher order processes. The program has been adapted from the one made by J. Kanzaki for the paper [8].

There are 80 Feynman diagrams contributing to the process. Exactly half of them, in which the initial e^- and e^+ do not annihilate (non-annihilation diagrams), are shown in Fig. 15. The remaining 40 diagrams are obtained from them simply by exchanging the final e^- line and the initial e^+ line.

One of the advantages of the HELAS system is that this simple crossing property of the Feynman diagram is made manifest in the computer program. In the following program, the non-annihilation diagrams as listed in Fig. 15 are computed first, and then the annihilation diagrams are evaluated by repeating exactly the same sequence of the computation after exchanging the two initial wavefunctions. At the end of the computation, the sum of the non-annihilation amplitudes and that of the annihilation amplitudes are added with a relative minus sign which reflects the exchange of the fermionic operator.

It may be a good exercise to read the program. In step I, 6 external wavefunctions are calculated by 6 CALL lines. In step II_1, 4 off-shell vector lines and 7 off-shell fermion lines are evaluated from 6 external wavefunctions. In step II_2, 7 off-shell vector, 4 off-shell fermion, and 1 off-shell scalar lines are evaluated from one external line and one off-shell line, which were prepared in the previous steps. In step III, each diagram of Fig. 15 is calculated by calling just one HELAS subroutine. The contribution of each diagram is named after the diagram number in Fig. 15. It should be noted here that because the HELAS subroutine W3W3XX computes the sum of the diagrams t24, t25 and t26, the amplitudes AMP26A and AMP26Z stand for the sum of the three amplitudes. Finally, the sum of all the non-annihilation diagrams is called AMPT. This same procedure is repeated to calculate the sum of all the annihilation diagrams AMPS, simply by exchanging the initial e^+ wavefunction SEPI and the final e^- wavefunction SEMF. The total amplitude AMP is obtained by adding the two contributions with a relative minus sign.

This program calculates accurately the matrix elements when the final e^- is not too collinear to the initial e^- beam. When one wishes to compute accurately the amplitudes in the collinear configuration, then the special subroutine JEEXXX should be used in place of the standard subroutine JIOXXX in the line which calculates the off-shell photon line JAEM from the initial and final e^-'s in the non-annihilation diagram. Details about the use of the special subroutine JEEXXX are found in Appendix B.4.

As demonstrated in Appendix B.4 for the process $e^-e^+ \to e^-\bar{\nu}_e W^+$, the BRS invariance [4] of the helicity amplitudes provides us with an excellent test of the program. In the present case, we may compute the helicity amplitude for the process $e^-e^+ \to e^-\bar{\nu}_e \chi^+ Z$, whose Feynman diagrams are listed in Fig. 16 for the non-annihilation channel. As in the example of Appendix B.4, the helicity amplitude for the 'scalar' W^+ production should agree exactly with that for the production of the associated Goldstone boson χ^+. The following program was checked by this method in the paper [8]. We leave this test to the readers as the last exercise of this manual.

```
C***********************************************************************
C Subroutine to calculate the helicity amplitude for the process        *
C                                                                       *
C    e-(pemi,nemi) + e+(pepi,nepi) ->   e-(pemf,nemf) + nubar(pneb,nneb) *
C                                     + W-(pwp,nwp) + Z(pz,nz)          *
C                                                                       *
C Inputs: external particle momenta      pemi,pepi,pemf,pneb,pwp,pz     *
C         external particle helicities   nemi,nepi,nemf,nneb,nwp,nz     *
C         coupling constants and masses  sw2,hm,em,wm,zm,wwid,zwid      *
C                                                                       *
C Output: helicity amplitude             AMP                            *
C***********************************************************************
C Coded by:  J. Kanzaki                                                 *
C***********************************************************************
      SUBROUTINE EEENWZ(PEMI,PEPI,PEMF,PNEB,PWP,PZ,
     &                  NEMI,NEPI,NEMF,NNEB,NWP,NZ,
     &                  SW2,HM,EM,WM,ZM,WWID,ZWID,
     &                  AMP)
C
      IMPLICIT COMPLEX (A)
C
      REAL    PEMI(0:3),PEPI(0:3),PEMF(0:3),PNEB(0:3),PWP(0:3),PZ(0:3)
      INTEGER NEMI,NEPI,NEMF,NNEB,NWP,NZ
      REAL    SW2,HM,EM,WM,ZM,WWID,ZWID
      COMPLEX AMP
C
      COMPLEX SETMP
      REAL    GW,GWWA,GWWZ
      REAL    GAL(2),GAU(2),GAD(2),GWF(2),
     &        GZN(2),GZL(2),GZU(2),GZD(2),G1(2)
      REAL    GWWH,GZZH,GHHH,GWWHH,GZZHH,GHHHH
      COMPLEX SEMI(6),SEPI(6)
      COMPLEX SEMF(6),SNEB(6),CZ(6),CWP(6)
      COMPLEX JAEM(6),JZEM(6),JWEP(6),JWZW(6),JZZH(6)
      COMPLEX SEZEMF(6),SNZNEB(6),SEZEMI(6),SEZEPI(6)
      COMPLEX SNWEMF(6),SEWNEB(6),SNWEPI(6)
      COMPLEX JZZEMF(6),JAZEMF(6),JWWEMF(6),JWZNEB(6)
      COMPLEX JZZEMI(6),JAZEMI(6),JWZEPI(6)
      COMPLEX S3WEMF(6),S3ZNEB(6),S3ANEB(6),S3ZEPI(6)
C
C Prepare coupling constants
C
      CALL COUP1X(SW2 , GW,GWWA,GWWZ)
      CALL COUP2X(SW2 , GAL,GAU,GAD,GWF,GZN,GZL,GZU,GZD,G1)
      CALL COUP3X(SW2,ZM,HM ,
```

```
     &           GWWH,GZZH,GHHH,GWWHH,GZZHH,GHHHH)
C
C Prepare external particle wave functions                     (step I)
C
      CALL IXXXXX(PEMI,EM,NEMI,  +1 , SEMI)
      CALL OXXXXX(PEPI,EM,NEPI,  -1 , SEPI)
      CALL OXXXXX(PEMF,EM,NEMF,  +1 , SEMF)
      CALL IXXXXX(PNEB,0.,NNEB,  -1 , SNEB)
      CALL VXXXXX(PWP, WM,NWP,   +1 , CWP )
      CALL VXXXXX(PZ,  ZM,NZ,    +1 , CZ  )
C
C Sum over non-annihilation diagrams AMPT and annihilation diagrams AMPS
C
      DO 1 I=1,2
         IF(I.EQ.2) THEN
            DO 2 J=1,6
            SETMP = SEPI(J)
            SEPI(J) = SEMF(J)
            SEMF(J) = SETMP
    2       CONTINUE
         END IF
C
C Prepare off-shell vector lines from external particles       (step II_1)
C
      CALL JIOXXX(SEMI,SEMF,GAL,0.,0.,     JAEM)
      CALL JIOXXX(SEMI,SEMF,GZL,ZM,ZWID,   JZEM)
      CALL JIOXXX(SNEB,SEPI,GWF,WM,WWID,   JWEP)
      CALL JVVXXX(CWP,CZ,GWWZ,WM,WWID,     JWZW)
C
C Prepare off-shell fermion lines from external particles      (step II_1)
C
      CALL FVOXXX(SEMF,CZ,GZL,EM,0.,       SEZEMF)
      CALL FVOXXX(SEMF,CWP,GWF,0.,0.,      SNWEMF)
      CALL FVIXXX(SNEB,CZ,GZN,0.,0.,       SNZNEB)
      CALL FVIXXX(SNEB,CWP,GWF,EM,0.,      SEWNEB)
      CALL FVIXXX(SEMI,CZ,GZL,EM,0.,       SEZEMI)
      CALL FVOXXX(SEPI,CZ,GZL,EM,0.,       SEZEPI)
      CALL FVOXXX(SEPI,CWP,GWF,0.,0.,      SNWEPI)
C
C Prepare off-shell vector lines from off-shell lines          (step II_2)
C
      CALL JIOXXX(SEMI,SEZEMF,GZL,ZM,ZWID, JZZEMF)
      CALL JIOXXX(SEMI,SEZEMF,GAL,0.,0.,   JAZEMF)
      CALL JIOXXX(SEMI,SNWEMF,GWF,WM,WWID, JWWEMF)
      CALL JIOXXX(SNZNEB,SEPI,GWF,WM,WWID, JWZNEB)
      CALL JIOXXX(SEZEMI,SEMF,GZL,ZM,ZWID, JZZEMI)
      CALL JIOXXX(SEZEMI,SEMF,GAL,0.,0.,   JAZEMI)
      CALL JIOXXX(SNEB,SEZEPI,GWF,WM,WWID, JWZEPI)
C
C Prepare off-shell fermion lines from off-shell lines         (step II_2)
C
      CALL FVIXXX(SEMI,JWEP,GWF,0.,0.,     S3WEMF)
      CALL FVOXXX(SEPI,JZEM,GZL,EM,0.,     S3ZNEB)
      CALL FVOXXX(SEPI,JAEM,GAL,EM,0.,     S3ANEB)
      CALL FVIXXX(SNEB,JZEM,GZN,0.,0.,     S3ZEPI)
C
C Prepare off-shell scalar lines from off-shell lines          (step II_2)
C
      CALL HVVXXX(CZ,JZEM,GZZH,HM,0.,      JZZH)
```

```
C
C Calculates amplitudes                                          (step III)
C
      CALL IOVXXX(SEWNEB,SEZEPI,JAEM,GAL,              AMP1A)
      CALL IOVXXX(SEWNEB,SEZEPI,JZEM,GZL,              AMP1Z)
      CALL IOVXXX(SEWNEB,S3ANEB,CZ,GZL,                AMP2A)
      CALL IOVXXX(SEWNEB,S3ZNEB,CZ,GZL,                AMP2Z)
      CALL IOVXXX(SNZNEB,S3ANEB,CWP,GWF,               AMP3A)
      CALL IOVXXX(SNZNEB,S3ZNEB,CWP,GWF,               AMP3Z)
      CALL IOVXXX(SNEB,S3ANEB,JWZW,GWF,                AMP4A)
      CALL IOVXXX(SNEB,S3ZNEB,JWZW,GWF,                AMP4Z)
      CALL VVVXXX(JWZNEB,CWP,JAEM,GWWA,                AMP5A)
      CALL VVVXXX(JWZNEB,CWP,JZEM,GWWZ,                AMP5Z)
      CALL VVVXXX(JWZEPI,CWP,JAEM,GWWA,                AMP6A)
      CALL VVVXXX(JWZEPI,CWP,JZEM,GWWZ,                AMP6Z)
      CALL IOVXXX(SNZNEB,SNWEPI,JZEM,GZN,              AMP7)
      CALL IOVXXX(S3ZEPI,SNWEPI,CZ,GZN,                AMP8)
      CALL IOVXXX(S3ZEPI,SEZEPI,CWP,GWF,               AMP9)
      CALL IOVXXX(S3ZEPI,SEPI,JWZW,GWF,                AMP10)
      CALL IOVXXX(SNEB,SNWEPI,JZZEMF,GZN,              AMP11)
      CALL IOVXXX(SNEB,SNWEPI,JZZEMI,GZN,              AMP12)
      CALL IOVXXX(SEZEMI,SNWEMF,JWEP,GWF,              AMP13)
      CALL IOVXXX(S3WEMF,SNWEMF,CZ,GZN,                AMP14)
      CALL IOVXXX(S3WEMF,SEZEMF,CWP,GWF,               AMP15)
      CALL IOVXXX(S3WEMF,SEMF,JWZW,GWF,                AMP16)
      CALL VVVXXX(JWEP,CWP,JAZEMF,GWWA,                AMP17A)
      CALL VVVXXX(JWEP,CWP,JZZEMF,GWWZ,                AMP17Z)
      CALL VVVXXX(JWEP,CWP,JAZEMI,GWWA,                AMP18A)
      CALL VVVXXX(JWEP,CWP,JZZEMI,GWWZ,                AMP18Z)
      CALL IOVXXX(SEWNEB,SEPI,JAZEMF,GAL,              AMP19A)
      CALL IOVXXX(SEWNEB,SEPI,JZZEMF,GZL,              AMP19Z)
      CALL IOVXXX(SEWNEB,SEPI,JAZEMI,GAL,              AMP20A)
      CALL IOVXXX(SEWNEB,SEPI,JZZEMI,GZL,              AMP20Z)
      CALL IOVXXX(SNZNEB,SEPI,JWWEMF,GWF,              AMP21)
      CALL IOVXXX(SNEB,SEZEPI,JWWEMF,GWF,              AMP22)
      CALL VVVXXX(JWEP,JWWEMF,CZ,GWWZ,                 AMP23)
      CALL W3W3XX(JWEP,CZ,CWP,JAEM,GWWZ,GWWA,WM,WWID,  AMP26A)
      CALL W3W3XX(JWEP,CZ,CWP,JZEM,GWWZ,GWWZ,WM,WWID,  AMP26Z)
      CALL VVSXXX(CWP,JWEP,JZZH,GWWH,                  AMP27)
C
C Sum over all the diagrams
C
      AMPSUM = AMP1A  +AMP1Z  +AMP2A  +AMP2Z
     &        +AMP3A  +AMP3Z  +AMP4A  +AMP4Z
     &        +AMP5A  +AMP5Z  +AMP6A  +AMP6Z
     &        +AMP7   +AMP8   +AMP9   +AMP10
     &        +AMP11  +AMP12  +AMP13  +AMP14
     &        +AMP15  +AMP16  +AMP17A +AMP17Z
     &        +AMP18A +AMP18Z +AMP19A +AMP19Z
     &        +AMP20A +AMP20Z +AMP21  +AMP22
     &        +AMP23  +AMP26A +AMP26Z +AMP27
C
C Non-annihilation diagrams AMPT and annihilation diagrams AMPS
C
        IF(I.EQ.1) THEN
           AMPT = AMPSUM
        ELSE
           AMPS = AMPSUM
        ENDIF
```

```
    1 CONTINUE
C
C Sum of non-annihilation and annihilation diagrams
C
      AMP = AMPT - AMPS
C
      RETURN
      END
```

Appendix C

HELAS.LIST

C.1 HELAS.LIST1

The subroutine list HELAS.LIST1 contains the full list of the subroutines as well as their brief descriptions. The same comments are also included in the HELAS.FOR itself. HELAS.LIST1 is just the extraction of all comment lines form HELAS.FOR. However, once you understood the basic strategy of HELAS, it will be sufficient to have a copy of HELAS.LIST1 when you do the actual programming. If you further get accustomed to it, then HELAS.LIST2 will be sufficient. We hope you a good luck.

```
*******************************************************************
***      SUBROUTINE LIST 1 for HELAS.FOR and HELAS_CHECK.FOR      ***
***               coded by H. Murayama & I. Watanabe              ***
***               ver.  2.3         3rd   Apr.  1992              ***
*******************************************************************

The subroutines are named as follows.

External Lines:
 | f >                    : fermion (flow-IN)            ==>  IXXXXX
 < f |                    : fermion (flow-OUT)           ==>  OXXXXX
 epsilon^mu , epsilon*^mu : vector boson (initial,final) ==>  VXXXXX
 S                        : scalar boson (initial,final) ==>  SXXXXX

Vertices:
 < f' V f >           : amplitude        of   FFV vertex ==>  IOVXXX
 | f' V f >           : flow-in  fermion from FFV vertex ==>  FVIXXX
 < f V f' |           : flow-out fermion from FFV vertex ==>  FVOXXX
 J^mu(< f' | V | f >) : vector   current from FFV vertex ==>  JIOXXX
                      : W3       current from FFV vertex ==>  J3XXXX
 < f' S f >           : amplitude        of   FFS vertex ==>  IOSXXX
 | f' S f >           : flow-in  fermion from FFS vertex ==>  FSIXXX
 < f S f' |           : flow-out fermion from FFS vertex ==>  FSOXXX
 J(< f' | S | f >)    : scalar   current from FFS vertex ==>  HIOXXX
 Gamma(V1,V2,V3)        : amplitude      of   VVV vertex ==>  VVVXXX
 J^mu(V':V1,V2)         : vector current from VVV vertex ==>  JVVXXX
 Gamma(V1,V2,S)         : amplitude      of   VVS vertex ==>  VVSXXX
 J^mu(V':V,S)           : vector current from VVS vertex ==>  JVSXXX
 J(S:V1,V2)             : scalar current from VVS vertex ==>  HVVXXX
 Gamma(V,S1,S2)         : amplitude      of   VSS vertex ==>  VSSXXX
 J^mu(V:S1,S2)          : vector current from VSS vertex ==>  JSSXXX
 J(S':V,S)              : scalar current from VSS vertex ==>  HVSXXX
```

```
  Gamma(S1,S2,S3)         : amplitude      of   SSS vertex ==>  SSSXXX
  J(S':S1,S2)             : scalar current from SSS vertex ==>  HSSXXX
  Gamma(WM,WP,WM,WP) : amplitude of   4-point W+/W- vertex ==>  WWWWXX
  J^mu(W':W1,W2,W3)  : W current from 4-point W+/W- vertex ==>  JWWWXX
  Gamma(WM,W3,WP,W3) : amplitude of   4-point W/W3  vertex ==>  W3W3XX
  J^mu(W':W1,W2,W3)  : W current from 4-point W/W3  vertex ==>  JW3WXX
  Gamma(V1,V2,S1,S2)   : amplitude        of   VVSS vertex ==>  VVSSXX
  J^mu(V':V,S1,S2)     : vector current   from VVSS vertex ==>  JVSSXX
  J(S':V1,V2,S)        : scalar current   from VVSS vertex ==>  HVVSXX
  Gamma(S1,S2,S3,S4)   : amplitude        of   SSSS vertex ==>  SSSSXX
  J(S':S1,S2,S3)       : scalar current   from SSSS vertex ==>  HSSSXX

 Special Vertices:
  | e' A e- >              : initial electron with photon ==>  EAIXXX
  < e+ A e' |              : initial positron with photon ==>  EAOXXX
  J^mu(< e+ | A | e->)     : t-channel photon from e-/e+  ==>  JEEXXX

 Utilities for Momentum Manipulations:
  P^mu(energy,mass,costh,phi)          : set up 4-momentum ==>  MOMNTX
  P1^mu & P2^mu   : set up two 4-momenta in 1 2 rest frame ==>  MOM2CX
  P_boosted                  : Lorentz boost of 4-momentum ==>  BOOSTX
  P_rotated                       : rotation of 4-momentum ==>  ROTXXX

 Standard Model Coupling Constants:
  for VVV,VVVV vertices                                    ==>  COUP1X
  for FFV vertices                                         ==>  COUP2X
  for VVS,SSS,VVSS,SSSS vertices                           ==>  COUP3X
  for FFS vertices                                         ==>  COUP4X

**********************************************************************

    SUBROUTINE IXXXXX(P,FMASS,NHEL,NSF , FI)

This subroutine computes a fermion wavefunction with the flowing-IN
fermion number.

INPUT:
      real    P(0:3)         : four-momentum of fermion
      real    FMASS          : mass          of fermion
      integer NHEL = -1 or 1 : helicity      of fermion
      integer NSF  = -1 or 1 : +1 for particle, -1 for anti-particle

OUTPUT:
      complex FI(6)          : fermion wavefunction              |FI>

----------------------------------------------------------------------

    SUBROUTINE OXXXXX(P,FMASS,NHEL,NSF , FO)

This subroutine computes a fermion wavefunction with the flowing-OUT
fermion number.

INPUT:
      real    P(0:3)         : four-momentum of fermion
      real    FMASS          : mass          of fermion
      integer NHEL = -1 or 1 : helicity      of fermion
      integer NSF  = -1 or 1 : +1 for particle, -1 for anti-particle

OUTPUT:
```

```
      complex FO(6)          : fermion wavefunction              <FO|

----------------------------------------------------------------------

    SUBROUTINE VXXXXX(P,VMASS,NHEL,NSV , VC)

This subroutine computes a VECTOR wavefunction.

INPUT:
      real    P(0:3)         : four-momentum of vector boson
      real    VMASS          : mass          of vector boson
      integer NHEL = -1, 0, 1: helicity      of vector boson
                               (0 is forbidden if VMASS=0.0)
      integer NSV  = -1 or 1 : +1 for final, -1 for initial

OUTPUT:
      complex VC(6)          : vector wavefunction       epsilon^mu(V)

----------------------------------------------------------------------

    SUBROUTINE SXXXXX(P,NSS , SC)

This subroutine computes a complex SCALAR wavefunction.

INPUT:
      real    P(0:3)         : four-momentum of scalar boson
      integer NSS  = -1 or 1 : +1 for final, -1 for initial

OUTPUT:
      complex SC(3)          : scalar wavefunction                   S

======================================================================

    SUBROUTINE IOVXXX(FI,FO,VC,G , VERTEX)

This subroutine computes an amplitude of the fermion-fermion-vector
coupling.

INPUT:
      complex FI(6)          : flow-in  fermion                  |FI>
      complex FO(6)          : flow-out fermion                  <FO|
      complex VC(6)          : input    vector                     V
      real    G(2)           : coupling constants                GVF

OUTPUT:
      complex VERTEX         : amplitude                    <FO|V|FI>

----------------------------------------------------------------------

    SUBROUTINE FVIXXX(FI,VC,G,FMASS,FWIDTH , FVI)

This subroutine computes an off-shell fermion wavefunction from a
flowing-IN external fermion and a vector boson.

INPUT:
      complex FI(6)          : flow-in  fermion                  |FI>
      complex VC(6)          : input    vector                     V
      real    G(2)           : coupling constants                GVF
      real    FMASS          : mass  of OUTPUT fermion F'
```

```
      real    FWIDTH         : width of OUTPUT fermion F'

OUTPUT:
      complex FVI(6)         : off-shell fermion             |F',V,FI>

----------------------------------------------------------------------

    SUBROUTINE FVOXXX(FO,VC,G,FMASS,FWIDTH , FVO)

This subroutine computes an off-shell fermion wavefunction from a
flowing-OUT external fermion and a vector boson.

INPUT:
      complex FO(6)          : flow-out fermion                   <FO|
      complex VC(6)          : input    vector                      V
      real    G(2)           : coupling constants                 GVF
      real    FMASS          : mass  of OUTPUT fermion F'
      real    FWIDTH         : width of OUTPUT fermion F'

OUTPUT:
      complex FVO(6)         : off-shell fermion             <FO,V,F'|

----------------------------------------------------------------------

    SUBROUTINE JIOXXX(FI,FO,G,VMASS,VWIDTH , JIO)

This subroutine computes an off-shell vector current from an external
fermion pair.  The vector boson propagator is given in Feynman gauge
for a massless vector and in unitary gauge for a massive vector.

INPUT:
      complex FI(6)          : flow-in  fermion                   |FI>
      complex FO(6)          : flow-out fermion                   <FO|
      real    G(2)           : coupling constants                 GVF
      real    VMASS          : mass  of OUTPUT vector V
      real    VWIDTH         : width of OUTPUT vector V

OUTPUT:
      complex JIO(6)         : vector current          J^mu(<FO|V|FI>)

----------------------------------------------------------------------

    SUBROUTINE J3XXXX(FI,FO,GAF,GZF,ZMASS,ZWIDTH , J3)

This subroutine computes the sum of photon and Z currents with the
suitable weights ( J(W3) = cos(theta_W) J(Z) + sin(theta_W) J(A) ).
The output J3 is useful as an input of VVVXXX, JVVXXX or W3W3XX.
The photon propagator is given in Feynman gauge, and the Z propagator
is given in unitary gauge.

INPUT:
      complex FI(6)          : flow-in  fermion                   |FI>
      complex FO(6)          : flow-out fermion                   <FO|
      real    GAF(2)         : FI couplings with A                GAF
      real    GZF(2)         : FI couplings with Z                GZF
      real    ZMASS          : mass  of Z
      real    ZWIDTH         : width of Z

OUTPUT:
```

```
      complex J3(6)         : W3 current           J^mu(<FO|W3|FI>)

=====================================================================

    SUBROUTINE IOSXXX(FI,FO,SC,GC , VERTEX)

This subroutine computes an amplitude of the fermion-fermion-scalar
coupling.

INPUT:
      complex FI(6)         : flow-in  fermion                   |FI>
      complex FO(6)         : flow-out fermion                   <FO|
      complex SC(3)         : input    scalar                      S
      complex GC(2)         : coupling constants                 GCHF

OUTPUT:
      complex VERTEX        : amplitude                      <FO|S|FI>

---------------------------------------------------------------------

    SUBROUTINE FSIXXX(FI,SC,GC,FMASS,FWIDTH , FSI)

This subroutine computes an off-shell fermion wavefunction from a
flowing-IN external fermion and a scalar boson.

INPUT:
      complex FI(6)         : flow-in  fermion                   |FI>
      complex SC(3)         : input    scalar                      S
      complex GC(2)         : coupling constants                 GCHF
      real    FMASS         : mass  of OUTPUT fermion F'
      real    FWIDTH        : width of OUTPUT fermion F'

OUTPUT:
      complex FSI(6)        : off-shell fermion             |F',S,FI>

---------------------------------------------------------------------

    SUBROUTINE FSOXXX(FO,SC,GC,FMASS,FWIDTH , FSO)

This subroutine computes an off-shell fermion wavefunction from a
flowing-OUT external fermion and a scalar boson.

INPUT:
      complex FO(6)         : flow-out fermion                   <FO|
      complex SC(6)         : input    scalar                      S
      complex GC(2)         : coupling constants                 GCHF
      real    FMASS         : mass  of OUTPUT fermion F'
      real    FWIDTH        : width of OUTPUT fermion F'

OUTPUT:
      complex FSO(6)        : off-shell fermion             <FO,S,F'|

---------------------------------------------------------------------

    SUBROUTINE HIOXXX(FI,FO,GC,SMASS,SWIDTH , HIO)

This subroutine computes an off-shell scalar current from an external
fermion pair.
```

```
INPUT:
      complex FI(6)          : flow-in  fermion                   |FI>
      complex FO(6)          : flow-out fermion                   <FO|
      complex GC(2)          : coupling constants                 GCHF
      real    SMASS          : mass  of OUTPUT scalar S
      real    SWIDTH         : width of OUTPUT scalar S

OUTPUT:
      complex HIO(3)         : scalar current           J(<FI|S|FO>)

======================================================================

    SUBROUTINE VVVXXX(WM,WP,W3,G , VERTEX)

This subroutine computes an amplitude of the three-point coupling of
the gauge bosons.

INPUT:
      complex WM(6)          : vector               flow-OUT W-
      complex WP(6)          : vector               flow-OUT W+
      complex W3(6)          : vector               J3 or A    or Z
      real    G              : coupling constant    GW or GWWA or GWWZ

OUTPUT:
      complex VERTEX         : amplitude               Gamma(WM,WP,W3)

----------------------------------------------------------------------

    SUBROUTINE JVVXXX(V1,V2,G,VMASS,VWIDTH , JVV)

This subroutine computes an off-shell vector current from the three-
point gauge boson coupling.  The vector propagator is given in Feynman
gauge for a massless vector and in unitary gauge for a massive vector.

INPUT:
      complex V1(6)          : first  vector                        V1
      complex V2(6)          : second vector                        V2
      real    G              : coupling constant (see the table below)
      real    VMASS          : mass  of OUTPUT vector V
      real    VWIDTH         : width of OUTPUT vector V

The possible sets of the inputs are as follows:
   -------------------------------------------------------------------
   |   V1   |   V2   |  JVV   |       G       |   VMASS  |  VWIDTH   |
   -------------------------------------------------------------------
   |   W-   |   W+   |  A/Z   |  GWWA/GWWZ    | 0./ZMASS | 0./ZWIDTH |
   | W3/A/Z |   W-   |  W+    | GW/GWWA/GWWZ |   WMASS  |  WWIDTH   |
   |   W+   | W3/A/Z |  W-    | GW/GWWA/GWWZ |   WMASS  |  WWIDTH   |
   -------------------------------------------------------------------
where all the bosons are defined by the flowing-OUT quantum number.

OUTPUT:
      complex JVV(6)         : vector current            J^mu(V:V1,V2)

----------------------------------------------------------------------

    SUBROUTINE GGGXXX(GA,GB,GC,G , VERTEX)

This subroutine computes an amplitude of the three-point coupling of
```

```
the gauge bosons.

INPUT:
      complex GA(6)          : vector               flow-OUT W-
      complex GB(6)          : vector               flow-OUT W+
      complex GC(6)          : vector               J3 or A    or Z
      real    G              : coupling constant    GW or GWWA or GWWZ

OUTPUT:
      complex VERTEX         : amplitude               Gamma(WM,WP,W3)

----------------------------------------------------------------------

    SUBROUTINE JGGXXX(G1,G2,G , JGG)

This subroutine computes an off-shell vector current from the three-
point gauge boson coupling.  The vector propagator is given in Feynman
gauge for a massless vector and in unitary gauge for a massive vector.

INPUT:
      complex V1(6)          : first  vector                        V1
      complex V2(6)          : second vector                        V2
      real    G              : coupling constant (see the table below)

The possible sets of the inputs are as follows:
   -----------------------------------------------------------------
   |   V1   |   V2   |  JVV   |      G       |   VMASS  |  VWIDTH   |
   -----------------------------------------------------------------
   |   W-   |   W+   |  A/Z   |  GWWA/GWWZ   | 0./ZMASS | 0./ZWIDTH |
   | W3/A/Z |   W-   |  W+    | GW/GWWA/GWWZ |   WMASS  |  WWIDTH   |
   |   W+   | W3/A/Z |  W-    | GW/GWWA/GWWZ |   WMASS  |  WWIDTH   |
   -----------------------------------------------------------------
where all the bosons are defined by the flowing-OUT quantum number.

OUTPUT:
      complex JGG(6)         : vector current            J^mu(V:V1,V2)

======================================================================

    SUBROUTINE VVSXXX(V1,V2,SC,GC , VERTEX)

This subroutine computes an amplitude of the vector-vector-scalar
coupling.

INPUT:
      complex V1(6)          : first  vector                        V1
      complex V2(6)          : second vector                        V2
      complex SC(3)          : input  scalar                        S
      complex GC             : coupling constant                  GVVH

OUTPUT:
      complex VERTEX         : amplitude                Gamma(V1,V2,S)

----------------------------------------------------------------------

    SUBROUTINE JVSXXX(VC,SC,GC,VMASS,VWIDTH , JVS)

This subroutine computes an off-shell vector current from the vector-
vector-scalar coupling.  The vector propagator is given in Feynman
```

```
gauge for a massless vector and in unitary gauge for a massive vector.

INPUT:
      complex VC(6)          : input vector                         V
      complex SC(3)          : input scalar                         S
      complex G              : coupling constant                 GVVH
      real    VMASS          : mass  of OUTPUT vector V'
      real    VWIDTH         : width of OUTPUT vector V'

OUTPUT:
      complex JVS(6)         : vector current            J^mu(V':V,S)

----------------------------------------------------------------------

    SUBROUTINE HVVXXX(V1,V2,GC,SMASS,SWIDTH , HVV)

This subroutine computes an off-shell scalar current from the vector-
vector-scalar coupling.

INPUT:
      complex V1(6)          : first  vector                       V1
      complex V2(6)          : second vector                       V2
      complex G              : coupling constant                 GVVH
      real    SMASS          : mass  of OUTPUT scalar S
      real    SWIDTH         : width of OUTPUT scalar S

OUTPUT:
      complex HVV(3)         : off-shell scalar current     J(S:V1,V2)

======================================================================

    SUBROUTINE VSSXXX(VC,S1,S2,GC , VERTEX)

This subroutine computes an amplitude from the vector-scalar-scalar
coupling.  The coupling is absent in the minimal SM in unitary gauge.

      complex VC(6)          : input  vector                        V
      complex S1(3)          : first  scalar                       S1
      complex S2(3)          : second scalar                       S2
      complex G              : coupling constant (S1 charge)

Examples of the coupling constant G for SUSY particles are as follows:
  -----------------------------------------------------------
  |    S1    | (Q,I3) of S1  ||   V=A   |   V=Z   |   V=W   |
  -----------------------------------------------------------
  | nu~_L    | (  0  , +1/2) ||   ---   |  GZN(1) |  GWF(1) |
  | e~_L     | ( -1  , -1/2) ||  GAL(1) |  GZL(1) |  GWF(1) |
  | u~_L     | (+2/3 , +1/2) ||  GAU(1) |  GZU(1) |  GWF(1) |
  | d~_L     | (-1/3 , -1/2) ||  GAD(1) |  GZD(1) |  GWF(1) |
  -----------------------------------------------------------
  | e~_R-bar | ( +1  ,  0  ) || -GAL(2) | -GZL(2) | -GWF(2) |
  | u~_R-bar | (-2/3 ,  0  ) || -GAU(2) | -GZU(2) | -GWF(2) |
  | d~_R-bar | (+1/3 ,  0  ) || -GAD(2) | -GZD(2) | -GWF(2) |
  -----------------------------------------------------------
where the S1 charge is defined by the flowing-OUT quantum number.

OUTPUT:
      complex VERTEX         : amplitude                Gamma(V,S1,S2)
```

```
---------------------------------------------------------------------

    SUBROUTINE JSSXXX(S1,S2,GC,VMASS,VWIDTH , JSS)

This subroutine computes an off-shell vector current from the vector-
scalar-scalar coupling.  The coupling is absent in the minimal SM in
unitary gauge.  The propagator is given in Feynman gauge for a
massless vector and in unitary gauge for a massive vector.

INPUT:
      complex S1(3)          : first  scalar                        S1
      complex S2(3)          : second scalar                        S2
      complex G              : coupling constant (S1 charge)
      real    VMASS          : mass  of OUTPUT vector V
      real    VWIDTH         : width of OUTPUT vector V

Examples of the coupling constant G for SUSY particles are as follows:
  -----------------------------------------------------------
  |    S1    | (Q,I3) of S1  ||   V=A   |   V=Z   |   V=W   |
  -----------------------------------------------------------
  | nu~_L    | (  0  , +1/2) ||   ---   |  GZN(1) |  GWF(1) |
  | e~_L     | ( -1  , -1/2) ||  GAL(1) |  GZL(1) |  GWF(1) |
  | u~_L     | (+2/3 , +1/2) ||  GAU(1) |  GZU(1) |  GWF(1) |
  | d~_L     | (-1/3 , -1/2) ||  GAD(1) |  GZD(1) |  GWF(1) |
  -----------------------------------------------------------
  | e~_R-bar | ( +1  ,  0  ) || -GAL(2) | -GZL(2) | -GWF(2) |
  | u~_R-bar | (-2/3 ,  0  ) || -GAU(2) | -GZU(2) | -GWF(2) |
  | d~_R-bar | (+1/3 ,  0  ) || -GAD(2) | -GZD(2) | -GWF(2) |
  -----------------------------------------------------------
where the S1 charge is defined by the flowing-OUT quantum number.

OUTPUT:
      complex JSS(6)         : vector current            J^mu(V:S1,S2)

---------------------------------------------------------------------

    SUBROUTINE HVSXXX(VC,SC,GC,SMASS,SWIDTH , HVS)

This subroutine computes an off-shell scalar current from the vector-
scalar-scalar coupling.  The coupling is absent in the minimal SM in
unitary gauge.

INPUT:
      complex VC(6)          : input vector                          V
      complex SC(3)          : input scalar                          S
      complex G              : coupling constant (S charge)
      real    SMASS          : mass  of OUTPUT scalar S'
      real    SWIDTH         : width of OUTPUT scalar S'

Examples of the coupling constant G for SUSY particles are as follows:
  -----------------------------------------------------------
  |    S1    | (Q,I3) of S1  ||   V=A   |   V=Z   |   V=W   |
  -----------------------------------------------------------
  | nu~_L    | (  0  , +1/2) ||   ---   |  GZN(1) |  GWF(1) |
  | e~_L     | ( -1  , -1/2) ||  GAL(1) |  GZL(1) |  GWF(1) |
  | u~_L     | (+2/3 , +1/2) ||  GAU(1) |  GZU(1) |  GWF(1) |
  | d~_L     | (-1/3 , -1/2) ||  GAD(1) |  GZD(1) |  GWF(1) |
  -----------------------------------------------------------
  | e~_R-bar | ( +1  ,  0  ) || -GAL(2) | -GZL(2) | -GWF(2) |
```

```
  | u~_R-bar | (-2/3 ,  0  ) || -GAU(2) | -GZU(2) | -GWF(2) |
  | d~_R-bar | (+1/3 ,  0  ) || -GAD(2) | -GZD(2) | -GWF(2) |
  ----------------------------------------------------------
where the SC charge is defined by the flowing-OUT quantum number.

OUTPUT:
      complex HVS(3)         : scalar current               J(S':V,S)

=======================================================================

    SUBROUTINE SSSXXX(S1,S2,S3,GC , VERTEX)

This subroutine computes an amplitude of the three-scalar coupling.

INPUT:
      complex S1(3)          : first  scalar                       S1
      complex S2(3)          : second scalar                       S2
      complex S3(3)          : third  scalar                       S3
      complex G              : coupling constant                 GHHH

OUTPUT:
      complex VERTEX         : amplitude              Gamma(S1,S2,S3)

-----------------------------------------------------------------------

    SUBROUTINE HSSXXX(S1,S2,GC,SMASS,SWIDTH , HSS)

This subroutine computes an off-shell scalar current from the three-
scalar coupling.

INPUT:
      complex S1(3)          : first  scalar                       S1
      complex S2(3)          : second scalar                       S2
      complex G              : coupling constant                 GHHH
      real    SMASS          : mass  of OUTPUT scalar S'
      real    SWIDTH         : width of OUTPUT scalar S'

OUTPUT:
      complex HSS(3)         : scalar current             J(S':S1,S2)

=======================================================================

    SUBROUTINE WWWWXX(WM1,WP1,WM2,WP2,GWWA,GWWZ,ZMASS,ZWIDTH , VERTEX)

This subroutine computes an amplitude of the four-point W-/W+
coupling, including the contributions of photon and Z exchanges.  The
photon propagator is given in Feynman gauge and the Z propagator is
given in unitary gauge.

INPUT:
      complex WM1(0:3)       : first  flow-OUT W-                 WM1
      complex WP1(0:3)       : first  flow-OUT W+                 WP1
      complex WM2(0:3)       : second flow-OUT W-                 WM2
      complex WP2(0:3)       : second flow-OUT W+                 WP2
      real    GWWA           : coupling constant of W and A      GWWA
      real    GWWZ           : coupling constant of W and Z      GWWZ
      real    ZMASS          : mass  of Z
      real    ZWIDTH         : width of Z
```

```
OUTPUT:
      complex VERTEX         : amplitude       Gamma(WM1,WP1,WM2,WP2)

----------------------------------------------------------------------

    SUBROUTINE JWWWXX(W1,W2,W3,GWWA,GWWZ,ZMASS,ZWIDTH,WMASS,WWIDTH ,
   &                  JWWW)

This subroutine computes an off-shell W+/W- current from the four-
point gauge boson coupling, including the contributions of photon and
Z exchanges.  The vector propagators for the output W and the internal
Z bosons are given in unitary gauge, and that of the internal photon
is given in Feynman gauge.

INPUT:
      complex W1(6)          : first  vector                        W1
      complex W2(6)          : second vector                        W2
      complex W3(6)          : third  vector                        W3
      real    GWWA           : coupling constant of W and A       GWWA
      real    GWWZ           : coupling constant of W and Z       GWWZ
      real    ZMASS          : mass  of internal Z
      real    ZWIDTH         : width of internal Z
      real    WMASS          : mass  of OUTPUT W
      real    WWIDTH         : width of OUTPUT W

The possible sets of the inputs are as follows:
  -------------------------------------------------------------------
  |  W1  |  W2  |  W3  |GWWA|GWWZ|ZMASS|ZWIDTH|WMASS|WWIDTH || JWWW |
  -------------------------------------------------------------------
  |  W-  |  W+  |  W-  |GWWA|GWWZ|ZMASS|ZWIDTH|WMASS|WWIDTH ||  W+  |
  |  W+  |  W-  |  W+  |GWWA|GWWZ|ZMASS|ZWIDTH|WMASS|WWIDTH ||  W-  |
  -------------------------------------------------------------------
where all the bosons are defined by the flowing-OUT quantum number.

OUTPUT:
      complex JWWW(6)        : W current             J^mu(W':W1,W2,W3)

----------------------------------------------------------------------

    SUBROUTINE W3W3XX(WM,W31,WP,W32,G31,G32,WMASS,WWIDTH , VERTEX)

This subroutine computes an amplitude of the four-point coupling of
the W-, W+ and two W3/Z/A.  The amplitude includes the contributions
of W exchange diagrams.  The internal W propagator is given in unitary
gauge.  If one sets WMASS=0.0, then the gggg vertex is given (see sect
2.9.1 of the manual).

INPUT:
      complex WM(0:3)        : flow-OUT W-                         WM
      complex W31(0:3)       : first    W3/Z/A                     W31
      complex WP(0:3)        : flow-OUT W+                         WP
      complex W32(0:3)       : second   W3/Z/A                     W32
      real    G31            : coupling of W31 with W-/W+
      real    G32            : coupling of W32 with W-/W+
                                                 (see the table below)
      real    WMASS          : mass  of W
      real    WWIDTH         : width of W

The possible sets of the inputs are as follows:
```

WM	W31	WP	W32	G31	G32
W-	W3	W+	W3	GW	GW
W-	W3	W+	Z	GW	GWWZ
W-	W3	W+	A	GW	GWWA
W-	Z	W+	Z	GWWZ	GWWZ
W-	Z	W+	A	GWWZ	GWWA
W-	A	W+	A	GWWA	GWWA

where all the bosons are defined by the flowing-OUT quantum number.

```
OUTPUT:
      complex VERTEX          : amplitude          Gamma(WM,W31,WP,W32)
```

```
    SUBROUTINE JW3WXX(W1,W2,W3,G1,G2,WMASS,WWIDTH,VMASS,VWIDTH , JW3W)
```

This subroutine computes an off-shell W+, W-, W3, Z or photon current
from the four-point gauge boson coupling, including the contributions
of W exchange diagrams. The vector propagator is given in Feynman
gauge for a photon and in unitary gauge for W and Z bosons. If one
sets WMASS=0.0, then the ggg-->g current is given (see sect 2.9.1 of
the manual).

```
INPUT:
      complex W1(6)           : first  vector                         W1
      complex W2(6)           : second vector                         W2
      complex W3(6)           : third  vector                         W3
      real    G1              : first  coupling constant
      real    G2              : second coupling constant
                                                   (see the table below)
      real    WMASS           : mass  of internal W
      real    WWIDTH          : width of internal W
      real    VMASS           : mass  of OUTPUT W'
      real    VWIDTH          : width of OUTPUT W'
```

The possible sets of the inputs are as follows:

W1	W2	W3	G1	G2	WMASS	WWIDTH	VMASS	VWIDTH	JW3W
W-	W3	W+	GW	GWWZ	WMASS	WWIDTH	ZMASS	ZWIDTH	Z
W-	W3	W+	GW	GWWA	WMASS	WWIDTH	0.	0.	A
W-	Z	W+	GWWZ	GWWZ	WMASS	WWIDTH	ZMASS	ZWIDTH	Z
W-	Z	W+	GWWZ	GWWA	WMASS	WWIDTH	0.	0.	A
W-	A	W+	GWWA	GWWZ	WMASS	WWIDTH	ZMASS	ZWIDTH	Z
W-	A	W+	GWWA	GWWA	WMASS	WWIDTH	0.	0.	A
W3	W-	W3	GW	GW	WMASS	WWIDTH	WMASS	WWIDTH	W+
W3	W+	W3	GW	GW	WMASS	WWIDTH	WMASS	WWIDTH	W-
W3	W-	Z	GW	GWWZ	WMASS	WWIDTH	WMASS	WWIDTH	W+
W3	W+	Z	GW	GWWZ	WMASS	WWIDTH	WMASS	WWIDTH	W-
W3	W-	A	GW	GWWA	WMASS	WWIDTH	WMASS	WWIDTH	W+
W3	W+	A	GW	GWWA	WMASS	WWIDTH	WMASS	WWIDTH	W-
Z	W-	Z	GWWZ	GWWZ	WMASS	WWIDTH	WMASS	WWIDTH	W+
Z	W+	Z	GWWZ	GWWZ	WMASS	WWIDTH	WMASS	WWIDTH	W-
Z	W-	A	GWWZ	GWWA	WMASS	WWIDTH	WMASS	WWIDTH	W+
Z	W+	A	GWWZ	GWWA	WMASS	WWIDTH	WMASS	WWIDTH	W-

```
  |  A   |  W-  |  A   |GWWA|GWWA|WMASS|WWIDTH|WMASS|WWIDTH ||  W+  |
  |  A   |  W+  |  A   |GWWA|GWWA|WMASS|WWIDTH|WMASS|WWIDTH ||  W-  |
  -------------------------------------------------------------------
where all the bosons are defined by the flowing-OUT quantum number.

OUTPUT:
      complex JW3W(6)        : W current            J^mu(W':W1,W2,W3)

----------------------------------------------------------------------

    SUBROUTINE GGGGXX(GA,GB,GC,GD,G , VERTEX)

This subroutine computes an amplitude of the four-point W-/W+
coupling, including the contributions of photon and Z exchanges.  The
photon propagator is given in Feynman gauge and the Z propagator is
given in unitary gauge.

INPUT:
      complex GA(0:3)       : first  flow-OUT W-                 GA
      complex GB(0:3)       : first  flow-OUT W+                 GB
      complex GC(0:3)       : second flow-OUT W-                 GC
      complex GD(0:3)       : second flow-OUT W+                 GD
      real    G             : coupling constant of W and A        G

OUTPUT:
      complex VERTEX         : amplitude       Gamma(GA,GB,GC,GD)

----------------------------------------------------------------------

    SUBROUTINE JGGGXX(G1,G2,G3,G , JGGG)

This subroutine computes an off-shell W+/W- current from the four-
point gauge boson coupling, including the contributions of photon and
Z exchanges.  The vector propagators for the output W and the internal
Z bosons are given in unitary gauge, and that of the internal photon
is given in Feynman gauge.

INPUT:
      complex G1(6)          : first  vector                       G1
      complex G2(6)          : second vector                       G2
      complex G3(6)          : third  vector                       G3
      real    G              : coupling constant                    G

The possible sets of the inputs are as follows:
  -------------------------------------------------------------------
  |  W1  |  W2  |  W3  |GWWA|GWWZ|ZMASS|ZWIDTH|WMASS|WWIDTH || JWWW |
  -------------------------------------------------------------------
  |  W-  |  W+  |  W-  |GWWA|GWWZ|ZMASS|ZWIDTH|WMASS|WWIDTH ||  W+  |
  |  W+  |  W-  |  W+  |GWWA|GWWZ|ZMASS|ZWIDTH|WMASS|WWIDTH ||  W-  |
  -------------------------------------------------------------------
where all the bosons are defined by the flowing-OUT quantum number.

OUTPUT:
      complex JGGG(6)        : gluon current        J^mu(G':G1,G2,G3)

======================================================================

    SUBROUTINE VVSSXX(V1,V2,S1,S2,GC , VERTEX)
```

```
This subroutine computes an amplitude of the vector-vector-scalar-
scalar coupling.

INPUT:
      complex V1(6)          : first  vector                       V1
      complex V2(6)          : second vector                       V2
      complex S1(3)          : first  scalar                       S1
      complex S2(3)          : second scalar                       S2
      complex G              : coupling constant                GVVHH

OUTPUT:
      complex VERTEX         : amplitude            Gamma(V1,V2,S1,S2)

----------------------------------------------------------------------

    SUBROUTINE JVSSXX(VC,S1,S2,GC,VMASS,VWIDTH , JVSS)

This subroutine computes an off-shell vector current from the vector-
vector-scalar-scalar coupling.  The vector propagator is given in
Feynman gauge for a massless vector and in unitary gauge for a massive
vector.

INPUT:
      complex VC(6)          : input  vector                       V
      complex S1(3)          : first  scalar                       S1
      complex S2(3)          : second scalar                       S2
      complex G              : coupling constant                GVVHH
      real    VMASS          : mass  of OUTPUT vector V'
      real    VWIDTH         : width of OUTPUT vector V'

OUTPUT:
      complex JVSS(6)        : vector current         J^mu(V':V,S1,S2)

----------------------------------------------------------------------

    SUBROUTINE HVVSXX(V1,V2,SC,GC,SMASS,SWIDTH , HVVS)

This subroutine computes an off-shell scalar current of the vector-
vector-scalar-scalar coupling.

INPUT:
      complex V1(6)          : first  vector                       V1
      complex V2(6)          : second vector                       V2
      complex SC(3)          : input  scalar                       S
      complex G              : coupling constant                GVVHH
      real    SMASS          : mass  of OUTPUT scalar S'
      real    SWIDTH         : width of OUTPUT scalar S'

OUTPUT:
      complex HVVS(3)        : scalar current            J(S':V1,V2,S)

======================================================================

    SUBROUTINE SSSSXX(S1,S2,S3,S4,GC , VERTEX)

This subroutine computes an amplitude of the four-scalar coupling.

INPUT:
      complex S1(3)          : first  scalar                       S1
```

```
      complex S2(3)         : second scalar                       S2
      complex S3(3)         : third  scalar                       S3
      complex S4(3)         : fourth scalar                       S4
      complex G             : coupling constant                GHHHH

OUTPUT:
      complex VERTEX        : amplitude           Gamma(S1,S2,S3,S4)

----------------------------------------------------------------------

    SUBROUTINE HSSSXX(S1,S2,S3,GC,SMASS,SWIDTH , HSSS)

This subroutine computes an off-shell scalar current from the four-
scalar coupling.

INPUT:
      complex S1(3)         : first  scalar                       S1
      complex S2(3)         : second scalar                       S2
      complex S3(3)         : third  scalar                       S3
      complex G             : coupling constant                GHHHH
      real    SMASS         : mass  of OUTPUT scalar S'
      real    SWIDTH        : width of OUTPUT scalar S'

OUTPUT:
      complex HSSS(3)       : scalar current          J(S':S1,S2,S3)

======================================================================

    SUBROUTINE EAIXXX(EB,EA,SHLF,CHLF,PHI,NHE,NHA , EAI)

This subroutine computes an off-shell electron wavefunction after
emitting a photon from the electron beam, with a special care for the
small angle region.  The momenta are measured in the laboratory frame,
where the e- beam is along the positive z axis.

INPUT:
      real    EB            : energy (GeV)    of beam  e-
      real    EA            : energy (GeV)    of final photon
      real    SHLF          : sin(theta/2)    of final photon
      real    CHLF          : cos(theta/2)    of final photon
      real    PHI           : azimuthal angle of final photon
      integer NHE  = -1 or 1 : helicity        of beam  e-
      integer NHA  = -1 or 1 : helicity        of final photon

OUTPUT:
      complex EAI(6)        : off-shell electron            |e',A,e>

----------------------------------------------------------------------

    SUBROUTINE EAOXXX(EB,EA,SHLF,CHLF,PHI,NHE,NHA , EAO)

This subroutine computes an off-shell positron wavefunction after
emitting a photon from the positron beam, with a special care for the
small angle region.  The momenta are measured in the laboratory frame,
where the e+ beam is along the negative z axis.

INPUT:
      real    EB            : energy (GeV)    of beam  e+
      real    EA            : energy (GeV)    of final photon
```

```
      real    SHLF           : sin(theta/2)    of final photon
      real    CHLF           : cos(theta/2)    of final photon
      real    PHI            : azimuthal angle of final photon
      integer NHE  = -1 or 1 : helicity        of beam  e+
      integer NHA  = -1 or 1 : helicity        of final photon

OUTPUT:
      complex EAO(6)         : off-shell positron            <e,A,e'|

--------------------------------------------------------------------

    SUBROUTINE JEEXXX(EB,EF,SHLF,CHLF,PHI,NHB,NHF,NSF , JEE)

This subroutine computes an off-shell photon wavefunction emitted from
the electron or positron beam, with a special care for the small angle
region.  The momenta are measured in the laboratory frame, where the
e- (e+) beam is along the positive (negative) z axis.

INPUT:
      real    EB             : energy (GeV)    of beam  e-/e+
      real    EF             : energy (GeV)    of final e-/e+
      real    SHLF           : sin(theta/2)    of final e-/e+
      real    CHLF           : cos(theta/2)    of final e-/e+
      real    PHI            : azimuthal angle of final e-/e+
      integer NHB  = -1 or 1 : helicity        of beam  e-/e+
      integer NHF  = -1 or 1 : helicity        of final e-/e+
      integer NSF  = -1 or 1 : +1 for electron, -1 for positron

OUTPUT:
      complex JEE(6)         : off-shell photon          J^mu(<e|A|e>)

********************************************************************

    SUBROUTINE MOMNTX(ENERGY,MASS,COSTH,PHI , P)

This subroutine sets up a four-momentum from the four inputs.

INPUT:
      real    ENERGY         : energy
      real    MASS           : mass
      real    COSTH          : cos(theta)
      real    PHI            : azimuthal angle

OUTPUT:
      real    P(0:3)         : four-momentum

--------------------------------------------------------------------

    SUBROUTINE MOM2CX(ESUM,MASS1,MASS2,COSTH1,PHI1 , P1,P2)

This subroutine sets up two four-momenta in the two particle rest
frame.

INPUT:
      real    ESUM           : energy sum of particle 1 and 2
      real    MASS1          : mass            of particle 1
      real    MASS2          : mass            of particle 2
      real    COSTH1         : cos(theta)      of particle 1
      real    PHI1           : azimuthal angle of particle 1
```

```
OUTPUT:
      real    P1(0:3)        : four-momentum of particle 1
      real    P2(0:3)        : four-momentum of particle 2

======================================================================

    SUBROUTINE BOOSTX(P,Q , PBOOST)

This subroutine performs the Lorentz boost of a four-momentum.  The
momentum P is assumed to be given in the rest frame of Q.  PBOOST is
the momentum P boosted to the frame in which Q is given.  Q must be a
timelike momentum.

INPUT:
      real    P(0:3)         : four-momentum P in the Q rest  frame
      real    Q(0:3)         : four-momentum Q in the boosted frame

OUTPUT:
      real    PBOOST(0:3)    : four-momentum P in the boosted frame

----------------------------------------------------------------------

    SUBROUTINE ROTXXX(P,Q , PROT)

This subroutine performs the spacial rotation of a four-momentum.
The momentum P is assumed to be given in the frame where the spacial
component of Q points the positive z-axis.  PROT is the momentum P
rotated to the frame where Q is given.

INPUT:
      real    P(0:3)         : four-momentum P in Q(1)=Q(2)=0 frame
      real    Q(0:3)         : four-momentum Q in the rotated frame

OUTPUT:
      real    PROT(0:3)      : four-momentum P in the rotated frame

**********************************************************************

    SUBROUTINE COUP1X(SW2 , GW,GWWA,GWWZ)

This subroutine sets up the coupling constants of the gauge bosons in
the STANDARD MODEL.

INPUT:
      real    SW2            : square of sine of the weak angle

OUTPUT:
      real    GW             : weak coupling constant
      real    GWWA           : dimensionLESS coupling of W-,W+,A
      real    GWWZ           : dimensionLESS coupling of W-,W+,Z

----------------------------------------------------------------------

    SUBROUTINE COUP2X(SW2 , GAL,GAU,GAD,GWF,GZN,GZL,GZU,GZD,G1)

This subroutine sets up the coupling constants for the fermion-
fermion-vector vertices in the STANDARD MODEL.  The array of the
couplings specifies the chirality of the flowing-IN fermion.  G??(1)
```

```
denotes a left-handed coupling, and G??(2) a right-handed coupling.

INPUT:
      real    SW2            : square of sine of the weak angle

OUTPUT:
      real    GAL(2)         : coupling with A of charged leptons
      real    GAU(2)         : coupling with A of up-type quarks
      real    GAD(2)         : coupling with A of down-type quarks
      real    GWF(2)         : coupling with W-,W+ of fermions
      real    GZN(2)         : coupling with Z of neutrinos
      real    GZL(2)         : coupling with Z of charged leptons
      real    GZU(2)         : coupling with Z of up-type quarks
      real    GZD(2)         : coupling with Z of down-type quarks
      real    G1(2)          : unit coupling of fermions

----------------------------------------------------------------------

    SUBROUTINE COUP3X(SW2,ZMASS,HMASS ,
   &                  GWWH,GZZH,GHHH,GWWHH,GZZHH,GHHHH)

This subroutine sets up the coupling constants of the gauge bosons and
Higgs boson in the STANDARD MODEL.

INPUT:
      real    SW2            : square of sine of the weak angle
      real    ZMASS          : mass of Z
      real    HMASS          : mass of Higgs

OUTPUT:
      real    GWWH           : dimensionFUL  coupling of W-,W+,H
      real    GZZH           : dimensionFUL  coupling of Z, Z, H
      real    GHHH           : dimensionFUL  coupling of H, H, H
      real    GWWHH          : dimensionFUL  coupling of W-,W+,H, H
      real    GZZHH          : dimensionFUL  coupling of Z, Z, H, H
      real    GHHHH          : dimensionLESS coupling of H, H, H, H

----------------------------------------------------------------------

    SUBROUTINE COUP4X(SW2,ZMASS,FMASS , GCHF)

This subroutine sets up the coupling constant for the fermion-fermion-
Higgs vertex in the STANDARD MODEL.  The coupling is COMPLEX and the
array of the coupling specifies the chirality of the flowing-IN
fermion.  GCHF(1) denotes a left-handed coupling, and GCHF(2) a right-
handed coupling.

INPUT:
      real    SW2            : square of sine of the weak angle
      real    ZMASS          : Z       mass
      real    FMASS          : fermion mass

OUTPUT:
      complex GCHF(2)        : coupling of fermion and Higgs
```

C.2 HELAS.LIST2

HELAS.LIST2 is just the list of the SUBROUTINE sentences in HELAS.FOR.

```
 ******************************************************************
 ***      SUBROUTINE LIST 2 for HELAS.FOR and HELAS_CHECK.FOR      ***
 ***              coded by H. Murayama & I. Watanabe              ***
 ***              ver.  2.3        3rd  Apr.   1992               ***
 ******************************************************************

    SUBROUTINE IXXXXX(P,FMASS,NHEL,NSF , FI)
    SUBROUTINE OXXXXX(P,FMASS,NHEL,NSF , FO)
...................................................................
    SUBROUTINE VXXXXX(P,VMASS,NHEL,NSV , VC)
...................................................................
    SUBROUTINE SXXXXX(P,NSS , SC)
===================================================================
    SUBROUTINE IOVXXX(FI,FO,VC,G , VERTEX)
    SUBROUTINE FVIXXX(FI,VC,G,FMASS,FWIDTH , FVI)
    SUBROUTINE FVOXXX(FO,VC,G,FMASS,FWIDTH , FVO)
    SUBROUTINE JIOXXX(FI,FO,G,VMASS,VWIDTH , JIO)
    SUBROUTINE J3XXXX(FI,FO,GAF,GZF,ZMASS,ZWIDTH , J3)
-------------------------------------------------------------------
    SUBROUTINE IOSXXX(FI,FO,SC,GC , VERTEX)
    SUBROUTINE FSIXXX(FI,SC,GC,FMASS,FWIDTH , FSI)
    SUBROUTINE FSOXXX(FO,SC,GC,FMASS,FWIDTH , FSO)
    SUBROUTINE HIOXXX(FI,FO,GC,SMASS,SWIDTH , HIO)
-------------------------------------------------------------------
    SUBROUTINE VVVXXX(WM,WP,W3,G , VERTEX)
    SUBROUTINE JVVXXX(V1,V2,G,VMASS,VWIDTH , JVV)
    SUBROUTINE GGGXXX(GA,GB,GC,G , VERTEX)
    SUBROUTINE JGGXXX(G1,G2,G , JGG)
...................................................................
    SUBROUTINE VVSXXX(V1,V2,SC,GC , VERTEX)
    SUBROUTINE JVSXXX(VC,SC,GC,VMASS,VWIDTH , JVS)
    SUBROUTINE HVVXXX(V1,V2,GC,SMASS,SWIDTH , HVV)
...................................................................
    SUBROUTINE VSSXXX(VC,S1,S2,GC , VERTEX)
    SUBROUTINE JSSXXX(S1,S2,GC,VMASS,VWIDTH , JSS)
    SUBROUTINE HVSXXX(VC,SC,GC,SMASS,SWIDTH , HVS)
...................................................................
    SUBROUTINE SSSXXX(S1,S2,S3,GC , VERTEX)
    SUBROUTINE HSSXXX(S1,S2,GC,SMASS,SWIDTH , HSS)
-------------------------------------------------------------------
    SUBROUTINE WWWWXX(WM1,WP1,WM2,WP2,GWWA,GWWZ,ZMASS,ZWIDTH , VERTEX)
    SUBROUTINE JWWWXX(W1,W2,W3,GWWA,GWWZ,ZMASS,ZWIDTH,WMASS,WWIDTH ,
   &                  JWWW)
    SUBROUTINE W3W3XX(WM,W31,WP,W32,G31,G32,WMASS,WWIDTH , VERTEX)
    SUBROUTINE JW3WXX(W1,W2,W3,G1,G2,WMASS,WWIDTH,VMASS,VWIDTH , JW3W)
    SUBROUTINE GGGGXX(GA,GB,GC,GD,G , VERTEX)
    SUBROUTINE JGGGXX(G1,G2,G3,G , JGGG)
...................................................................
    SUBROUTINE VVSSXX(V1,V2,S1,S2,GC , VERTEX)
    SUBROUTINE JVSSXX(VC,S1,S2,GC,VMASS,VWIDTH , JVSS)
    SUBROUTINE HVVSXX(V1,V2,SC,GC,SMASS,SWIDTH , HVVS)
...................................................................
    SUBROUTINE SSSSXX(S1,S2,S3,S4,GC , VERTEX)
    SUBROUTINE HSSSXX(S1,S2,S3,GC,SMASS,SWIDTH , HSSS)
===================================================================
    SUBROUTINE EAIXXX(EB,EA,SHLF,CHLF,PHI,NHE,NHA , EAI)
    SUBROUTINE EAOXXX(EB,EA,SHLF,CHLF,PHI,NHE,NHA , EAO)
    SUBROUTINE JEEXXX(EB,EF,SHLF,CHLF,PHI,NHB,NHF,NSF , JEE)
===================================================================
```

```
    SUBROUTINE MOMNTX(ENERGY,MASS,COSTH,PHI , P)
    SUBROUTINE MOM2CX(ESUM,MASS1,MASS2,COSTH1,PHI1 , P1,P2)
.......................................................................
    SUBROUTINE BOOSTX(P,Q , PBOOST)
    SUBROUTINE ROTXXX(P,Q , PROT)
=======================================================================
    SUBROUTINE COUPSM
    SUBROUTINE COUPMSSM
```