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BSM Physics in Sphaleron Vertices

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Zusammenfassung:

In nicht-pertubativen Sphaleron oder Instanton Prozessen wird der Wechselwirkungsvertex durch die chirale und $B+L$ elektroschwache Anomalie bestimmt. Für den Fall, dass Erweiterungen des Standardmodells $SU(2)$ geladene Teilchen enthalten, werden diese Teilchen Einfluss auf die $B+L$ verletzenden Prozesse nehmen. Handelt es sich bei diesen Teilchen um Teilchen in chiralen Darstellungen, so erhält man auschließlich erweiterte SM Vertices und reproduziert keinen SM ähnlichen Vertex. Für nicht-chirale Zustände hingegen können sowohl der SM Prozess mit Quarks und Leptonen, als auch erweiterte Sphaleron Interaktionen mit weiteren Teilchen vorhergesagt werden. Wir schätzen ab, wie sich die Raten solcher Prozesse an Hochenergie-Teilchenbeschleunigern verhalten. Wie zu erwarten, aufgrund des Entkopplungstheorem, verhält sich der SM ähnliche Prozess der neuen Theorie genau wie der des SM. Für einen bestimmten Massenbereich und Beschleuniger mit ausreichenden Energien sind die Prozesse mit neuen Teilchen potentiell um einige Größenordnungen größer als die des SM ähnlichen Falls. Folglich kann ein Bezug von $B+L$ verletzenden Wechselwirkungen zu Physik jenseits des Standardmodells hergestellt werden.

Abstract:

The vertex of nonperturbative sphaleron/instanton processes is determined by the chiral and $B+L$ electroweak anomalies. If new particles in extensions of the Standard model are charged under $SU(2)$, they will take part in these $B+L$ violating processes. In case of new states being in the chiral representations, the vertex consists of the maximal amount of Weyl fermions only fixed by the chiral charge violation and no SM-like vertices are reproduced. The decoupling theorem does not hold for chiral extensions of $SU(2)$. However, for non-chiral states, one predicts modified sphaleron vertices as well as the usual SM interaction with quarks and leptons.. We estimate how the rates of these electroweak $SU(2)$-sphaleron processes behave at high-energy colliders. As expected from decoupling, we see that the SM-like processes in the new theory with non-chiral fermions behave as in the SM. The BSM vertices are potentially enhanced by several order of magntiudes for a certain mass range and powerful enough accelerators. Thus, $B+L$ violating processes are linked to BSM physics.

Contents

1 Introduction

The Standard Model (SM) of Particle Physics has been a great success so far. With the discovery of the Higgs boson in 2012 by ATLAS [1] and CMS [2], the particle content of the SM is completed. Besides the Higgs boson, it consists of three quark and three lepton families as well as force carriers. Those are the gluons g for the strong force, the $W^\pm$ and Z^0 bosons for the weak force and the photon γ for the electromagnetic force. Although the SM classifies all known elementary particles and all experiments have confirmed their interactions via the strong, weak and electromagnetic force very well, there are hints for physics beyond the Standard Model (BSM). For example, the SM cannot describe the asymmetry of matter over anti-matter sufficiently. Several models try to explain the processes that could have happened in the Early Universe to create that asymmetry like Electroweak Baryogenesis [3] or Baryogenesis through Leptogenesis [4, 5]. Such theories always include baryon and lepton number violating processes. Especially the sphaleron process, a non-pertubative effect at the quantum level of the SM, plays a special role there.

Another big mystery is the large amount of non-baryonic matter in the Universe. Several astrophysical observations indicate that about 27% of the energy density of the Universe is a new type of cold dark matter (DM) which cannot be described by any SM particle. So far, all astrophysical measurements of DM are based on gravitational effects, either through dynamical effects [6, 7, 8, 9], through light deflection by gravitional lenses, or by looking at the gravitational potential of galaxy clusters. If we assume that the current relic abundance of DM is produced thermally, we obtain a thermally averaged annihilation cross section of the order of $\langle \sigma v \rangle \approx 3 \times 10^{-26}$ cm^{3}s^{-1}. Considering the simple case of an s-channel self-annihilation of DM through an exchange of a gauge boson to SM particles, the cross section is about $\langle \sigma v \rangle \sim G_F^2 m_{\text{WIMP}}^2$. The correct relic density is then obtained for masses in the range $m_{\text{DM}} \sim$ few GeV - TeV. Hence, the main focus on DM searches is on weakly interacting massive particles (WIMP) especially tested in direct detection experiments. Although strong limits on the mass and cross sections of DM particles charged under the weak SM group $SU(2)$ has been set, it is still one of the favoured DM candidates.

A general approach to describe BSM physics is to introduce new particles with some connection to the SM either via new mediators or with couplings to the SM gauge bosons. Starting with a specific model, one tries to identify channels to observe the new physics in certain experiments, *e.g.* in collider experiments.

The search for new physics we focus on in this thesis is different from the general approach.

We don't start with a specific model describing new physics and test its observability in collider experiments. Our approach is to start with a not yet measured SM process, namely the sphaleron process, and investigate its influence by new physics in form of new $SU(2)$ Weyl fermions. Such particles are considered *e.g.* in dark matter models or in SUSY models. New physics could change the rates as well as the vertex of these processes in crucial ways depending on the mass and chiral nature of the new fermions as well as the energy scale of the process.

The thesis is structured as follows. In chapter 2, the theoretical background for understanding fermion number violating sphaleron processes is described. First, we explain the general concept of symmetries in quantum field theories, followed by applying it to the SM. In chapter 3 an introduction into sphaleron and instanton calculations is given. In Chapter 4, we discuss in which way we include new particles in the sphaleron vertex and which rates result in such processes. Here, we focus on particles with masses below and above the vacuum expectation value $v = 246$ GeV and the special case of chiral fermions. Chapter 5 concludes in which way sphaleron processes change by allowing for additional $SU(2)$ Weyl fermions.

The work presented in this thesis is based on a project with David G. Cerdeño, Carlos Tamarit and Kazuki Sakurai and will be published soon.

2 Symmetries and the Standard Model

2.1 Symmetries in Quantum Field Theories

All the information to describe a Quantum Field Theory (QFT) is encoded in the Lagrangian. Among others, it tells you in which way the particles interact with each other. Furthermore, their equations of motion can be derived from it and it describes if and how particles acquire a mass. Symmetries play a key role in constructing such a Lagrangian. A symmetry of a Lagrangian $\mathcal{L}(\phi_i, \partial_\mu \phi_i)$ of the fields ϕ_i is defined as a field transformation that changes $\mathcal{L}$ at most by a total derivative such that the action

$$S = \int d^4x \mathcal{L}(\phi_i, \partial_\mu \phi_i) \tag{2.1}$$

changes at most only by a surface term and consequently the equations of motion stay invariant. The equations of motion

$$\frac{\partial \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x))}{\partial \phi_i(x)} = \partial_\mu \frac{\partial \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x))}{\partial(\partial_\mu \phi_i(x))} \tag{2.2}$$

can be derived from the Euler-Lagrange equations of a Lagrangian by varying the action.

As an illustrative example of how to work with a Lagrangian, consider the Lagrangian of quantum electrodynamics (QED)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left(i\gamma^\mu D_\mu - m \right) \psi \tag{2.3}$$

with the electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ where A_μ is the covariant four-potential of the electromagnetic field. ψ and $\bar{\psi} = \psi^\dagger \gamma^0$ are the spinor field of a spin-$\frac{1}{2}$ particle, e.g.

electrons or quarks, and its Dirac adjoint, respectively, and $D_\mu = \partial_\mu + igA_\mu$ is the gauge covariant derivative [1] with the coupling constant g and the mass m of the particle. Using the Euler-Lagrange equations for the four-potential A_μ

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\mu} = 0 \tag{2.4}$$

we can get the inhomogeneous Maxwell equations

$$\partial_\nu F^{\nu\mu} = g\bar{\psi}\gamma^\mu\psi = gJ^\mu \tag{2.5}$$

with the current density $J^\mu = \psi\bar{\gamma}^\mu\psi$ by identifying the components of the electric field $E^i = -F^{0i}$ and magnetic field $\epsilon^{ijk}B^k = -F^{ij}$. The homogeneous Maxwell equations can be written as

$$\partial_\nu \tilde{F}^{\mu\nu} = 0 \tag{2.6}$$

with $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. Moreover, from the QED Lagrangian one can read off the QED interaction vertex $g\tilde{\psi}\gamma^\mu A_\mu\psi$. In QED, being a local $U(1)$ gauge theory, direct mass terms $m\bar{\psi}\psi$ are allowed. Embedding the electron and quark fields into the $SU(2)\times U(1)$ theory of the electroweak sector, these direct mass terms are forbidden due to the chiral nature of the theory. More about that can be found in sec. 2.2.

2.1.1 Noether's Theorem

The Noether Theorem relates symmetries of a theory to its conservation laws. A continuous symmetry can be written infinitesimally as

$$\phi_i \to \phi_i + \epsilon\delta\phi_i + \mathcal{O}(\epsilon^2)\ . \tag{2.7}$$

The Lagrangian becomes

$$\mathcal{L} \to \mathcal{L} + \epsilon\delta\mathcal{L} + \mathcal{O}(\epsilon^2) \tag{2.8}$$

where $\delta\mathcal{L} = \partial_\mu F^\mu$ is a total derivative of some F^μ. $\delta\mathcal{L}$ can be written as

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) \\ &= \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi \right] + \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \right] \delta\phi\ . \end{aligned} \tag{2.9}$$

[1] We will see later in sec. why we consider such a gauge covariant derivative instead of the ordinary ∂_μ derivative

The last term on the right-hand side vanishes using the equations of motion in eq. 2.2. This is summarized by Noether's Theorem. It says that the so called "Noether current"

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - F^\mu \tag{2.10}$$

is conserved

$$\partial_\mu J^\mu = 0 \tag{2.11}$$

if the transformation is a symmetry and hence, the on-shell condition (= equations of motions) can be used. This results in a conserved charge

$$Q = \int_{\mathbb{R}^3} d^3x J^0(x) \tag{2.12}$$

since

$$\begin{aligned} \partial_0 Q &= \int_{\mathbb{R}^3} d^3x \partial_0 J^0 \\ &= \int_{\mathbb{R}^3} d^3x \partial_\mu J^\mu - \int_{\mathbb{R}^3} d^3x \partial_i J^i \\ &= -\int_{\mathbb{R}^3} d^3x \partial_i J^i = 0 \end{aligned} \tag{2.13}$$

assuming that the spatial components $J^i,\ i = 1, 2, 3$, fall off fast for large x.

2.1.2 Group theory in QFT

The mathematical structure behind symmetries in physics is group theory as described, *e.g.* in [10] and [11]. We categorize groups in Abelian and Non-Abelian groups depending on whether their group elements g_i commute or not. Elements of a group G in a certain representation $U(g_i)$ act as unitary operators on the fields of the Lagrangian in our theory. If it leaves the equations of motion invariant, it is a symmetry of the theory. Furthermore, transformations can be be classified into global and local transformations. Later on, we discuss gauge transformations which can be seen as making a global symmetry local. QFTs are based on continuously generated groups with infinitesimal group elements in a certain representation U_d with dimension d

$$U_d(g(\alpha_i)) = \mathbb{I} + \delta\alpha_i \frac{\partial U_d(g(\alpha_i))}{\partial \alpha_i}\bigg|_{\alpha_i=0} + ... \tag{2.14}$$

with $\delta\alpha_i = \frac{\alpha_i}{N}$. "Continuously" means that that the group is parametrized by a set of continuous parameters α_i, for $i = 1, ..., n$ and $n = \dim(G)$. Group elements can then be written as

$g(\alpha_i)$. Such groups are called **Lie groups**. These groups are generated by generators

$$T_i \equiv -i \frac{\partial U_d}{\partial \alpha_i}\bigg|_{\alpha_i=0} \tag{2.15}$$

as one can see in the following. A finite transformation $U_d(g(\alpha_i))$ can be obtained by an infinite number of infinitesimal transformations

$$\begin{aligned} U(g(\alpha_i)) &= \lim_{N\to\infty}(1 + i\delta\alpha_i T_i)^N = \lim_{N\to\infty}(1 + \frac{\alpha_i}{N}T_i)^N \\ &= e^{i\alpha_i T_i} \ . \end{aligned} \tag{2.16}$$

The generators T_i form the Lie algebra $Lie(G)$ of the Lie group G. They span the vector space of the Lie algebra and the commutation relation of these generators form the structure of the Lie group. This can be seen by looking at the multiplication of two group elements in a certain representation

$$e^{i\alpha_i T_i} e^{i\beta_j T_j} = e^{i\delta_k T_k} \tag{2.17}$$

which should also be an element of the group $e^{i\delta_k T_k}$ due to the closure of the group. By expanding the exponential and keeping only terms up to second order in α and β, one obtains

$$e^{i\alpha_i T_i} e^{i\beta_j T_j} = e^{i(\alpha_i T_i + \beta_j T_j) - \frac{1}{2}[\alpha_i T_i, \beta_j T_j]} \ . \tag{2.18}$$

This is the famous **Baker-Campbell-Hausdorff** formula. It tells us that the commutation relation of our generators determines the way group elements interact with each other. For Abelian groups the commutator is obviously zero whereas in general, we have

$$[T_i, T_j] = i f_{ijk} T_k \tag{2.19}$$

with the structure constant f_{ijk}. Summarized, one can say that the generators, under a specific commutation relation defined by the structure constant, form the Lie Algebra of the group, and the commutation structure forms the structure of the Lie group. Besides, the generators satisfy the commutator identity

$$[T_a, [T_b, T_c]] + [T_b, [T_c, T_a]] + [T_c, [T_a, T_b]] = 0 \tag{2.20}$$

which implies that the structure constants obey

$$f_{ade} f_{bcd} + f_{bde} f_{cad} + f_{cde} f_{abd} = 0 \tag{2.21}$$

which is the so called Jacobi identity. Coming back to QFT, once we have specified a (local) symmetry group, the fields appearing in the Lagrangian of the theory transform according to a certain finite-dimensional unitary representation of this group. A finite-dimensional unitary representation of the symmetry group's Lie algebra with dimension d consists of a set of $d \times d$ Hermitian matrices T_a. A special form of representations are the irreducible[2] representations denoted by T_r^a. In a certain basis for irreducible representations, the trace of the product of two generator matrices is proportional to the identity

$$\mathrm{Tr}[T_r^a T_r^b] = C(r)\delta^{ab} \tag{2.22}$$

where $C(r)$ is a constant for each representation r. Using the commutation relations in eq. 2.19 the structure constant can be written as

$$f^{abc} = -\frac{i}{C(r)}\mathrm{Tr}\{[T_r^a, T_r^b]T_r^c\} \ . \tag{2.23}$$

There are two very important representations, namely the fundamental and the adjoint representation denoted by $r = f$ and $r = \text{adj.}$, respectively. The representation matrices of the adjoint representation, present for any simple Lie algebra, are given by the structure constants

$$(T^b_{\text{adj.}})_{ac} = if^{abc} \tag{2.24}$$

and for $SU(N)$ its dimension is $d(\text{adj.}) = N^2 - 1$. So the dimension of the group and the adjoint representation coincide. This includes that the matrices of the adjoint representation are $(N^2-1)\times(N^2-1)$ matrices. Yet another characteristic operator of representations is the quadratic operator

$$T^2 = T^a T^a \tag{2.25}$$

which commutes with all group generators

$$\begin{aligned}[T^b, T^a T^a] &= (if^{bac}T^c)T^a + T^a(if^{bac}T^c) \\ &= if^{bac}\{T^c, T^a\} = 0 \ .\end{aligned} \tag{2.26}$$

In the last step, we used that the product of an antisymmetric f^{bac} and symmetric object $\{T^c, T^a\}$ vanishes. Being an invariant of the algebra, the operator takes a constant value on each irreducible representation

$$T_r^a T_r^a = C_2(r)\cdot \mathbb{I}, \tag{2.27}$$

where $\mathbb{I}$ is a $d(r) \times d(r)$ unit matrix and $C_2(r)$ is called the **quadratic Casimir operator**. For the adjoint representation, we have

$$f^{acd}f^{bcd} = C_2(\text{adj.})\delta^{ab} \tag{2.28}$$

[2]We will not specify this expression. Further explanations can be found in the literature, *e.g.* [10]

Using eq. 2.22 and eq. 2.27, we find

$$d(r)C_2(r) = d(\text{adj.})C(r) \ . \tag{2.29}$$

For the fundamental representation of $SU(N)$, we have

$$\text{Tr}[T_f^a T_f^b] = \frac{1}{2}\delta^{ab} \tag{2.30}$$

and hence we get

$$C(f) = \frac{1}{2}, \quad C_2(f) = \frac{N^2-1}{2N} \tag{2.31}$$

using eq. 2.29 and the fact that $d(f) = N$ for the fundamental representation. For the adjoint representation of $SU(N)$ we have

$$C_2(\text{adj.}) = C(\text{adj.}) = N \ . \tag{2.32}$$

With these group-theoretic concepts in mind, we are ready to continue with computations in non-Abelian gauge theories.

2.1.3 Global and Local Transformations - The Abelian Case

Global Transformations

Before going to non-Abelian gauge theories, we first regard spacetime independent Abelian **global** and **local** $U(1)$ transformations of the unitary group with $U(\alpha) \equiv e^{-i\alpha}$ and the unitary condition

$$U^\dagger = U^{-1} \ . \tag{2.33}$$

We start with the Dirac Lagrangian of the form

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{2.34}$$

and show that only by imposing symmetry constraints on our Lagrangian, we will end up with the full QED Lagrangian of eq. 2.3. It is straightforward to prove that the Lagrangian in 2.34 is invariant under the global transformation

$$\psi(x) \to e^{-i\alpha}\psi(x) \simeq \psi - i\alpha\psi = \psi + \alpha\delta\psi + \mathcal{O}(\alpha^2) \tag{2.35}$$

with $\delta\psi = -i\psi$. Noether's theorem tells us that this gives rise to a conserved current of the form

$$J^\mu = \bar{\psi}\gamma^\mu\psi \ . \tag{2.36}$$

This can be easily checked

$$
\begin{aligned}
\partial_\mu J^\mu &= (\partial_\mu \bar{\psi})\gamma^\mu \psi + \bar{\psi}\gamma^\mu \partial_\mu \psi \\
&= (im\bar{\psi})\psi + \bar{\psi}(-im\psi) \\
&= 0
\end{aligned} \tag{2.37}
$$

using the equations of motion

$$
\begin{aligned}
(i\gamma^\mu \partial_\mu - m)\psi = 0 \\
-i\partial_\mu \bar{\psi}\gamma^\mu - m\bar{\psi} = 0 \ .
\end{aligned} \tag{2.38}
$$

The corresponding charge is

$$
Q = \int d^3x \bar{\psi}(x)\gamma^0 \psi(x) = \int d^3x \psi^\dagger(x)\psi(x) \tag{2.39}
$$

which corresponds to the conserved electromagnetic charge in QED.

Local Transformations

Whereas global transformations act on fields of the Lagrangian in the same way at every point in spacetime, local transformations allow independent symmetry transformations at every point in spacetime. Hence, our transformation matrix $U(\alpha(x)) = U(x)$ is spacetime dependent. The procedure in the following is called gauging a theory meaning that we make a global symmetry local. Starting with the fact that the Lagrangian in eq. 2.34 is invariant under global $U(1)$ transformations like in eq. 2.35, we promote this to local tranformations

$$
\psi(x) \to U(x)\psi(x) := e^{-ie\alpha(x)}\psi(x) \ . \tag{2.40}
$$

For the mass term $m\bar{\psi}\psi$ in eq. 2.3 this does not cause any problems. Nevertheless, difficulties arise when we look at terms including the ordinary derivative $\partial_\mu \psi$ formally defined as

$$
n^\mu \partial_\mu \psi = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\psi(x + \epsilon n) - \psi(x)\right] \tag{2.41}
$$

because the two objects in eq. 2.41 transform differently under 2.40

$$
\psi(x) \to U(x)\psi(x), \quad \text{but} \quad \psi(x + n\epsilon) \to U(x + n\epsilon)\psi(x + n\epsilon) \ . \tag{2.42}
$$

To compensate the difference in the phase transformation from one point x to the neighbouring points $x + n\epsilon$, we introduce the so called Wilson-line or comparator such that under 2.40

$$
C(y, x)\psi(x) \to U(y)C(y, x)\psi(x) \ . \tag{2.43}
$$

The so called covariant derivative can then defined via

$$n^\mu D_\mu \psi = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\psi(x + \epsilon n) - C(x + n\epsilon, x)\psi(x) \right] \tag{2.44}$$

to obtain the appropriate transformation behaviour

$$D_\mu \psi(x) \to U(x) D_\mu \psi(x) \ . \tag{2.45}$$

The requirement 2.43 implies that

$$C(y, x) \to U(y) C(y, x) U^{-1}(x) \ . \tag{2.46}$$

We further impose that $C(y, y) = \mathbb{I}$ and consider $C(y, x)$ to be a pure phase. By Taylor expanding

$$C(x + \epsilon n, x) = 1 - ieA_\mu(x)\epsilon n^\mu + \mathcal{O}(\epsilon^2) \tag{2.47}$$

with some vector field $A_\mu(x)$ appearing as the infinitesimal limit of a comparator of local symmetry transformations, we obtain

$$D_\mu \psi = \partial_\mu \psi + ieA_\mu(x)\psi(x) \tag{2.48}$$

for the covariant derivative. By expanding $U(x + \epsilon n) = U(x) + \epsilon n^\mu \partial_\mu U(x)$ and considering the transformation like in eq. 2.46, we obtain for the infinitesimal behaviour of $C(x + n\epsilon)$

$$1 - ieA_\mu(x)n^\mu \epsilon \to (U(x) + \epsilon n^\mu \partial_\mu U(x))(1 - ieA_\mu(x)n^\mu \epsilon)U^{-1}(x) \tag{2.49}$$

up to order of ϵ and hence, for the vector field

$$A_\mu(x) \to U(x) A_\mu(x) U^{-1}(x) + \frac{i}{e}(\partial_\mu U(x))U^{-1}(x) \ . \tag{2.50}$$

For the tranformation we consider in eq. 2.40, this yields $A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$. So only by imposing a local symmetry under $U(1)$, we obtained a vector field $A_\mu(x)$. Being a local vector field, $A_\mu(x)$ has its own dynamics. To complete the construction of a locally invariant Lagrangian, we have to find a kinetic term for $A_\mu(x)$. This should be a term consisting of A_μ's and derivatives only. Therefore, we play with the covariant derivative of 2.48 to find such a term. Since we know that ψ transforms like 2.40 and the derivative transforms in the same way as seen in 2.45, every higher order derivative transforms in the same way, *e.g.* the second derivative. Hence, the commutator of the covariant derivative transforms as

$$[D_\mu, D_\nu]\psi(x) \to U(x)[D_\mu, D_\nu]\psi \ . \tag{2.51}$$

However, the commutator of the covariant derivatives itself is not a derivative anymore

$$\begin{aligned}[D_\mu, D_\nu]\psi(x) &= [\partial_\mu + ieA_\mu(x), \partial_\nu + ieA_\nu(x)]\psi(x) \\ &= ie\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ie[A_\mu(x), A_\nu(x)]\right)\psi(x) \ .\end{aligned} \tag{2.52}$$

This multiplicative factor is the field strength defined as

$$F_{\mu\nu} := \frac{1}{ie}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu] \ . \tag{2.53}$$

In case of the transformations 2.40 in the Abelian $U(1)$ theory, the commutator term in 2.53 vanishes and we get the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.54}$$

invariant under $U(1)$ transformations of the form $A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$. The full QED Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi \tag{2.55}$$

as already seen in 2.3.

2.1.4 Non-Abelian Gauge Theories

A more general contruction of a theory can be done by considering more general group symmetries. In non-Abelian gauge theories, we consider gauge transformations of the form

$$U = \exp\left(-ig\sum_{a=1}^{n} \alpha^a T^a\right) \tag{2.56}$$

where U is an element of the Lie group G in a certain representation determined by the explicit form of T^a and n is again the dimension of the Lie group. In sec. 2.1.2, we already discussed some general properties of the basis elements T^a of the Lie Algebra $Lie(G)$ generating the Lie group. For the Abelian group $G = U(1)$ these generators are simply $T^a \equiv T \in \mathbb{R}$ with $[T, T] = 0$. In the non-Abelian case, the generators follow the results obtained in sec. 2.1.2. In the following, we only consider $SU(N)$ and especially $SU(2)$. The matrix representations $U(g) \in \mathbb{C}^{d,d}$ of the group elements g of $SU(N)$ have the properties

$$U^\dagger = U^{-1} \quad \text{and} \quad \det U = 1 \tag{2.57}$$

Hence, the generators $T^a \in \mathbb{C}^{d,d}$ of the underlying Lie algebra $Lie(G)$ have to be of the form

$$T^{a\dagger} = T^a \quad \text{and} \quad \mathrm{Tr}T^a = 0 \ . \tag{2.58}$$

We study again a matter Lagrangian with fields $\psi(x)$ transforming in a unitary representation $U(g)$, with $g \in G$, of the group G leaving the Lagrangian $\mathcal{L}$ invariant under global transformations

$$\psi(x) \to U(g)\psi(x), \quad U^\dagger = U^{-1} \ . \tag{2.59}$$

As examples, we look at the fundamental and adjoint representation which we will use later on in the $SU(2)$ case. For the **fundamental** representation of $SU(N)$, we work with a $\mathbb{C}^N$-valued Dirac spinor $\psi(x)$ field with

$$\forall x : \psi(x) \equiv \psi_i(x) = \begin{pmatrix} \psi_1(x) \\ ... \\ \psi_N(x) \end{pmatrix}, \quad \psi_i(x) \to U_{ij}(g)\psi_j(x) \tag{2.60}$$

where the ψ_i are Weyl spinors and N is the dimension of the representation $d(f) = N$. The Lagrangian of eq. 2.34

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi = \sum_{i=1}^{N} \bar{\psi}_i(i\gamma^\mu \partial_\mu - m)\psi_i \tag{2.61}$$

is invariant under global $SU(N)$ transformations

$$\begin{aligned} \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \to \bar{\psi}_i U^\dagger(g)_{ij}(i\gamma^\mu \partial_\mu - m)U_{jk}\psi_k &= \bar{\psi}_i \underbrace{U^\dagger_{ij} U_{jk}}_{\delta_{ik}}(i\gamma^\mu \partial_\mu - m)\psi_k \\ &= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \end{aligned} \tag{2.62}$$

In the **adjoint** representation of $SU(N)$, we deal with spinor fields of the form

$$\forall x : \psi(x) \equiv \psi_{ij}(x) \in \Big(\mathbb{C}^{N,N} | \psi^\dagger(x) = \psi(x), \mathrm{Tr}\psi(x) = 0\Big) \tag{2.63}$$

transforming like

$$\psi(x)_{il} \to U(g)_{ij}\psi_{jk}(U^{-1}(g))_{kl} \ . \tag{2.64}$$

In this case, we consider the Lagrangian

$$\mathcal{L} = \mathrm{Tr}\bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{2.65}$$

which is invariant under global transformations because

$$\begin{aligned}
\mathrm{Tr}\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \to \mathrm{Tr}\left((U\psi U^{-1})^\dagger\gamma^0(i\gamma^\mu\partial_\mu - m)U\psi U^{-1}\right) \\
&= \mathrm{Tr}\left(U\bar{\psi}U^\dagger(i\gamma^\mu\partial_\mu - m)U\psi U^{-1}\right) \\
&= \mathrm{Tr}\left(\bar{\psi}\underbrace{U^\dagger U}_{\mathbb{I}}(i\gamma^\mu\partial_\mu - m)\psi\underbrace{U^{-1}U}_{\mathbb{I}}\right) \\
&= \mathrm{Tr}\left(\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi\right)
\end{aligned} \tag{2.66}$$

For gauging the theory, we repeat all the steps involved in the Abelian case of $U(1)$. For simplicity, we only consider the fundamental representation of $SU(N)$. We proceed to the spacetime dependend transformation

$$\begin{aligned}
\psi(x) \to U(x)\psi(x) \quad \text{with} \\
U = \exp\left(-ig\sum_{a=1}^{n}\alpha^a(x)T^a\right)
\end{aligned} \tag{2.67}$$

and consider the Wilson-line

$$C(x+n\epsilon, x) = \mathbb{I} - ig\sum_{a=1}^{n}A_\mu^a(x)T^a\epsilon n^\mu + \mathcal{O}(\epsilon^2) \tag{2.68}$$

with n vector fields $A_\mu^a(x)$. We further define an $N\times N$ matrix-valued vector field

$$A_\mu(x) \equiv \sum_a A_\mu^a T^a\ . \tag{2.69}$$

The covariant derivative is constructed just like in the Abelian case in eq. 2.48 through

$$D_\mu\psi(x) = \partial_\mu\psi(x) + ig\sum_a A_\mu^a T^a\psi(x) \tag{2.70}$$

where $T^a\psi(x) \equiv T_{ij}^a\psi_j(x)$. The gauge transformation for $A_\mu(x)$ is still like in eq. 2.50 but in the non-Abelian case we have $U(x)A_\mu(x)U^{-1}(x) \neq A_\mu(x)$. By expanding

$$U(x) = \mathbb{I} - ig\sum_a\alpha^a(x)T^a + \mathcal{O}((\alpha^a(x))^2) \tag{2.71}$$

we obtain

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha^a(x)T^a - ig\sum_a\alpha^a(x)[T^a, A_\mu(x)] \tag{2.72}$$

where $[T^a, A_\mu(x)] = [T^a, A^b_\mu(x)T^b] = A^b_\mu(x)[T^a, T^b] = if^{abc}A^b_\mu(x)T^c$ using eq. 2.19. Finally, we have

$$A^c_\mu(x) \to A^c_\mu(x) + \partial_\mu \alpha^c(x) + gf^{abc}\alpha^a(x)A^b_\mu(x) \ . \tag{2.73}$$

For the field strength $G_{\mu\nu} = \frac{1}{ig}[D_\mu, D_\nu] \equiv G^a_{\mu\nu}(x)T^a$, one obtains

$$\begin{aligned} G_{\mu\nu}(x) &= \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig[A_\mu(x), A_\nu(x)], \\ G^a_{\mu\nu}(x) &= \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) - gf^{abc}A^b_\mu(x)A^c_\nu(x). \end{aligned} \tag{2.74}$$

Since the field strength transforms in the adjoint representation as

$$G_{\mu\nu}(x) \to U(x)G_{\mu\nu}(x)U^{-1}(x), \tag{2.75}$$

it is not invariant due to the non-commuting generators in all terms of eq. 2.75. Thus, we proceed like in the adjoint fermion case and consider the trace over the product of field strengths which is invariant under the transformation

$$\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}) \to \mathrm{Tr}(UG_{\mu\nu}\underbrace{U^{-1}U}_{\mathbb{I}}G^{\mu\nu}U^{-1}) = \mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}\underbrace{U^{-1}U}_{\mathbb{I}}) = \mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}) \ . \tag{2.76}$$

With eq. 2.30, we can write the invariant Lagrangian $\mathcal{L}$ as

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}) + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \\ &= -\frac{1}{4}\sum_a G^a_{\mu\nu}G^{\mu\nu a} + \bar{\psi}_i(i\gamma^\mu\partial_\mu - m)\psi_i - g\bar{\psi}_i\gamma^\mu A^a_\mu T^a_{ij}\psi_j \ . \end{aligned} \tag{2.77}$$

The crucial difference between the Abelian $U(1)$ theory and the non-Abelian case of $SU(N)$ is the cubic and quartic gauge field interaction term contained in the field strength term of eq. 2.77 compared to eq. 2.55 where we have kinetic terms only. Everything we have considered so far is the basis of the SM. The only thing we have taken into account is proposing some symmetry constraints to our Dirac Lagrangian in 2.34. However, we will also see that experimental constraints determine the structure of the full SM Lagrangian. For example, it turns out that direct mass terms like in eq. 2.77 are not allowed in the SM. An additional mechanism generates the masses through breaking a symmetry at a certain scale. This scenario is described by the electroweak sector of the SM.

2.2 The Electroweak Sector and the Higgs Mechanism

The electroweak theory is the theory of the unification of two local gauge symmetries $U(1)$ and $SU(2)$ into a larger $SU(2) \times U(1)$ group and the spontaneous symmetry breaking to one $U(1)$ at a certain scale v introduced by Glashow, Weinberg, and Salam (GWS) [12, 13, 14]. In the following, we sketch the GWS theory of electroweak interations and the Higgs mechanism. A detailed discussion can be found for example in [11]. We begin with a $SU(2) \times U(1)$ gauge theory with a scalar Higgs field ϕ in the spinor representation of $SU(2)$ transforming like

$$\phi \to \exp(-i\alpha^a \tau^a) \exp(-i\beta/2)\phi \tag{2.78}$$

with $\tau^a = \sigma^a/2$. The symmetry breaking occurs at a certain scale where ϕ acquires a vacuum expectation value of the form

$$\langle\phi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} . \tag{2.79}$$

Then a gauge transformation like in eq. 2.78 with

$$\alpha^1 = \alpha^2 = 0, \quad \alpha^3 = \beta \tag{2.80}$$

leaves $\langle\phi\rangle$ invariant and the massless gauge boson is a combination of the two generators of this transformation whereas the massive bosons are combinations of $\tau^{1,2}$. These gauge bosons acquire a mass from the Higgs field. This can be worked out straightforwardly. The covariant derivative of the Higgs is

$$D_\mu\phi = (\partial_\mu + igA^a_\mu\tau^a + i\frac{1}{2}g'B_\mu)\phi \tag{2.81}$$

where A^a_μ and B_μ are the $SU(2)$ and $U(1)$ gauge boson fields, respectively. Inserting 2.79 in the kinetic term $\frac{1}{2}(D_\mu\phi)^2$ of the Higgs field and explicitly evaluating the matrix products with $\tau^a = \sigma^a/2$, we get

$$\Delta\mathcal{L} = \frac{1}{2}\frac{v^2}{4}\left[g^2(A^1_\mu)^2 + g^2(A^2_\mu)^2 + (-gA^3_\mu + g'B_\mu)^2\right] \tag{2.82}$$

which yields us the mass terms of the massive vector bosons. These can be expressed as

$$\begin{aligned} W^\pm_\mu &= \frac{1}{\sqrt{2}}(A^1_\mu \mp iA^2_\mu) \quad \text{with mass} \quad m_W = g\frac{v}{2}; \\ Z^0_\mu &= \frac{1}{\sqrt{g^2+g'^2}}\left(gA^3_\mu - g'B_\mu\right) \quad \text{with mass} \quad m_Z = \sqrt{g^2+g'^2}\frac{v}{2} . \end{aligned} \tag{2.83}$$

The massless vector field is

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(g'A^3_\mu + gB_\mu\right) . \tag{2.84}$$

We can rewrite the covariant derivative in eq. 2.81 in terms of the mass eigenstates

$$\begin{aligned} D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}\left(W_\mu^+ T^+ + W_\mu^- T^-\right) - i\frac{1}{\sqrt{g^2+g'^2}} Z_\mu \left(g^2 T^3 - g'^2 Y\right) \\ - i\frac{gg'}{\sqrt{g^2+g'^2}} A_\mu \left(T^3 + Y\right) \end{aligned} \tag{2.85}$$

where we inserted a $U(1)$ charge Y by hand called hypercharge and $T^\pm = \left(T^1 \pm iT^2\right) = \left(\tau^1 \pm i\tau^2\right) = \frac{1}{2}(\sigma^1 \pm \sigma^2)$. In eq. 2.85, we can see explicitly what we have observed before in eq. 2.80, namely that the massless gauge boson A_μ couples to a combination of the gauge generators $T^3 + Y$ and the theory remains symmetric under transformations like in eq. 2.80 with the gauge field A_μ. So we break down $SU(2) \times U(1)$ to a $U(1)$ where the Higgs field acquires a vacuum expectation value v and we obtain three massive and one massless vector gauge bosons. In our new $U(1)$, we identify $T^3 + Y$ with the electric charge quantum number

$$Q = T^3 + Y \tag{2.86}$$

and the coefficient of the electromagnetic interaction through A_μ with the electron charge e

$$e = \frac{gg'}{\sqrt{g^2+g'^2}} \ . \tag{2.87}$$

We can further simplify the covariant derivative in eq. 2.85 by introducing a weak mixing angle θ_W appearing in the change of basis from (A^3, B) to (Z^0, A)

$$\begin{pmatrix} Z^0 \\ A \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A^3 \\ B \end{pmatrix} \ . \tag{2.88}$$

By identifying

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}, \quad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}} \tag{2.89}$$

we can write the covariant derivative of eq. 2.85 like

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}\left(W_\mu^+ T^+ + W_\mu^- T^-\right) - i\frac{g}{\cos\theta_W} Z_\mu \left(T^3 - \sin\theta_W Q\right) - ieA_\mu Q \tag{2.90}$$

with $g = \frac{e}{\sin\theta_W}$. All weak processes can now be written in terms of the electric charge e, the weak mixing angle θ_W and the W boson mass since we obtain $m_W = m_Z \cos\theta_W$ from eq. 2.83.

We continue with the question how fermions couple to the gauge bosons. For one, the covariant derivative determines the coupling of the W and Z^0 bosons to the fermion fields once we specified the quantum numbers of the fermions through eq. 2.86. For another, it turns out that nature does not treat all fermions couple equally. We must take into account that the W boson only couples to the left-handed helicity states of the quarks and leptons. Therefore, we split the derivative term of the Dirac fields in the Lagrangian 2.77 into left-handed and right-handed fields

$$\bar{\psi} i\gamma^\mu \partial_\mu \psi = \bar{\psi}_L i\gamma^\mu \partial_\mu \psi_L + \bar{\psi}_R i\gamma^\mu \partial_\mu \psi_R \ . \tag{2.91}$$

Hence, we can treat the couplings to left- and right-handed fields separately. This implies that the covariant derivative in eq. 2.90 acts differently on both helicity states or in other words, the left-handed fields are assigned to be $SU(2)$ doublets and the right-handed states are singlets under $SU(2)$. So the weak sector of the SM is a chiral theory. To reproduce the correct electric charge Q, we have to get the correct sum of T^3 and Y values for the fields. In case of the right-handed fields with $T^3 = 0$, the hypercharge equals the electric charge $Y = Q$. We write the left-handed fields as

$$E_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L , \quad Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L , \tag{2.92}$$

with $Y = -1/2$ and $Y = 1/6$, respectively and with $T^3 = \pm 1/2$ for the upper and lower components, respectively. For the fermion kinetic terms, we obtain

$$\mathcal{L} = \bar{E}_L(i\gamma^\mu D_\mu)E_L + \bar{e}_R(i\gamma^\mu D_\mu)e_R + \bar{Q}_L(i\gamma^\mu D_\mu)Q_L + \bar{u}_R(i\gamma^\mu D_\mu)u_R + \bar{d}_R(i\gamma^\mu D_\mu)d_R \ . \tag{2.93}$$

with the covariant derivative given by eq. 2.48 with T^a and Y acting on the fields depending on the particular representation. In the SM, we don't take into account right-handed neutrinos since they neither couple to the $SU(2)$ gauge bosons nor to the $U(1)$ gauge boson.

Fermion Mass Terms

Now that we have separated left- and right-handed fermions and considered them to live in different representations, we end up with the problem of finding a global gauge invariant mass term for the fermion fields. Terms like

$$\Delta\mathcal{L} = -m_e \left(\bar{e}_L e_R + \bar{e}_R e_L\right) \tag{2.94}$$

are forbidden since e_L and e_R transform differently under $SU(2)$ and have different $U(1)$ charges. The problem is solved by introducing the Higgs field ϕ here as well. Being a spinor under $SU(2)$ and having $Y = 1/2$, the Higgs field provides the missing properties to define a gauge-invariant term given by

$$\Delta\mathcal{L} = -y_e \bar{E}_L \phi e_R + \text{h.c.} \ . \tag{2.95}$$

It is easy to prove that all Y sum up to zero in this term and $SU(2)$ transformations leave the Lagrangian invariant. We call y_i the Yukawa couplings. After the electroweak symmetry breaking, *i.e.* replacing $\phi \to \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}$, we obtain the mass terms for the fermions, *e.g.* for electrons we have

$$\Delta\mathcal{L} = -\frac{1}{\sqrt{2}} y_e \bar{e}_L e_R + \text{h.c.} \tag{2.96}$$

with the mass $m_e = \frac{1}{\sqrt{2}} y_e v$. This procedure is done for the quarks in the same way.

The Higgs Potential

The only incredient to complete the electroweak theory with a spontaneous symmetry breaking is to actually have a Higgs field potential with a non-vanishing vacuum expectation value. A renormalizable Lagrangian resulting in a non-zero vacuum expectation value v is the one of the form

$$\mathcal{L}_H = (D_\mu \phi)^\dagger D^\mu \phi + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \tag{2.97}$$

with the minimum $v = \left(\frac{\mu^2}{\lambda}\right)^{1/2}$. We will work in the unitarity gauge and parametrize the scalar field like

$$\phi(x) = U(x) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \tag{2.98}$$

with a real-valued field $h(x)$ with $\langle h(x) \rangle = 0$ and a $SU(2)$ gauge transformation $U(x)$ which produces the most general complex-valued two-component spinor. After eliminating $U(x)$ by a gauge transformation and inserting it in the Lagrangian in eq. 2.97, the potential energy terms in the unitarity gauge become

$$\begin{aligned} \mathcal{L}_V &= -\mu^2 h^2 - \lambda v h^3 - \frac{1}{4}\lambda h^4 \\ &= -\frac{1}{2} m_h^2 h^2 - \sqrt{\frac{\lambda}{2}} m_h h^3 - \frac{1}{4}\lambda h^4 \end{aligned} \tag{2.99}$$

where we identified the Higgs boson mass $m_h = \sqrt{2}\mu = \sqrt{2\lambda} v$.

2.3 Cross Section Calculations

The total cross section of a $2 \to n$ process with initial particles p_a and p_b and $s = p_a^2 + p_b^2$ is

$$\sigma_n = \frac{1}{F} I_n(s), \tag{2.100}$$

where

$$F = 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2} \tag{2.101}$$

is the flux factor which can be written as

$$F = 2\lambda^{1/2}(s, m_a^2, m_b^2) \tag{2.102}$$

for initial particles in $z-$direction with the Källén function $\lambda(x,y,z) \equiv x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$. The other term is of the form

$$I_n(s) = \int \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i} (2\pi)^4 \delta^{(4)} \left(p_a + p_b - \sum_i p_i \right) |\mathcal{M}(\vec{p}_i)|^2 \tag{2.103}$$

and contains the integration over the phase space as well as the square of the matrix element $|\mathcal{M}(\vec{p}_i)|^2$.

Phase Space Integration

The phase space is the $3n-4$ dimensional surface of the $3n$ dimensional momentum space of the final state momentum vectors $\vec{p}_i$ obtained by constraining the momentum space by the four-momentum conservation of the process

$$\begin{aligned} E_a + E_b &= \sum_{i=1}^{n} E_i \\ \vec{p}_a + \vec{p}_b &= \sum_{i=1}^{n} \vec{p}_i \\ \text{with} \quad E_i^2 &= |\vec{p}|_i^2 + m_i^2, \quad i = a, b, 1, ..., n \ . \end{aligned} \tag{2.104}$$

where m_i are the fixed particle masses. The delta distribution $\delta^{(4)}\left(p_a + p_b - \sum_i p_i\right)$ in eq. 2.103 takes care of the four-momentum conservation. In principle, we could consider the factor $\prod_i (2E_i)^{-1}$ to be part of $|\mathcal{M}(\vec{p}_i)|^2$ but we separate it since the quantity $\frac{d^3 p_i}{2E_i}$ is Lorentz invariant. This can be seen for example from a boost in $z-$direction

$$\begin{aligned} dp_x &= dp'_x \\ dp_y &= dp'_y \\ dp_z &= \gamma(dp'_z + v dE') \\ &= \gamma dp'_z (1 + v p'_z / E') \\ &= dp'_z E / E' \end{aligned} \tag{2.105}$$

using that $dE'/dp'_z = \frac{d}{dp'_z}(\sqrt{|\vec{p}_i|^2 + m_i^2}) = p'_z/E'$ and $E = \gamma(E' + v p'_z)$. Hence, we have

$$\frac{d^3 p'_i}{E'} = \frac{d^3 p}{E} \ . \tag{2.106}$$

By using the property of the delta function

$$
\begin{aligned}
\delta(f(x)) &= \frac{1}{|f'(x_0)|}\delta(x - x_0) \\
f(x_0) &= 0,
\end{aligned}
\tag{2.107}
$$

and writing $p^2 = (p_0)^2 - |\vec{p}|^2 = (p_0)^2 - E^2 + m^2$ we can easily show that

$$
\frac{d^3p}{2E} = \int d^4p\delta(p^2 - m^2)\Theta(p_0)\ . \tag{2.108}
$$

Therefore, we can write 2.103 equivalently in the form

$$
I_n(s) = \frac{1}{(2\pi)^{3n-4}} \int \prod_{i=1}^{n} d^4p_i\delta\left(p_i^2 - m_i^2\right)\Theta(p_i^0)\delta^{(4)}\left(p_a + p_b - \sum_i^n p_i\right)|\mathcal{M}(\vec{p}_i)|^2\ . \tag{2.109}
$$

Usually, the square of the matrix element $|\mathcal{M}(\vec{p}_i)|^2$ is of course momentum dependent. Nevertheless, let's consider the case $|\mathcal{M}|^2 = 1$ for now. The integral I_n becomes a pure phase space integral

$$
R_n(s) = \frac{1}{(2\pi)^{3n-4}} \int \prod_{i=1}^{n} \frac{d^3p_i}{2E_i}\delta^{(4)}(p - \sum p_i)\ . \tag{2.110}
$$

The mass dimension of the pure phase space integral is $2n-4 = 2(n-2)$ where $2n$ comes from the product $\left[\prod_{i=1}^{n}\frac{d^3p_i}{2E_i}\right]_m = 2n$ and -4 from the delta distribution $\left[\delta^{(4)}(p - \sum_i p_i)\right]_m = -4$. Whereas for $n = 2$ the phase space is dimensionless, for $n > 2$ it is energy dependent. We also consider the non-covariant phase space integral $R_n(p^\mu)$, defined by

$$
R_n(p^\mu) = \frac{1}{(2\pi)^{3n-4}} \int \prod_{i=1}^{n} d^3p_i\delta^{(4)}(p - \sum p_i) \tag{2.111}
$$

where we have chosen $|\mathcal{M}|^2 = \prod_i(2E_i)$. To get a feel for how phase space integrations are performed and the final results look like, we consider the simple $n = 2$ case. Since the phase space is Lorentz invariant, we are allowed to compute the integral in a preferred frame, *i.e.* the rest frame of the two-body system, $P = p_1 + p_2 = (\sqrt{s}, 0, 0, 0)$, of the final particles p_1

and p_2. We can get

$$
\begin{aligned}
R_2(\sqrt{s}) &= \frac{1}{(2\pi)^2}\int \frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\delta^{(4)}(P-p_1-p_2) \\
&= \frac{1}{(2\pi)^2}\int \frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\delta(\sqrt{s}-E_1-E_2)\delta^{(3)}(\vec{p}_1+\vec{p}_2) \\
&= \frac{1}{(2\pi)^2}\int \frac{d^3p}{2\sqrt{m_1^2+\vec{p}^2}}\frac{1}{2\sqrt{m_2^2+\vec{p}^2}}\delta\left(\sqrt{s}-\sqrt{m_1^2+\vec{p}^2}-\sqrt{m_2^2+\vec{p}^2}\right) \\
&= \frac{1}{(2\pi)^2}\int \frac{p^2\ dp\ d\cos\theta\ d\phi}{2\sqrt{m_1^2+\vec{p}^2}}\frac{1}{2\sqrt{m_2^2+\vec{p}^2}}\delta\left(\sqrt{s}-\sqrt{m_1^2+\vec{p}^2}-\sqrt{m_2^2+\vec{p}^2}\right)\ . \quad (2.112)
\end{aligned}
$$

By solving the delta function δ for $p=\frac{\sqrt{s}}{2}\beta=\frac{\sqrt{s}}{2}\sqrt{1-\frac{2(m_1^2+m_2^2)}{s}+\frac{2(m_1^2-m_2^2)^2}{s^2}}$ and using eq. 2.107, this can be simplified to

$$
R_2(\sqrt{s}) = \frac{1}{(2\pi)^2}\int \frac{p^2dpd\cos\theta d\phi}{2\sqrt{m_1^2+\vec{p}^2}}\frac{1}{2\sqrt{m_2^2+\vec{p}^2}}\frac{\delta(p-\beta\sqrt{s}/2)}{(p/E_1)+(p/E_2)}\ . \quad (2.113)
$$

Finally, we obtain

$$
R_2(\sqrt{s}) = \frac{\beta}{8\pi}\int \frac{d\cos\theta}{2}\frac{d\phi}{2\pi} = \frac{\beta}{8\pi} \quad (2.114)
$$

by integrating over the full momentum space. For massless particles, we have $\beta=1$ and hence get $R_2(\sqrt{s})=\frac{1}{8\pi}$ which is dimensionless as expected.

Matrix Element

Another integral part of cross section calculations is the matrix element $\mathcal{M}(\vec{p}_i)$ or in particular the square of the matrix element $|\mathcal{M}(\vec{p}_i)|^2$ as seen in eq. 2.100. The matrix element is also Lorentz invariant[3]. It is obtained directly from the Feynman rules of the corresponding theory. All vertices in Feynman diagrams are proportional to the gauge coupling of the considered theory. Its exact form can be directly obtained from the corresponding Lagrangian of the interaction. Hence, we have $|M|^2\propto g^x$ with a certain power x of the coupling dependent on the number of vertices in the diagram. Knowing the exact power is often already sufficient to say something about the order of magnitude of the interaction rate and provides information about leading and subleading diagrams of the process. This rough estimate is also useful for comparisons of cross sections of different processes. Another important part of matrix element calculations are the momentum dependent external particles. For every external fermion, we obtain a Dirac spinor $u^s(p), \bar{v}^s(p)$ for initial particles and anti-particles, respectively, or its Dirac conjugated for the final states. By squaring the matrix element, we get pairs of these Dirac spinors proportional to the momentum of the external particles, *i.e.* $\bar{u}u\propto\gamma_\mu p^\mu$. If we

[3]Only the prefactor 2.101 of 2.100 is not Lorentz invariant. It's only invariant to boosts in the $z-$direction, but in general has the transformation properties of a cross sectional area as expected.

have several external particles, we have to contract them in all possible ways like $p_a \cdot p_b$ for particles a and b. Nevertheless, every term in the final result will roughly be proportional to the external momenta of the fermions, *i.e.* $|M|^2 \propto g^x \prod_{i=1}^{n} p_i$ for n external particles.

3 Instanton and Sphaleron Theory

3.1 Instantons

3.1.1 Euclidean Formulation

The theory of instantons is the study of semiclassical approximations of the path-integral that defines a QFT with gauge fields. The approximation relies on saddle-point expansions around classical extrema of the Euclidean action.

Therefore, we first formulate our non-Abelian gauge theory in Euclidean space-time, labeled by hats like $\hat{v}$. In Minkowski space we distinguish between the covariant and contravariant vectors, v_μ and v^μ, respectively, with $\mu = 0, 1, 2, 3$. In Euclidean space, there is no such distinction. It doesn't matter if we write upper or lower indices, $v_\mu = v^\mu$ with $\mu = 0, 1, 2, 3$. The spatial coordinates are not changed going from Minkowski to Euclidean space $\hat{x}_i = x^i, i = 1, 2, 3$. For the time coordinate, we have

$$\hat{x}_0 = -ix_0 \tag{3.1}$$

such that for Euclidean four-vectors, we get $\hat{x}_\mu = (-ix_0, x_i)$. From the chain rule

$$\frac{\partial}{\partial \hat{x}_0} = \frac{\partial x_0}{\partial \hat{x}_0} \frac{\partial}{\partial x_0} = i \frac{\partial}{\partial x_0} \tag{3.2}$$

we obtain the transformation to the Euclidean version of the derivative $\hat{\partial}_\mu = (i\partial_0, \partial_i)$. In order to find a reasonable tranformation for the covariant derivative $D_\mu = \partial_\mu + igA_\mu^a T^a$, the vector potential A_μ has to transform in the same way as ∂_μ and hence, we have

$$\hat{A}_\mu = (iA_0, A_i), \tag{3.3}$$

$$\hat{D}_\mu = (iD_0, D_i) \ . \tag{3.4}$$

For the non-Abelian gauge field strength tensor $G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gf^{abc} A_\mu^b A_\nu^c$ we discussed in detail in sec. 2.1.4 one obtains

$$\hat{G}_{0\nu}^a = iG_{0\nu}, \quad \hat{G}_{ij} = G_{ij} \ \ (i, j = 1, 2, 3) \ . \tag{3.5}$$

For the pure gauge action, we finally find

$$\begin{aligned} iS &= -\hat{S} \quad \text{with} \\ S &= \int d^4x \left[-\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \right] \qquad (3.6) \\ \hat{S} &= \int d^4\hat{x} \left[\frac{1}{4} \hat{G}^a_{\mu\nu} \hat{G}^a_{\mu\nu} \right] \ . \qquad (3.7) \end{aligned}$$

In the following, we only consider Euclidean space-time and drop the hat over the Euclidean version for notational simplicity. We will make clear, whenever we go back to the Minkowski case again.

3.1.2 Finiteness of the Action and Vacuum Structure of $SU(N)$

For the Euclidean action like in eq. 3.6 to be finite, the action integral has to converge. The convergence of the integral is controlled by the behaviour of the gauge vector field A_μ. That means that the field strength $G_{\mu\nu}$ must fall off faster than $\mathcal{O}(1/r^2)$ for $r \to \infty$ where r is the radial variable in Euclidean four-space $r^2 = x_0^2 + \vec{x}^2$. Nevertheless, this does not imply that A_μ must decrease faster than $1/r$ but merely that A_μ must be of the form

$$A_\mu = iS\partial_\mu S^\dagger + \mathcal{O}(1/r^2) \qquad (3.8)$$

where S is a unitary unimodular matrix mapping the four-space to the considered group (in our case $SU(2)$) depending on the angular variables only. In that case of A_μ having a purely gauge form, the field strength $G_{\mu\nu}$ is vanishing for large r as wanted. Since gauge transformations are functions mapping the Euclidean space to the gauge group space, every finite-action field configuration is associated with a angular-only-dependend function mapping the three-dimensional hypersphere S^3 into the gauge group of the theory. Topologically, $SU(2)$ is related to the three-dimensional sphere S^3 and therefore, finite-action field configurations are related to mappings of S^3 to S^3. While preserving finite action, we are still allowed to perform a continuous gauge transformation $U(r)$ on A_μ

$$A_\mu \to U^\dagger A_\mu U + iU^\dagger \partial_\mu U \qquad (3.9)$$

and hence

$$S \to U^\dagger S + \mathcal{O}(1/r^2) \ . \qquad (3.10)$$

The regularity of $U(r)$ implies that it has to be independent of the angles at the origin, and hence constant. Therefore, $U(r \to \infty)$ has to be a continuous deformation of constant $U(r)$ at the origin, in particular a continuous deformation of the identity mapping. When classifying all physically inequivalent $S^3 \to S^3$ mappings, we consider all possible mappings modulo gauge transformations $U(r)$ of the form of eq. 3.9 to obtain equivalence classes of mappings. In other words, two mappings are considered equivalent if they can be related by a transformation $U(r = \infty)$ which is a continuous deformation of the identity, *i.e.* if they can be continuously deformed into each other. These equivalence classes are called homotopy classes and elements of the same class are called homotopic. As a result, one can say that for calculations with finite field configurations, one has to find the corresponding homotopy classes for the physically interesting gauge group.

In $SU(2)$, matrices can be parametrized as

$$M = A + i\vec{B}\vec{\sigma}, \quad M \in SU(2) \tag{3.11}$$

where the four real parameters A and $\vec{B}$ satisfy $A^2 + |\vec{B}|^2 = 1$ such that $M^\dagger M = 1$ and $\det M = 1$. Since $SU(2)$ is S^3, we study homotopy classes of mappings from S^3 to S^3. In general, there exists an infinte number of different classes of mappings $S^3 \to G$ if G is a non-Abelian simple Lie group. Mappings belonging to different homotopy classes cannot be continuously deformed into another. As we will see later, every class is characterized by an integer topological value n. As a result, we have an infinite amount of field configurations A_μ such that the field strength $G_{\mu\nu}$ vanishes for large r. A derivation of the desired solutions can be found in [15]. Here, we will only mention the standard mappings from S^3 to S^3. The trivial mapping is

$$S_0(x) = \mathbb{I} \tag{3.12}$$

Another identity mapping is

$$S_1(x) = \frac{x_0 + i\vec{x}\vec{\sigma}}{\sqrt{x^2}} = \frac{i\tau_\mu^+ x_\mu}{\sqrt{x^2}} \tag{3.13}$$

where $x^2 = r^2 = x_0^2 + \vec{x}^2$ and $i\tau_\mu^+ x_\mu = x_0 + i\vec{x}\vec{\sigma}$ with $\tau_\mu^\pm = (\mp i, \vec{\sigma})$. The general form of these standard mappings is

$$S_n(x) = (S_1(x))^n \tag{3.14}$$

where n is the topological integer charge mentioned above called winding number. Topologically spoken, the winding number measures the number of times the hypersphere at infinity is wrapped around the corresponding group G. Formally, one can define it as

$$n = -\frac{1}{24\pi^2} \oint_{S^3} d^3\theta_i \mathrm{Tr}\left[\epsilon^{\nu\rho\sigma} S(\partial_\nu S^\dagger) S(\partial_\rho S^\dagger) S(\partial_\sigma S^\dagger)\right] \tag{3.15}$$

where θ_i are three angles parametrizing the hypersphere S^3. The winding number defined in eq. 3.15 is a homotopy invariant quantity, see for example [16]. In [16], it is also shown that eq. 3.15 can be written in an gauge-invariant integral representation [17]

$$n = \frac{g^2}{32\pi^2} \int d^4x G^a_{\mu\nu} \tilde{G}^{a\mu\nu} \tag{3.16}$$

with $\tilde{G}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\gamma\delta} G^a_{\gamma\delta}$. This can be verified by rewriting $G^a_{\mu\nu}\tilde{G}^{a\mu\nu}$ in the form of a total derivative

$$G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} = \partial_\mu K_\mu \tag{3.17}$$

$$K_\mu = 2\epsilon_{\mu\nu\gamma\delta}\left(A^a_\nu \partial_\gamma A^a_\delta + \frac{1}{3} g \epsilon^{abc} A^a_\nu A^b_\gamma A^c_\delta\right) \tag{3.18}$$

so that the volume integral in eq. 3.16 can be transformed into an integral over the surface of a large sphere in four-dimensional space and A^a_μ is of the form as in eq. 3.8. The vector K_μ is called Chern-Simons current and the corresponding Chern-Simons charge or number is

$$N_{\mathrm{CS}} = \frac{g^2}{32\pi^2} \int K_0(x) d^3x \; . \tag{3.19}$$

One can show that (transparently seen in the so called topological gauge, with $A_0 = 0$ and $A_i \to 0$ faster than $1/r$ for $|\vec{x}| \to \infty$)

$$N_{\mathrm{CS}}(t_1) - N_{\mathrm{CS}}(t_0) = \int_{t_0}^{t_1} dt \int d^3x \partial_\mu K_\mu = n \; . \tag{3.20}$$

It is obvious that we get $n = 0$ for the trivial mapping in eq. 3.12. For the mapping in eq. 3.13, we use the fact that the integrand of eq. 3.15 is constant and evaluate it only at the north pole of the unit hypersphere, $x_4 = 1, x_i = 0$. This results in

$$\begin{aligned}
n &= -\frac{1}{24\pi^2} \oint_{S^3} d^3\theta_i \mathrm{Tr}\left[\epsilon^{\nu\rho\sigma} S(\partial_\nu S^\dagger) S(\partial_\rho S^\dagger) S(\partial_\sigma S^\dagger)\right] \\
&= -\frac{1}{24\pi^2} \oint_{S^3} d^3\theta_i \mathrm{Tr}\left[\epsilon^{ijk} S(\partial_i S^\dagger) S(\partial_j S^\dagger) S(\partial_k S^\dagger)\right] \\
&= -\frac{1}{24\pi^2} \oint_{S^3} d^3\theta_i \mathrm{Tr}\left[\epsilon^{ijk} (-i\sigma_i)(-i\sigma_j)(-i\sigma_k)\right] \\
&= -\frac{i}{24\pi^2} \oint_{S^3} d^3\theta_i 2i\epsilon^{ijk}\epsilon_{ijk} \\
&= \frac{6}{12\pi^2} 2\pi^2 = 1 \; .
\end{aligned} \tag{3.21}$$

The winding number for all other standard mappings in eq. 3.14, can be found by observing that for

$$S = S_1 S_2 \tag{3.22}$$

we get

$$n = n_1 + n_2 \ . \tag{3.23}$$

To sum up, for a gauge field theory based on $SU(2)$ every field configuration of finite action in four-dimensional Euclidean space has an integer called the winding number associated with it. As seen in eq. 3.20, this integer value is the difference between the Chern-Simons numbers of two static configurations in the distant (Euclidean) future and the distant past. Hence, field configurations $A_\mu(x_4, \vec{x})$ satisfying eq. 3.8 with a certain $S = S_n$ can be intepreted as trajectories that interpolate between the states with $N_{\rm CS}$ and $N_{\rm CS}+n$. This is often visualized by the periodic potential as seen in Fig. 3.1. It can be shown that $N_{\rm CS}$ takes integer values

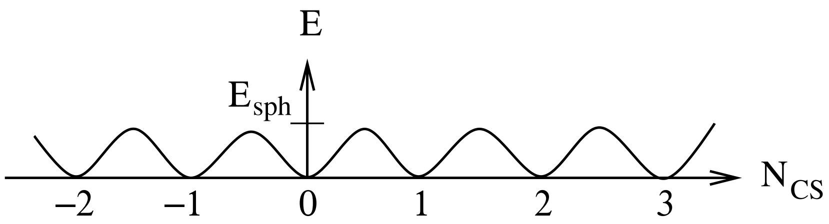


Figure 3.1: Structure of the gauge field configurations as a function of the Chern-Simons number $N_{\rm CS}$ [18].

only for configurations that are pure gauge transformations; these, being gauge-equivalent to zero, have vanishing field-strength and Hamiltonian, and therefore correspond to vacuum configurations with minimum energy and different winding number (topological vacua). A gauge transformation with winding number k, when applied to a field configuration with a given $N_{\rm CS} = n$, gives a new configuration of identical energy (as the latter is gauge invariant) and $N_{\rm CS} = k + n$. From this it follows that the energy is a periodic function of the Chern-Simons number. Finally, while $N_{\rm CS}$ is defined at constant time slices, the instanton solutions depend on time, and interpolate between static configurations with integer $N_{\rm CS}$, *i.e.* different topological vacua. So the instantons represent tunneling trajectories between energy minima.

We will see later that transitions between those vacua can be important in the context of baryon number violating processes used for example for baryogenesis models. A central point in this discussion will be to consider the top of the barrier between two such $n-$vacua states. These so called sphalerons wil be discussed in section 3.3.

3.1.3 Explicit Instanton Solution

Already without an explicit form of the field configuration A_μ, we can calculate the minimum of the Euclidean action in eq. 3.6. For positive values of the winding number in eq. 3.16, the action can be rewritten in the form

$$
\begin{aligned}
S &= \int d^4x \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} = \int d^4x \left[\frac{1}{4} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} + \frac{1}{8} (G^a_{\mu\nu} - \tilde{G}^a_{\mu\nu})^2 \right] \\
&= \frac{8\pi^2}{g^2} n + \frac{1}{8} \int d^4x (G^a_{\mu\nu} - \tilde{G}^a_{\mu\nu})^2
\end{aligned}
\tag{3.24}
$$

Therefore, for every positive n, the minimum of is attained for $G^a_{\mu\nu} = \tilde{G}^a_{\mu\nu}$, *i.e.* for fulfilling self-duality. In the following, we only consider the minimum solution for the case $n = 1$, the so called BPST (Belavin-Polyakov-Schwarz-Tyupkin) instanton solution [15]. In this case, the action is simply $S_0 = 8\pi^2/g^2$. For the BPST instanton, we insert eq. 3.13 into eq. 3.8 and use that $\tau^+_\mu \tau^-_\nu = \delta_{\mu\nu} + i\eta_{a\mu\nu}\sigma^a$ to get with

$$
A^a_\mu \xrightarrow[x\to\infty]{} \frac{2}{g} \eta_{a\mu\nu} \frac{x_\nu}{x^2}
\tag{3.25}
$$

the asymptotic behaviour of A^a_μ in terms of the 't Hooft symbols [19]

$$
\eta_{a\mu\nu} = \begin{cases} \epsilon_{aij} & \text{if } i,j = 1,2,3 \\ -\delta_{a\nu} & \text{if } \mu = 4 \\ \delta_{a\mu} & \text{if } \nu = 4 \\ 0 & \text{if } \mu = \nu = 4 \ . \end{cases}
\tag{3.26}
$$

Further explanations and identities of the 't Hooft symbols can be found in [19]. For a full solution in the whole x range, we assume the same angular dependence of the field for all x, *i.e.* we introduce a radial function $f(x^2)$ to obtain a solution of the form

$$
A^a_\mu(x) = \frac{2}{g} \eta_{a\mu\nu} x_\nu \frac{f(x^2)}{x^2}
\tag{3.27}
$$

for an instanton with its center at $x = 0$ where

$$
f(x^2) \xrightarrow[x^2\to\infty]{} 1, \quad f(x^2) \xrightarrow[x^2\to 0]{} \text{const} \times x^2 \ .
\tag{3.28}
$$

Plugging eq. 3.27 into the field strength tensor, we obtain

$$
G^a_{\mu\nu} = -\frac{4}{g} \left[\eta_{a\mu\nu} \frac{f(1-f)}{x^2} + \frac{x_\mu \eta_{a\nu\gamma} x_\gamma - x_\nu \eta_{a\mu\gamma} x_\gamma}{x^4} [f(1-f) - x^2 f'] \right]
\tag{3.29}
$$

where we use the 't Hooft relation $\epsilon_{abc}\eta_{b\mu\nu}\eta_{c\gamma\lambda} = \delta_{\mu\gamma}\eta_{a\nu\lambda} - \delta_{\mu\lambda}\eta_{a\nu\gamma} - \delta_{\nu\lambda}\eta_{a\mu\lambda} + \delta_{\nu\lambda}\eta_{a\mu\gamma}$ and f' denotes the differentation with respect to x^2. For the expression for the dual form $\tilde{G}^a_{\mu\nu}$, we use that $\epsilon_{\mu\nu\lambda\sigma}\eta_{a\gamma\sigma} = \delta_{\gamma\mu}\eta_{a\nu\lambda} - \delta_{\gamma\nu}\eta_{a\mu\lambda} + \delta_{\gamma\lambda}\eta_{\mu\nu}$ to obtain

$$\tilde{G}^a_{\mu\nu} = -\frac{4}{g}\left[\eta_{a\mu\nu}f' + \frac{x_\mu\eta_{a\nu\gamma}x_\gamma - x_\nu\eta_{a\mu\gamma}x_\gamma}{x^4}[f(1-f) - x^2 f']\right] \ . \tag{3.30}$$

For the condition of self-duality to be fulfilled to get a minimum action solution, the function $f(x^2)$ of the instanton solution has to fulfill $f(1-f) - x^2 f' = 0$. This yields to a function of the form

$$f(r^2) = \frac{x^2}{x^2 + \rho^2} \tag{3.31}$$

where ρ^2 is a constant of integration; ρ is called the instanton size, the instanton radius or the instanton scale, obviously with the dimension of a length. Finally, the final expression for an arbitrary center x_* of the instanton is

$$\begin{aligned} A^a_\mu &= \frac{2}{g}\eta_{a\mu\nu}\frac{(x-x_*)_\nu}{(x-x_*)^2 + \rho^2} \\ G^a_{\mu\nu} &= -\frac{4}{g}\eta_{a\mu\nu}\frac{\rho^2}{[(x-x_*)^2 + \rho^2]^2} \ . \end{aligned} \tag{3.32}$$

The parameters ρ and x_* are called collective coordinates. As we discussed in eq. 3.9, we are allowed to perform a gauge transformation of the form

$$\begin{aligned} g\frac{\tau^a}{2}\bar{A}^a_\mu &= U^+ g\frac{\tau^a}{2}A^a_\mu U + iU^+\partial_\mu U \ , \\ g\frac{\tau^a}{2}\bar{G}^a_{\mu\nu} &= U^+ g\frac{\tau^a}{2}G^a_{\mu\nu}U \end{aligned} \tag{3.33}$$

with

$$U = \frac{i\tau^+_\mu (x-x_*)_\mu}{\sqrt{(x-x_*)^2}} \tag{3.34}$$

and we obtain for the vector potential $\bar{A}^a_\mu$ and the field strength tensor $\bar{G}^a_{\mu\nu}$

$$\begin{aligned} \bar{A}^a_\mu &= \frac{2}{g}\bar{\eta}_{a\mu\nu}(x-x_*)_\nu\frac{\rho^2}{(x-x_*)^2[(x-x_*)^2+\rho^2]}, \\ \bar{G}^a_{\mu\nu} &= -\frac{8}{g}\left[\frac{(x-x_*)_\mu(x-x_*)_\rho}{(x-x_*)^2} - \frac{1}{4}\delta_{\mu\rho}\right]\bar{\eta}_{a\nu\rho}\frac{\rho^2}{[(x-x_*)^2+\rho^2]^2} - (\mu \leftrightarrow \nu) \ . \end{aligned} \tag{3.35}$$

This solution is called the instanton in the singular gauge since it has a singularity for $x \to x_*$ or likewise called the 't Hooft Ansatz being used in [19]. It becomes desirable to get gauge fields of this form. As we discussed before, considerations like in eq. 3.19 and 3.20 are only valid in a topological gauge in which $A_i \to 0$ faster than $1/r$ for $|\vec{x}| \to \infty$. This is not true for eq. 3.32 but holds for 3.35. Hence, for interpretating the instanton as interpolating between vacua, or associating vacua with N_{CS}, one has to consider such configurations. Another advantage of eq. 3.35 is that when studying the spectrum of fluctuations around an instanton background like one can see in the following, it is easier to work with a fast fall-off at infinity.

3.1.4 Zero Modes and Collective coordinates

As seen in eq. 3.32 and 3.35, the solution depends on the parameters x_* and ρ. These parameters are called collective coordinates and describe the instantons. Expanding the path integral around the instanton, we get $\exp(-S_0)\exp(-S')$ where the primed S' stands for fluctuations around instanton solution. Hence, we have an exponential suppression $\exp(-S_0) = \exp(-8\pi^2/g^2)$. The collective coordinates determine part of the pre-exponential structure of such considerations. Every collective coordinate is associated with a zero-mode of the gauge field operator mentioned further below and those zero-modes play a special role in calculating the instanton determinant as extensively discussed in [19]. This will be crucial in determining the instanton effective Lagrangian. ρ describes the scale of the instanton whereas x_* specifies its space-time location. In $SU(N)$, we have $4C_2(SU(N)) = 4N$ [20] collective coordinates where C_2 is the quadratic Casimir operator in the adjoint representation defined in eq. 2.27. So far, we have only found 5 collective coordinate for the $SU(2)$ instanton. So what are three remaining 3 ones for $SU(2)$? They are encoded in the 't Hooft symbols. Apart from dilatations (change in ρ) and translations through x_0, we have to take care of group specific transformations. In $SU(N)$ these are the global color rotations O_{ab} acting on the group index of the 't Hooft symbols in eq. 3.32 or eq. 3.35

$$\eta_{a\mu\nu} \to O_{ab}\eta_{b\mu\nu}, \quad \bar{\eta}_{a\mu\nu} \to O_{ab}\bar{\eta}_{b\mu\nu} \ . \tag{3.36}$$

Whereas the number of collective coordinates due to dilatations and translation is 5 for every gauge group, global color rotations increase with higher gauge groups. In $SU(2)$ we denote these rotations with three Eulerian angles θ, φ, ψ specifying the orientation of the instanton in color space.

In order to calculate the pre-exponential factor for instanton vacuum-vacuum transitions, we consider solutions around the extremum solution $A_\mu^{a,inst}$ of the instanton

$$A_\mu^a = A_\mu^{a,\mathrm{inst}} + a_\mu^a \ . \tag{3.37}$$

in the spirit of 't Hooft's famous paper [19]. By expanding the pure gauge action of eq. 3.7 with respect to a_μ^a, we get

$$\begin{aligned} S &= \frac{8\pi^2}{g^2} + \frac{1}{2}\int d^4x a_\mu^a \left[D^2 a_\mu^a - D_\mu D_\nu a_\nu^a + 2g\epsilon^{abc}G_{\mu\nu}^b a_\nu^c\right] \\ &= S_0 + \frac{1}{2}\int d^4x a_\mu^a L_{\mu\nu}^{ab}(A^{\mathrm{inst}})a_\nu^b \end{aligned} \tag{3.38}$$

where the instanton field $A_\mu^{a,\mathrm{inst}}$ is hidden in D_μ and $G_{\mu\nu}$. The operator $L_{\mu\nu}^{ab}$ contains these operators and hence, also depends on $A_\mu^{a,\text{inst}}$. From the form the action is written in eq. 3.38, we can see that calculating a vacuum-vacuum transition $\langle 0|0_T\rangle$ from t_0 to a time t_1 with $t_1 - t_0 = T$ reduces to the calculation of the determinant of the operator $L_{\mu\nu}^{ab}$. This can be done if the operator $L_{\mu\nu}^{ab}$ is invertible, *i.e.* if its kernel is non-trivial. However, for a field of the form $a_\mu^a = D_\mu \lambda^a$ the quadratic form of in eq. 3.38 vanishes. This is a consequence of gauge invariance and its treatment is discussed in the standard QFT literature, *e.g.* [11]. To work with these kind of integrals, we have to fix the gauge with a gauge fixing term

$$\Delta S_{\text{fix}} = \frac{1}{2}\int d^4x (D_\mu a_\mu^a)^2 = \frac{1}{2}\int d^4x a_\mu^a (\Delta\mathcal{L}_{\text{fix}})_{\mu\nu}^{ab} a_\nu^b \tag{3.39}$$

and introduce corresponding Faddeev-Popov ghost terms

$$\Delta S_{\text{gh}} = -\int d^4x \bar{\Phi}^a D^2 \Phi^a = \int d^4x \bar{\Phi}^a \mathcal{L}_{\text{gh}}^{ab} \Phi^b \tag{3.40}$$

to cancel unphysical polarizations of our Yang-Mills field. The complex and anticommuting fields Φ^a and $\bar{\Phi}^a$ are called Faddeev-Popov ghosts and anti-ghosts. Finally, the vacuum-vacuum transition amplitude can be written as

$$\langle 0|0_T\rangle_{\text{inst}} = [\det(\mathcal{L} + \Delta\mathcal{L}_{\text{fix}})]^{-1/2} \det(\mathcal{L}_{\text{gh}}) e^{-S_0} \ . \tag{3.41}$$

The first step in calculating amplitudes like eq. 3.41 is to regularize the expression. This is done by introducing a cut-off parameter μ for both the gauge and ghost field determinant and considering the ratio $\det(\mathcal{L} + \Delta\mathcal{L})/\det(\mathcal{L} + \Delta\mathcal{L} + M^2)$ instead of the operator $\det(\mathcal{L} + \Delta\mathcal{L})$

$$\langle 0|0_T\rangle_{\text{inst}}^{\text{Reg}} = \left[\frac{\det(\mathcal{L} + \Delta\mathcal{L})}{\det(\mathcal{L} + \Delta\mathcal{L} + M_0^2)}\right]^{-1/2} \frac{\det(\mathcal{L}_{\text{gh}})}{\det(\mathcal{L}_{\text{gh}} + M_0^2)} \exp(-S_0) \ . \tag{3.42}$$

Furthermore, all divergences have to be eliminated by renormalization of the coupling constant. It turns out that each zero mode leads in $[\det(\mathcal{L} + \Delta\mathcal{L})]^{-1/2}$ to a factor proportional to $\sqrt{S_0}$ [19, 17] plus an integral with respect to the corresponding collective coordinate. Since through regularization our amplitude is multiplied by $[\det(\mathcal{L}+\Delta\mathcal{L}+M_0^2)]^{1/2}$, every zero mode contributes to the amplitude with a factor M_0. As a result of these considerations, we get

$$\begin{aligned} \frac{\langle 0|0_T\rangle_{\text{inst}}^{\text{Reg}}}{\langle 0|0_T\rangle_{\text{p.th}}} &= \text{const'} \int d^4x_* d\rho \sin(\theta) d\theta d\psi d\varphi M_0^8 \sqrt{S_0}^8 \rho^3 \exp(-S_0) \\ &= \text{const} \int \frac{d^4x d\rho}{\rho^5} \left(\frac{8\pi^2}{g_0^2}\right)^4 \exp\left(-\frac{8\pi^2}{g^2} + 8\ln(M_0\rho) + \Phi_1\right) \end{aligned} \tag{3.43}$$

where $\langle 0|0_T\rangle_{\text{p.th}}$ is the corresponding perturbative case with $A_\mu^a = 0$ which cancels the ghost contribution. ρ^3 has been inserted on the basis of dimensional considerations and Φ_1 takes into account nonzero mode contributions. $g_0 \equiv g(M_0)$ is the bare coupling constant. Taking also into account nonzero modes up to one-loop corrections, we can rewrite the coupling in eq. 3.43 as

$$\frac{8\pi^2}{g^2(\rho)} = \frac{8\pi^2}{g_0^2} - 8\ln(\mu\rho) + \frac{2}{3}\ln(M_0\rho) = \frac{8\pi^2}{g_0^2} - \frac{22}{3}\ln(M_0\rho)\ . \tag{3.44}$$

Already here one can see that the coupling constant is the one we get in the procedure of renormalizing the coupling, *i.e.*

$$\frac{8\pi^2}{g^2(\rho)} = \frac{8\pi^2}{g^2(M_0)} - b\ln(M_0\rho), \tag{3.45}$$

where b is the one-loop beta function coefficient

$$\frac{d}{d\log\mu}\left(\frac{8\pi^2}{g^2(\mu)}\right) = b, \quad b = \frac{11}{3}C_2(r_g) - \frac{2}{3}\sum_{i=1}^{n_f} C(r_f) - \frac{1}{3}\sum_{i=1}^{n_s} C(r_s)\ . \tag{3.46}$$

C_2 and C have been defined in sec. 2.1.2. n_s and n_f stands for the number of complex scalars and fermions in a certain representation r_s and r_f respectively. The quadratic Casimir $C_2(r_g)$ for the gauge field representation $r_g = \text{adj.}$ is in the $SU(N)$ case N and therefore, we have

$$b = \frac{22}{3} \tag{3.47}$$

for the $SU(2)$ case without considering the fermion contribution. Together with 3.45, this is exactly what we've obtained in eq. 3.44. As a result, up to one-loop corrections the instanton contribution to the vacuum-vacuum transition for $SU(2)$ has the form

$$\begin{aligned}\frac{\langle 0|0_T\rangle^{\text{Reg}}_{\text{inst}}}{\langle 0|0_T\rangle_{\text{p.th}}} &= \text{const}\int \frac{d^4x d\rho}{\rho^5}\left[\frac{8\pi^2}{g^2(\rho)}\right]^4 \exp\left(-8\pi^2/g^2(\rho)\right) \\ &= \int \frac{d^4x d\rho}{\rho^5} d(\rho)\end{aligned} \tag{3.48}$$

with $g(\rho)$ given by eq. 3.45. Here, we have defined the instanton density

$$d(\rho) := \left[\frac{8\pi^2}{g^2(\rho)}\right]^4 \exp\left(-8\pi^2/g^2(\rho)\right) \tag{3.49}$$

3.2 Effective Instanton-Induced Lagrangian

3.2.1 Fermion Contribution

In addition to the factors obtained by pure gauge field dynamics, we can also take a look at the influence of fermion fields to the vacuum-vacuum transitions in the instanton background. Remember that we are still computing everything in Euclidean spacetime. Therefore, for the

Dirac Lagrangian we have to look at

$$\Delta S_{\text{Dirac}} = \int d^4x \bar{\psi}(-i\gamma_\mu D_\mu - im)\psi \ . \tag{3.50}$$

Like in the pure gauge field case, the determinant of the corresponding operator plays a big role. In case of Dirac fermions or equivalently pairs of Weyl fermions in a certain representation of the gauge group, the determinant of the Dirac operator $\det(-i\gamma_\mu D_\mu - im)$ is the central part of our calculations. Again, since the determinant of the operator is the product of its eigenvalues

$$\det(-i\gamma_\mu D_\mu - im) = \prod_n (\lambda_n - im), \tag{3.51}$$

the zero modes play a big part in determining the pre-exponential factors again. They are the modes with $\lambda_0 = 0$ and hence, the solution of

$$-i\gamma_\mu D_\mu u_0 = 0 \ . \tag{3.52}$$

Passing over to the two-component spinor notation χ_L, χ_R with

$$u_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \chi_L + \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \tau^+_\mu D_\mu \chi_L = 0, \quad \tau^-_\mu D_\mu \chi_R = 0, \tag{3.53}$$

we can find

$$-D^2_\mu \chi_L = 0, \quad -D^2_\mu \chi_R = -4\vec{\sigma}\vec{\tau}\frac{\rho^2}{[(x-x_*)^2+\rho^2]^2}\chi_R \tag{3.54}$$

for the squares of the Hermitian operator which is positive definite and therefore does not have vanishing eigenvalues. As a result, we get $\chi_L = 0$. In the above calculations, we have used the relations $\tau^+_\mu \tau^-_\nu = \delta_{\mu\nu} + i\eta_{a\mu\nu}\tau^a$ and $\tau^-_\mu \tau^+_\nu = \delta_{\mu\nu} + i\bar{\eta}_{a\mu\nu}\tau^a$, as well as the commutator relation $[D_\mu, D_\nu] = -(ig/2)\tau^a G^a_{\mu\nu}$ and the gauge field instanton solution of eq. 3.32. From that form, we can find the final solution for the zero mode normalized by $\int u^+ u dx = 1$

$$u_0(x) = \frac{1}{\pi}\frac{\rho}{((x-x_*)^2+\rho^2)^{3/2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\varphi, \quad \varphi^{\alpha m} = \frac{1}{\sqrt{2}}\epsilon^{\alpha m} \tag{3.55}$$

where $\alpha = 1,2$ and $m = 1,2$ are the spin and color indices, respectively and φ is satisfying $(\vec{\sigma} + \vec{\tau})\varphi = 0$. The corresponding solution in the singular gauge of eq. 3.35 is

$$u_0^{\text{sing}}(x) = \frac{1}{\pi}\frac{\rho}{((x-x_0)^2+\rho^2)^{3/2}}\frac{(x-x_0)_\mu\gamma_\mu}{\sqrt{x^2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\varphi \ . \tag{3.56}$$

Finally, in the Dirac case the contribution of the lowest modes to the determinant is simply a factor of m. Hence, by dimensional arguments, we get an additional factor of $m\rho$ for the instanton amplitude. For the determinant $\det(-i\gamma_\mu D_\mu - im)'$ omitting the zero-mode contribution (denoted by a prime), we get [19]

$$\det(-i\gamma_\mu D_\mu - im)' = \exp\left(-\frac{2}{3}\sum_{i=1}^{n_f} C(r_f)\ln(M_0\rho)\right) \tag{3.57}$$

which captures the leading log corrections and matches the well-known Callan-Symanzik one-loop beta function b in eq. 3.46 again of course.

With this knowledge, we are ready to calculate the instanton-induced effective Lagrangian for light fermions following Shifman *et al* [21]. In the following calculations, we consider the instanton solutions in the singular gauge, see eq. 3.35 for reasons mentioned in sec. 3.1.3. The effect of fermions can be captured by considering processes involving excitations around a single topological vacuum treated with the usual perturbative procedure of path integral calculations near a trivial background plus including contributions from the saddle-point expanded fields representing processes involving transitions between vacua. By assuming small-size instantons $\rho \ll R$ where R is the confinement radius, the contribution to the Green functions of the theory due to a single instanton centered at x_* can be mimicked with an effective Lagrangian of the form

$$\Delta\mathcal{L} = \frac{d\rho}{\rho^5}\sum_n C_n(\rho)O(x_*) \tag{3.58}$$

where the contribution of rapidly varying fields at the scale ρ is included in the coefficient $C_n(\rho)$ of the local operators that act in the space of the the slowly varying fields (dominant in the vacuum state). We start with one light quark q and extend this later to the multi-quark case. Furthermore, we only take into account the first two terms of the effective Lagrangian in eq. 3.58. The first term of the sum includes the unit operator $O_1 \equiv \mathbb{I}$, and the coefficient is exactly everything we have considered so far, namely the instanton density $d(\rho)$ and the mass insertion $(m\rho)$ due to the fermion zero modes

$$C_1\mathbb{I} = \text{const}\left[\frac{8\pi^2}{g^2(\rho)}\right]^4 e^{-8\pi^2/g^2(\rho)}(m_q\rho) = d(\rho)(m_q\rho)\ . \tag{3.59}$$

The next term contribution of the effective Lagrangian 3.58

$$C_2(\rho)\bar{q}(x_*)\Gamma q(x_*) \tag{3.60}$$

is of dimension 3 and includes the quark field operator $q(x_*)$ and a matrix Γ acting on the spin and color space. The coefficient C_2 can be extracted from calculations of the scattering amplitude of a quark with momentum p into a quark with momentum p', where $p, p' \ll \rho^{-1}$, on an instanton fluctuation with fixed center x_* and radius ρ

$$\langle p'|\Delta\mathcal{L}|p\rangle = \frac{d\rho}{\rho^5} C_q(\rho)\bar{v}(p')\Gamma v(p) e^{-ipx_* + ip'x_*} \tag{3.61}$$

where v is the spinor corresponding to the Lorentz and color state of the quark considered. As described in [21], for calculating the amplitude with the reduction formula

$$\begin{aligned}\langle p'|\Delta\mathcal{L}|p\rangle = i^2 \int dx dx' e^{ip'x' - ipx} \bar{v}^m_\alpha(p')(\hat{p}')_{\alpha\gamma} \\ \times \langle 0|T\{q^m_\gamma(x')\bar{q}^k_\beta(x)\}|0\rangle(\hat{p})_{\beta\delta} v^k_\delta(p)\end{aligned} \tag{3.62}$$

where q^m_γ represents the Dirac field of the m-th flavour, with Dirac index γ, we have to evaluate the quark Green function in the Euclidean spacetime. The Green'sr function is the inverse of the propagator in the instanton background, which can be expressed in terms of eigenfunctions $u_{(n)}$ of the Dirac operator in the background:

$$\begin{aligned}&\langle 0|T\{q^m_\gamma(x')\bar{q}^k_\beta(x)\}|0\rangle \\ &\xrightarrow[x_0 \to ix_0]{} \sum_n u^m_{(n)\gamma}(x') u^k_{(n)\beta}(x)(m - i\lambda_n)^{-1} C_1(\rho)\frac{d\rho}{\rho^5}\ .\end{aligned} \tag{3.63}$$

where the functions $u_{(n)}$ are the eigenfunctions of the Dirac operator 3.50 with the eigenvalues λ_n. For small masses, the sum in (3.63) is clearly dominated by the zero-mode contribution with $\epsilon_0 = 0$! By only considering the 't Hooft zero mode in eq. 3.56 we can find

$$\begin{aligned}&C_2(\rho)e^{-ip'x_* + ipx_*}\bar{v}\Gamma v \\ &= \int dx dx' e^{-ip'x' + ipx} \frac{C_1(\rho)}{m}(\bar{v}\hat{p}' u_0(x' - x_*))(u_0^+(x - x_*)\hat{p}v)\end{aligned} \tag{3.64}$$

where $\hat{p} = p_0\gamma_0 + \vec{p}\vec{\gamma}$. After performing the Fourier transformation

$$\int dx e^{-ipx}\psi_{(0)}(x) \xrightarrow[p\rho \ll 1]{} -2\pi i\rho \frac{1}{\hat{p}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \varphi \tag{3.65}$$

we finally arrive at

$$\begin{aligned}C_2(\rho)e^{-ip'x_* + ipx}\bar{v}\Gamma v &= \frac{C_1(\rho)}{m} 8\pi^2\rho^2 \bar{v}\varphi\varphi^\dagger v \\ &= d(\rho)8\pi^2\rho^3\bar{v}\varphi\varphi^\dagger v\end{aligned} \tag{3.66}$$

Now, we are left with calculating the density matrix $P_{mn,\gamma\delta} = (\varphi)_{m\gamma}(\varphi^\dagger)_{n\delta} = \frac{1}{4}\left[I_{mn}I^c_{\gamma\delta} - \sigma_{mn}\tau_{\gamma\delta}\right]$ of the state φ. As described in [21], averaging over the instanton orientation in the color space, and averaging over the possible ways of embedding a particular $SU(2)$ group into the $SU(3)$

yields $\langle P\rangle_{SU(3)} = \frac{2}{3}\frac{1}{4} I \times (I^c)_{SU(3)}$. Finally, we get

$$\langle 0|\Delta L|0\rangle = \frac{d\rho}{\rho^5} d(\rho) \left[m_q \rho - \frac{2}{3}\pi^2 \langle 0|[\bar{q}(1+\gamma_5)q]_\rho|0\rangle \rho^3 \right] \tag{3.67}$$

for the instanton density only including the contribution of a single light quark. Since we only consider $SU(2)$ color, the last step of embedding it into $SU(3)$ doesn't have to be done and no additional factor 2/3 is included in the second part of the expression:

$$\Delta\mathcal{L} = \frac{d\rho}{\rho^5} d(\rho)(m\rho - 2\pi^2\rho^3 \bar{q}_R q_L) \ . \tag{3.68}$$

Eq. 3.68 already tells us something about the form of the total effective Lagrangian with n_f fermions. We can see that a theory with N flavours, meaning $2N$ pairs of Weyl fermions charged under the gauge group, we have vertices with $2N$ and less fermions. For less fermions, every pair of Weyl fermions is substituted by a mass insertion of the form $m\rho$. Besides, a pair of Weyl fermions always comes with a factor of $(2\pi^2\rho^3)$. Considering all that, the Lagrangian is schematically of the form

$$\Delta L \supset \sum_{k=0}^{N} \int \frac{d\rho}{\rho^5} d(\rho)(2\pi^2\rho^3)^k \, \rho^{(N-k)} \times \Pi^{N-k}_{(a,b),a\neq b} m_{ab} \times \Pi^{k}_{(m,n),m\neq n} \psi_m \psi_n, \quad \rho m_{ab} \ll 1. \tag{3.69}$$

Above, $(\Pi^{N-k}_{(a,b),a\neq b})$ denotes a product over $N-k$ distinct pairs of indices (a,b) (with no index repetition within each pair), labelling different Weyl spinors. m_{ab} denotes a mass term connecting Weyl spinors a, b. In the SM with 12 left-handed Weyl fermions [1], *i.e.* 6 "flavours" however, gauge symmetry forbids that we have mass terms pairing up the left-handed fermions. For that reason, we don't have instanton effective Lagrangians with less than twelve chiral fermions. In the regime $m_{\rm SM}\rho < 1$, the Lagrangian with light SM fermions finally looks like

$$\Delta L^{\rm SM} \sim \int_0^{1/v} \frac{d\rho}{\rho^5} d(\rho) \left[(2\pi^2\rho^3)^6 \ \times \Pi^{12}_{m=1} \psi^{SM}_m \right] \tag{3.70}$$

The reason why we chose a cut-off for large $\rho < 1/v$ by the Higgs vev will be discussed in the following.

3.2.2 Instantons in the Higgs Regime

The ρ integral in eq. 3.70 converges for $\rho \to 0$ but diverges for $\rho \to \infty$. Therefore, to calculate effective couplings, we have to consider an IR-cutoff, given *e.g.* by the Higgs vev v. Whereas in unbroken gauge theories like QCD a IR-divergence is part of the theory, we don't expect such a divergence in the weak sector. In the low energy regime, the electrowak sector is determined

[1] We will see below why we consider exactly 12 left-handed Weyl fermions in the SM

by spontaneous symmetry breaking as discussed in sec. 2.2. So we expect the Higgs vev to control this divergence. It turns out that the Higgs cutoff $1/v$ is reasonable since instantons larger than than v are suppressed and its contribution to the effective coupling is negligible. In other words, in $SU(2)$ the small-size instanton approximation $\rho m \ll 1$ always holds for weak phenomena.

This can be easily justified by calculations [19, 22]. In order to do that, consider the Euclidean Higgs field Lagrangian [23]

$$\Delta\mathcal{L}_H = D_\mu\phi^\dagger D_\mu\phi + \lambda(\phi^\dagger\phi - \frac{1}{2}v^2)^2 \tag{3.71}$$

where the Higgs vev is $v = 246$ GeV. We follow [17] to derive the most basic results in a heuristic way. For simplicity, we set $\lambda \to 0$ such that the Euler-Lagrange equations for the scalar field are purely determined from the kinetic term in the Lagrangian

$$D^2\phi = 0 \ . \tag{3.72}$$

With the familiar instanton solution in pure gauge theory written as

$$A_\mu = \frac{ix^2}{x^2+\rho^2}S_1\partial_\mu S_1^\dagger \tag{3.73}$$

we find

$$\phi = \frac{v}{\sqrt{2}}\left(\frac{x^2}{x^2+\rho^2}\right)^{1/2} S_1 \tag{3.74}$$

for the solution of eq. 3.72 where we set $x_* = 0$ for simplicity. S_1 is the mapping introduced in 3.13. We can see that the Higgs vev is suppressed for $x \to 0$. For large x, *i.e.* in the low energy regime, the Higgs field converges towards $v/\sqrt{2}$ as expected. The action including the kinetic term in the Lagrangian can now be calculated as [19, 17]

$$\Delta S_H = \int d^4x\partial_\mu(\phi^\dagger D_\mu\phi) = -\pi^2v^2\rho^2 \ . \tag{3.75}$$

Finally, the extra term in the instanton density due to a nonvanishing vacuum expectation value of the Higgs has the form

$$\exp(-S_H) = \exp(-\pi^2v^2\rho^2) \ . \tag{3.76}$$

This damping term in the integral over ρ controls the IR divergence and shows explicitly that large ρ values are suppressed and only instantons with $\rho v \le 1$ contribute to the effective coupling of the Lagrangian. Therefore, the only valid value range for ρ is below $1/v$ and we are allowed to set a cutoff $1/v$.

3.3 Sphalerons

3.3.1 The Barrier Height of the Sphaleron and Unsuppressed Transitions

So far, we have only talked about the minima of the periodic potential of Fig. 3.1. We labeled the vacua with integer Chern-Simons numbers N_{CS} and identified the instanton solutions to be trajectories tunneling between such vacua through a barrier. The question is how high the minimum value of the barrier is. In order to find that out, we try to find a solutions for the top of the barrier, *i.e.* solutions of the static equations of motion for the full Lagrangian including gauge and Higgs fields, since the position at the top of the barrier is an unstable equilibrium configuration. Since sphalerons have the minimum possible energy barrier, they correspond to saddle points of the energy functional. In the $A_0 = 0$ gauge with $A_i \to 0$ fast enough, a reasonable ansatz for these so called sphaleron solutions is

$$A_i^a = \frac{1}{g}\epsilon_{iak}\frac{x_k}{r}f(r), \quad \phi = \frac{\vec{\sigma}\vec{x}}{r}h(r) \tag{3.77}$$

where $f(r) \to -2/r$ and $h \to v\sqrt{2}$ for $r \to \infty$ with $r = \sqrt{x^2}$ and $f, h \to 0$ for $r \to 0$ to avoid singularities. The condition for A_i is necessary to ensure that it's pure gauge at infinty. In order to find a maximum energy solution, it is sufficient to write down the energy functional, substitute the ansatz 3.77 into it and finally extremize it by solving the equations of motion. With eq. 3.77 substituted in the Lagrangian of the theory, we obtain

$$\mathcal{H} = 4\pi \int_0^\infty r^2 dr \left(\frac{1}{g^2}\left[f'^2 + \frac{2}{r^2}f^2 + \frac{2}{r}f^3 + \frac{1}{2}f^4 \right] + h'^2 + 2h^2\left[\frac{1}{r} + \frac{f}{2}\right]^2 \right) . \tag{3.78}$$

A detailed numerical calculation can be found in [24]. Here, we only consider an approximate estimate to get the right range. By rescaling the fields and r to dimensionless quantities

$$f = gvF, \quad h = vH, \quad r = R(gv)^{-1} \tag{3.79}$$

we can right the rescaled energy functional as

$$\mathcal{H} = 4\pi\frac{v}{g} \int_0^\infty R^2 dR \left(\left[F'^2 + \frac{2}{R^2}F^2 + \frac{2}{R}F^3 + \frac{1}{2}F^4 \right] + H'^2 + 2H^2\left[\frac{1}{R} + \frac{F}{2}\right]^2 \right), \tag{3.80}$$

where the prime denotes differentiation over R. Since for $R \to \infty$, we have $H \to 1$ and $F \to -2/R$ and for $R \to 0$ vanishing H and F, the integral in eq. 3.80 should be of order one and the barrier sphaleron energy E_{sph} is in the range [24]

$$E_{\mathrm{sph}} = \mathcal{O}(1) \times \frac{4\pi v}{g} \sim 7 - 14\mathrm{TeV} . \tag{3.81}$$

More precise calculations [25] predict an energy of $E_{\text{sph}} = 9$ TeV. Note also that the solution given in eq. 3.77 gives exactly the Chern-Simons number $N_{\text{CS}}(A_{\text{sph}}) = 1/2$, *i.e.* the top of the barrier lies just symmetrically between the two minima $N_{\text{CS}} = 0$ and $N_{\text{CS}} = 1$ like depicted in Fig. 3.1.

Whereas one might think that transitions are still suppressed by $\exp(-8\pi^2/g^2)$, recent developments have argued that processes potentially become unsuppressed somewhere above the sphaleron energy. This was discussed in detail in [25] by modelling sphaleron transitions by a one-dimensional Schrödinger equation. The Bloch wave function approach leads to a band structure taking into account the periodicity of the Chern-Simons potential, which finally results in a more and more unsuppressed transition fully disappearing above the sphaleron energy. Hence, it might be possible to observe such so called sphaleron processes at colliders providing the required energies in the $\mathcal{O}(10)$ TeV range [26]. Discovering such processes would be a remarkable confirmation of really theoretical considerations and would also be important for cosmology since sphaleron processes play an essential role in creating the baryon asymmetry of the Universe. In which way these processes violate baryon number is discussed in the following section.

3.3.2 Baryon Number Violation in the SM and the Chiral Anomaly

The topological structure of $SU(2)$ we investigated in the last sections is not a pure theoretical consideration only but can be applied to cosmology to describe the baryon asymmetry in the Universe. We will see that a change in the Chern-Simons number, *i.e.* jumping from one vacuum to another, implies a violation of lepton and baryon number. This is based on an anomaly of a symmetry which is conserved at the classical level but is broken at quantum level. We distinguish between Abelian and non-Abelian anomalies discussed by Adler, Bell and Jackiw [27] and Bardeen, Gross and Jackiw [28, 29]. Although one might think that it's the non-Abelian anomaly one has to consider for the fermion number violation in the electroweak sector, it's in fact the Abelian anomaly $U_{\text{A}}(1)$ which is responsible for baryon and lepton number violation in the SM as found by 't Hooft [30]. The fact that only left-handed fermions of the SM couple to the gauge field plays an essential role here. The anomaly is based on the covariant derivative of the fermion field in $SU(2)$ like in eq. 2.70

$$\mathcal{L}_{\text{ABJ}} = \bar{\psi} i \gamma_\mu D_\mu \psi, \quad \text{with} \quad D_\mu = \partial_\mu - \frac{i}{2} g A_\mu^a \sigma^a \ . \tag{3.82}$$

The Lagrangian is invariant under the global chiral transformation

$$\psi \to e^{i\gamma_5 \alpha} \psi \tag{3.83}$$

and consequently the corresponding axial current

$$J_{5\mu} := \bar{\psi}\gamma_\mu\gamma_5\psi \tag{3.84}$$

is conserved on the classical level $\partial_\mu J_{5\mu} = 0$. Nevertheless, on the quantum level, the axial current is anomalous,

$$\partial_\mu J_{5\mu} = \frac{g^2}{16\pi^2} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} \ . \tag{3.85}$$

Finally, in the instanton background, the chiral charge Q_5 is violated by

$$\begin{aligned}\Delta Q_5 = \int J_{50} d^3x &= \int_{-\infty}^{\infty} dt \partial_0 \int J_{50} \\ &= \int_{-\infty}^{\infty} dt \int \partial_\mu J_{5\mu} d^3x = \frac{g^2}{16\pi^2} \int d^4x G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} \\ &= 2n \sum_r n(r) C(r)\end{aligned} \tag{3.86}$$

using the winding number n from 3.16 and $C(r)$ from 2.22. Again, we consider a fast falloff of A_i for $\vec{x} \to \infty$. $n(r)$ stands for the number of fermions in the representation r. In the SM, we have three lepton doublets in the fundamental representation with $C(f) = 1/2$ and 3×3 doublets of quarks in the fundamental since $SU(2)$ counts all quark color flavour doublets to be distinct doublets of a representation. Hence, in the SM we get

$$\Delta Q_5 = 12 \ . \tag{3.87}$$

for the vacuum-vacuum transition with $n = 1$. As a result, in order to break the chiral charge by 12 units, the corresponding vertex of the process has to include 12 left-handed $SU(2)$ Weyl fermions. However, this is not what we mean by baryon and lepton number violation. In that case, we have to consider a second anomalous contribution. It comes from the anomaly of the vector current $\bar{\psi}\gamma_\mu\psi$ based on global $U(1)$ transformations of the form mentioned in sec. 2.1.3. One way to see this is considering fermion triangle diagrams with a global current $J^{B,L}_\mu$ and two $SU(2)$ gauge currents at the three vertices. Calculating such diagrams yields the electroweak $B + L$ anomaly

$$\begin{aligned}\partial_\mu J^{B+L}_\mu &= \frac{N_B g^2}{32\pi^2} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} + \frac{N_L g^2}{32\pi^2} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} \\ &= \frac{3g^2}{16\pi^2} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}\end{aligned} \tag{3.88}$$

where $N_B = N_L$ stands for the number of left-handed $SU(2)$ doublets charged under the global $U(1)_{B,L}$. This results in a $B + L$ violation of

$$\Delta(B + L) = 6n \tag{3.89}$$

and in case of the instanton with $n = 1$ the vertex of the process violates the baryon and lepton number by 6 units. The triangle diagrams mentioned above are proportional to the trace

$$\mathrm{Tr}\left[T_r^a T_r^b Q_x\right] = C(r)\delta^{ab}\sum_i Q_{x,i} \tag{3.90}$$

where $Q_{x,i}$ is the global charge of the i-th particle which is in our case the global B/L. In the SM, all left-handed fermion doublets with same sign charges contribute to the sum in 3.90 and with a non-vanishing sum $B+L$ is broken by sphalerons/instantons. Instead, if we consider vector fermions $X, \bar{X}$ in the fundamental representation of $SU(2)$ with opposite global charge, we see that their contribution cancels and hence, they don't contribute to the $B+L$ violating electroweak anomaly. But they still couple to the sphalerons via the chiral current! As a result, the chiral anomaly determines the number of fermions of our sphaleron vertex and the electroweak anomaly determines the $B+L$ violation of that vertex. Hence, we have a recipe for constructing the sphaleron vertex:

1. The $B+L$ anomaly gives us the kind of $B+L$ violation of our vertex. The SM contributes to it with six units where we break baryon and lepton number equally by $\Delta B = \Delta L = 3$ units.

2. The chiral current fixes the number of left-handed Weyl fermions involved in the sphaleron vertex. Since the SM provides 12 Weyl fermions in the fundamental representation with $C(f) = 1/2$, we have $\Delta Q_5 = 12$.

3. electric charge conservation (or charge neutrality)

4. $SU(3)$ invariance.

For the SM vertex with 12 left-handed Weyl fermions breaking $B+L$ by six units, we get

$$\mathcal{O}_{\mathrm{SM}} = \prod_{i=1,2,3} (u_L d_L d_L v_L)_i \tag{3.91}$$

where u_L and d_L are left-handed up- and down-type quarks, respectively and v_L are left-handed neutrinos. A typical illustration of the process is shown in Fig. 3.2.

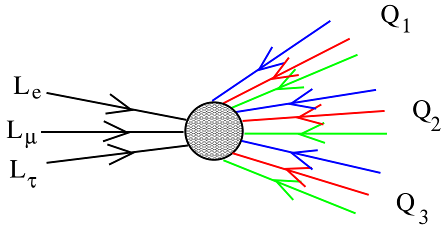


Figure 3.2: Illustrative picture of the sphaleron process [18]

4 Beyond the Standard Model Particles in Sphaleron Vertices

4.1 Rate Comparisons of SM and BSM Vertices

Goal of the thesis is to compare the usual SM sphaleron vertex to certain BSM scenarios, in other words to compare the rates of SM and BSM sphaleron processes on an order of magnitude basis. The rate of a process Γ is proportional to its matrix element squared $|\mathcal{M}|^2$ and its phase space PS, *i.e.* $\Gamma \propto |\mathcal{M}|^2 PS$. Thus, our order of magnitude estimates are of the form

$$\frac{\Gamma_{\mathrm{BSM}}}{\Gamma_{\mathrm{SM}}} \sim \frac{|\mathcal{M}_{\mathrm{BSM}}|^2}{|\mathcal{M}_{\mathrm{SM}}|^2} \frac{PS_{\mathrm{BSM}}}{PS_{\mathrm{SM}}} . \tag{4.1}$$

Energy and momentum dependent factors of the amplitude are absorbed in the expression of the phase. If we compare rates with the same particle content, the phase space is the same and the rate ratio comparison reduces to a comparison of the matrix elements squared. This is the case for the comparison of the SM vertex and the mass-insertion SM-like vertex. The matrix element can be constructed from the effective Lagrangians and its dimension and order of magnitude is determined from the number of particles included and from the effective coupling of the interaction. In our considerations, the effective coupling is calculated by the ρ-integral in the effective Lagrangian. Fermions contribute a product of spinor with mass dimension one to the matrix element. In a vertex with $n_f \geq 10$ fermions, we usually have to consider an average over the initial and final state projections of color and charge states. Since, we are only interested in the order of magnitude, we ignore this $\mathcal{O}(1)$ factor contribution and assume a fermion spinor contribution of $|E_i|$ with E being the energy of the i-th fermion in the final state. In case of additional W and Higgs bosons in the vertex and using the techniques of [23], we have to take into account an additional contribution to the ρ integral of

$$(\sqrt{2}\pi^2\rho^2 h)^{n_h} \times \left(\frac{4\pi^2\rho^2}{g}\right)^{n_W} \times \Pi^{n_W}_{(a,\mu)}[-\eta_{a\mu\nu}\partial^\nu W^a_\mu] \tag{4.2}$$

where h stands for the Higgs field contribution and W^a_μ is the W boson field. Whereas the Higgs field gives some factors of ρ and v, the W boson contributes with a factor

$$f_W(\vec{p}) = \frac{1}{m_W^2} \sum_{\mathrm{pol}} \eta_{a\mu\nu}\eta_{a\rho\sigma}\epsilon_\mu k_\nu \epsilon^*_\rho k^*_\sigma \tag{4.3}$$

consisting of polarization vectors and momenta in Euclidean spacetime where $\epsilon_\mu = (-i\epsilon^0, \epsilon^i)$ and $k_\mu = (iE_W, p_i)$. With $k_\mu = (iE_W, 0, 0, k_z)$ in the z-direction, the explicit polarizations are

$$\epsilon^1_\mu = (0,1,0,0), \quad \epsilon^2_\mu = (0,0,1,0), \quad \epsilon^L_\mu = (\frac{-ik_z}{m_W}, 0, 0, \frac{E_W}{m_W}) \ . \tag{4.4}$$

Finally, using

$$\eta_{a\mu\nu}\eta_{a\rho\sigma} = g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma}. \tag{4.5}$$

we obtain

$$f_W(\vec{p}) = \frac{4E_W^2 - m_W^2}{m_W^2} \tag{4.6}$$

which coincides with [31] and the factor will finally also be absorbed in the phase space calculation. The final phase space term for n_f fermions, n_W W bosons and n_h Higgs bosons can be written as

$$PS[n_f, n_h, n_W] = \int \left(\Pi_f \frac{d^3p_f}{2(2\pi)^3}\right)\left(\Pi_h \frac{d^3p_h}{2(2\pi)^3 E_h}\right)\left(\Pi_W \frac{d^3p_W}{2(2\pi)^3 E_W} f_W(\mathbf{p})\right), \tag{4.7}$$

$$f_W(\mathbf{p}) = \frac{4E_W^2 - m_W^2}{m_W^2} = \frac{3E_W^2 + |\mathbf{p}|^2}{m_W^2}. \tag{4.8}$$

We calculate the phase space contribution numerically following [32]. This gives us a flat phase space for massless particles and an approximately flat phase space for massive particles with masses much smaller thant the momentum scale of the process. This is given in our case since we consider energies of several TeV compared to particles in the GeV range up to $\mathcal{O}(10^2)$ GeV. The phase space volume for n massless particles is given by

$$V_n = \frac{1}{(2\pi)^{3n-4}} \left(\frac{\pi}{2}\right)^{n-1} \frac{(Q^2)^{n-2}}{(n-1)!(n-2)!} \tag{4.9}$$

which reproduces exactly the $n = 2$ case of eq. 2.114. $Q = \sqrt{s}$ is the total momentum of all final particles, or likewise the center of mass energy $\sqrt{s}$ of the incoming particles. The correction factors for the massive case can be found in [32]. The algorithm described in [32] not only gives us the phase space volume but also the single momenta of the final particles and hence, we can calculate the full phase space integral in the Monte Carlo way by replacing the integration over the energies by a number of random choices of the energies with the flat weight given above, corrected by the function $f_W(\mathbf{p})$. The processes we will look at in the following, are $qq \to$ final states collisions to link it to hadron collider physics. The final states depend on the model we look at, either pure SM sphaleron processes with 10 final SM fermions or processes including additional new particles.

4.2 Construction of Vertices and Rate Estimates

4.2.1 BSM Particles in the Low Mass Regime

Everything we dealt with so far in terms of the sphaleron effective Lagrangian holds for the case $m\rho < 1$ and it's straightforward to include BSM left-handed Weyl fermions with $m\rho < 1$ to the sphaleron vertex. For example, we can extend the SM with a Dirac fermion consisting of 2 left-handed $SU(2)$ charged Weyl fermions $\psi, \tilde{\psi}$ in the fundamental representation with $C(f) = 1/2$ following eq. 3.69. In that case the chiral charge of eq. 3.86 is violated by

$$\Delta Q_5 = 2 \sum_r n(r) C(r) = 2 \cdot 14 \cdot \frac{1}{2} = 14 \tag{4.10}$$

and the vertex consists of two additional particles. The $B + L$ violation remains the same since Dirac fermion charges cancel each other like discussed before. As a result, the vertex looks like

$$\mathcal{O}_{\text{SM+vector}} = \prod_{i=1,2,3} (u_L d_L d_L v_L)_i \tilde{\psi} \psi \ . \tag{4.11}$$

For the effective Lagrangian, we get

$$\begin{aligned} \Delta\mathcal{L} &= \int_0^{1/v} \frac{d\rho}{\rho^5} d(\rho) (2\pi^2\rho^3)^6 (\rho m_\psi) \Pi_{m=1}^{12} \psi_m^{SM} \exp(-\pi^2\rho^2 v^2) \quad \text{SM-like} \\ \Delta\mathcal{L} &= \int_0^{1/v} \frac{d\rho}{\rho^5} d(\rho) (2\pi^2\rho^3)^7 \Pi_{m=1}^{12} \psi_m^{SM} \tilde{\psi}\psi \exp(-\pi^2\rho^2 v^2) \quad \text{BSM vertex} \end{aligned} \tag{4.12}$$

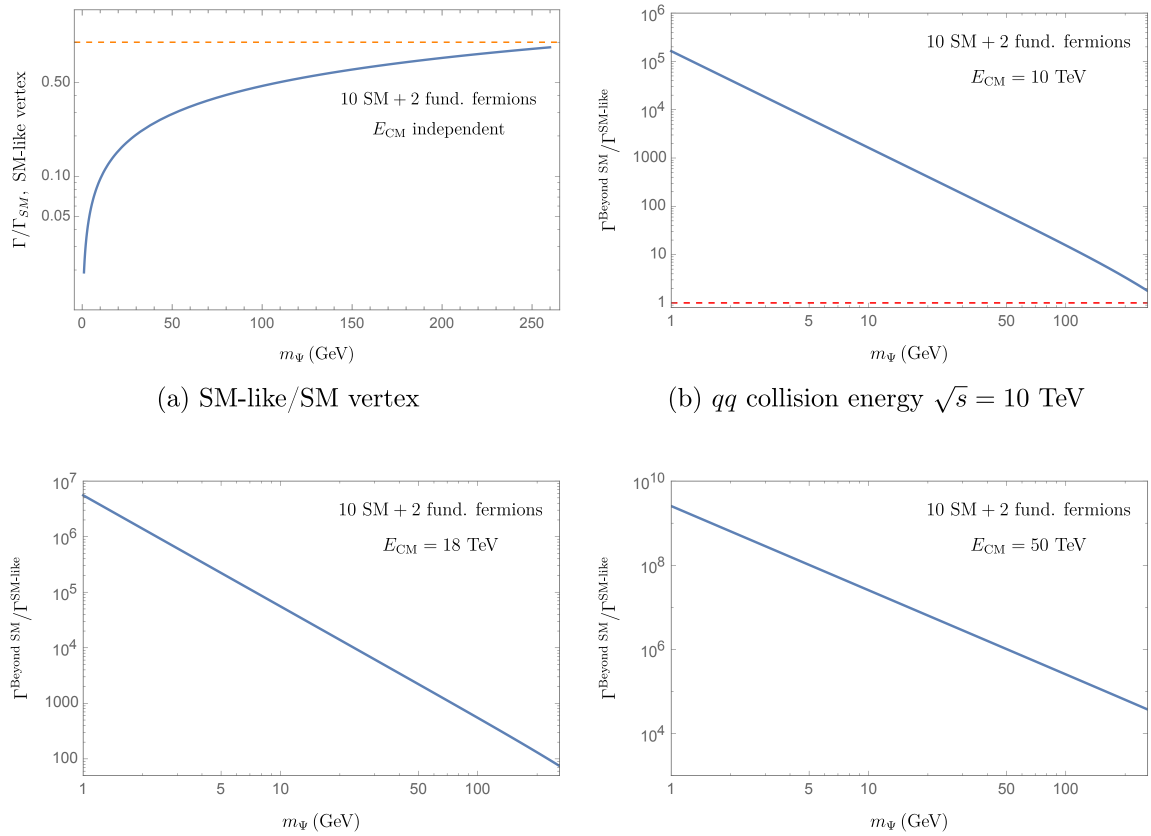
where we considered both the mass insertion vertex reproducing the SM case and the BSM vertex with new particles and adjusted the beta function of 3.45 to the considered amount of fermions n_f. That changes the instanton density $d(\rho)$ and the full effective Lagrangian gets another mass term of $(m\rho)^{-2/3}$. One might ask why we are allowed to consider a vertex with less fermions than the chiral anomaly told us to include? Weyl fermions have a chiral charge of $+1$, so an instanton vertex with $2N$ Weyl fermions violates the chiral charge by $2N$ units. When mass terms are present, the chiral symmetry is lost, but one can still treat the fermion masses as "spurion" fields with chiral charge -2. Hence, one can replace pairs of Weyl fermions by a mass term and still respect the chiral charge in the end. Combined with the corresponding phase space integral, we obtain for the SM-like ratio

$$\frac{\Gamma_{\text{SM-like}}}{\Gamma_{\text{SM}}} \propto \frac{\int_0^{1/v} \frac{d\rho}{\rho^5} d(\rho) (2\pi^2\rho^3)^6 (\rho m_\psi) \exp(-\pi^2\rho^2 v^2)}{\int_0^{1/v} \frac{d\rho}{\rho^5} d(\rho) (2\pi^2\rho^3)^6} \frac{PS[10,0,0]}{PS[10,0,0]} \tag{4.13}$$

and for the BSM/SM-like ratio

$$\frac{\Gamma_{\text{BSM}}}{\Gamma_{\text{SM-like}}} \propto \frac{\int_0^{1/v} \frac{d\rho}{\rho^5} d(\rho)(2\pi^2\rho^3)^7}{\int_0^{1/v} \frac{d\rho}{\rho^5} d(\rho)(2\pi^2\rho^3)^6} \frac{PS[12,0,0;m_\psi]}{PS[10,0,0]} \tag{4.14}$$

in the low mass regime $m_\psi \rho < 1$ of two new BSM left-handed $SU(2)$ Weyl fermions in the fundamental representation where the PS calculations also considers the mass m_ψ of the new fermions. The results are shown in Fig. 4.1. One can see that the mass of the new fermions m_ψ



(a) SM-like/SM vertex

(b) qq collision energy $\sqrt{s} = 10$ TeV

(c) qq collision energy $\sqrt{s} = 18$ TeV

(d) qq collision energy $\sqrt{s} = 50$ TeV

Figure 4.1: SM-like/SM (upper left) and BSM/SM-like rate ratios for two additional light fermions with masses $m_\psi < v$

suppresses the SM-like vertex and is going to zero for $m_\psi \to 0$. Hence, for almost massless new left-handed Weyl fermions we don't recover the SM-like vertex and one can only see the BSM vertex. This can also be seen by looking at the ratio $\frac{\Gamma_{\text{BSM}}}{\Gamma_{\text{SM-like}}}$ where the BSM vertex is largely enhanced compared to the SM-like case for several energy ranges for lower masses. So if the sphaleron processes are unsuppressed above $E_{\text{sph}} = 9$ TeV, we expect them to occur mostly at threshold and therefore at about 10 TeV. That's the first case considered. The higher energies are taken, for one, to cover also the possibility of higher thresholds of unsuppressed sphaleron processes and to see in which way the estimates depend on the center of mass energy of the colliding particles. For higher masses $m_\psi \to v$, we can already guess that the SM-like vertex converges towards the SM case. A very interesting case of new particles is also to consider chiral $SU(2)$ particles. Then, we proceed just like in the SM and have only a vertex with the maximum amount of Weyl fermions determined by the chiral anomaly and no mass insertions. In particular, in chiral theories one does not recover a SM-like interaction with 12 fermions. The only vertex we get is

$$\Delta L^{\text{SM+chiral}} \sim \int_0^{1/v} \frac{d\rho}{\rho^5} d(\rho) \left[(2\pi^2\rho^3)^7 \times (\Pi_{m=1}^{12} \psi_m^{SM}) \psi_1 \psi_2 \right]. \tag{4.15}$$

The interesting point is that one could actually test chirality of new $SU(2)$ fermions by looking for SM-like interactions. If no SM-like vertices occur, while we see vertices producing more particles than expected by the SM, one would conclude that we have at least one new $SU(2)$ chiral fermion!

4.2.2 The Heavy Fermion Case

Fermions in the Fundamental Representation

By looking at eq. 4.12, one might think that for very large energies the SM-like rate is increasing due to the dependence on the mass m_ψ. Here, one has to keep in mind that all calculations so far are only valid for instanton scales smaller than the fermion masses and not in the regime $m_\psi \rho \gg 1$. Thus, for heavy fermions, we have to calculate the effective Lagrangian differently. The question is if the SM vertex is recovered for heavy nonchiral fermions suggested by the decoupling theorem and if the rate is the same? In the chiral case, this is apparently not the case.

In the vector fermion case, decoupling would mean that the SM-like vertex in the SM+vector theory should coincide with the SM result. Although the effective Lagrangian considered so far is not valid for $\rho m_\psi >$, we can use it plus the decoupling argument to estimate modifactions to it in order to obtain an effective Lagrangian valid for $m_\psi \rho > 1$. An important role here plays the change of the gauge coupling g from the UV theory to the IR regime. We expect the gauge coupling in a UV theory to match the IR coupling at a certain scale. In other words, once we have integrated out the heavy fermions, we should produce an effective action in which the running gauge coupling coincides with the one in the theory without heavy fermions. Additional operators not corresponding to the IR theory should be suppressed by powers of the heavy mass $1/m_\psi$. So we consider the effective Lagrangian we have so far to be the result after integrating out the heavy fermions with a running coupling $g(\rho)$ corresponding to the IR theory. In this effective theory all fermions are light. So by constructing an effective Lagrangian for a theory with heavy fermions, we proceed in the same way as for the light fermion case and first look at the term $C_1\mathbb{I}$ as in eq. 3.59. In eq. 3.59, the proportionality to m_ψ comes from the determinant contribution of the zero-modes with $\lambda_0 = 0$ as seen in eq. 3.51, *i.e.* $\det(-i\gamma_\mu D_\mu - im_\psi) \propto m_\psi \det(-i\gamma_\mu D_\mu - im_\psi)'$ and the primed determinant is the one over higher modes again. But once we have integrated out the heavy fermions, we don't get a m_ψ contribution anymore. Therefore, the first term in the expansion 3.58 is

$$C_1\mathbb{I} = \text{const}\left[\frac{8\pi^2}{g^2_{\text{IR}}(\rho)}\right]^4 e^{-8\pi^2/g^2_{\text{IR}}(\rho)} \tag{4.16}$$

where $g_{\text{IR}}(\rho)$ is the effective coupling in the theory without heavy fermions. The corresponding version for the UV theory is

$$C_1\mathbb{I} = \text{const}\left[\frac{8\pi^2}{g^2_{\text{UV}}(\rho)}\right]^4 e^{-8\pi^2/g^2_{\text{UV}}(\rho)}(m_\psi\rho)^\alpha \tag{4.17}$$

with a certain yet unknown power of m_ψ^α. To match both contributions, we may use the fact that for the theory at one-loop we have a coupling behaviour like in eq. 3.45. Hence, the UV and IR theory differ by the beta coefficient defined in eq. 3.46

$$b_{\text{UV}} - b_{\text{IR}} = -\frac{2}{3} \tag{4.18}$$

since we have integrated out the heavy fermions in the IR case. Now, we can express the UV coupling in terms of the IR coupling up to one-loop

$$\frac{8\pi^2}{g^2_{\text{UV}}(\rho)} = \frac{8\pi^2}{g^2_{\text{IR}}(\rho)} + \frac{2}{3}\ln(\tilde{m}\rho), \tag{4.19}$$

with some mass scale $\tilde{m}$ where both couplings match. We identify this scale with the heavy fermion mass since decoupling tells us that below scales equal to the heavy mass m_ψ, the IR theory captures all the relevant physics. Thus, we get

$$C_I I = \text{const}\left[\frac{8\pi^2}{g^2_{\text{IR}}(\rho)} + \frac{2}{3}\ln(m_\psi\rho)\right]^{2N} e^{-8\pi^2/g^2_{\text{IR}}(\rho)}(m_\psi\rho)^{\alpha-2/3} + \text{(higher loop corrections)}. \tag{4.20}$$

To finally match both approaches in eq. 4.16 and 4.17, the power of α has to be $\alpha = 2/3$ and therefore, we obtain

$$C_1\mathbb{I} = \text{const}\left[\frac{8\pi^2}{g^2_{\text{UV}}(\rho)}\right]^4 e^{-8\pi^2/g^2_{\text{UV}}(\rho)}(m_\psi\rho)^{2/3}\ . \tag{4.21}$$

Comparing that result with the light fermion case, we see that they differ in a factor of $(m_\psi\rho)^{-1/3}$. For the C_2 term, we argue as follows. In the estimate of the fermionic Green's function in eq. 3.63, we had the expectation value

$$\langle 0|T\{q^m_\gamma(x')\bar{q}^k_\beta(x)\}|0\rangle \sim \sum_n \psi^m_{(n)\gamma}(x')\psi^k_{(n)\beta}(x)(m_\psi - i\epsilon_n)^{-1}C_I(\rho)\frac{d\rho}{\rho^5}. \tag{4.22}$$

For small masses, the sum was dominated by the zero-mode contribution $n = 0$. In the limit of large masses, we may neglect the eigenvalue contribution ϵ_n, and consider the sum to be mainly proportional to the zero mode contribution again. C_1 can now be obtained by eq. 4.21 with an additional power of $(m_\psi\rho)^{-1/3}$ compared to the light fermion case. By writing down the effective Lagrangian for the heavy fermion case, this extra factor remains in the new particle vertex and we obtain

$$\Delta\mathcal{L} = \frac{d\rho}{\rho^5}d(\rho)\left[(m_\psi\rho)^{2/3} - 2\pi^2\rho^3(m_\psi\rho)^{-1/3}\bar{q}_R q_L\right], \quad m_\psi\rho \gg 1, \text{ vector fermion.} \tag{4.23}$$

The generalization to more fermions is straightforward and we get the schematic form

$$\begin{aligned}\Delta L \supset &\sum_{k=0}^{N}\int\frac{d\rho}{\rho^5}d(\rho)(2\pi^2\rho^3)^k \times \Pi^{N-k}_{(a,b),a\neq b}(\rho m_{ab})^{1+b_{ab}} \times \Pi^k_{(m,n),m\neq n}\left[(\rho m_{mn})^{b_{mn}}\psi_m\psi_n\right], \\ b_{mn} = &\left\{\begin{array}{ll} 0, & \rho m_{mn} < 1, \\ -1/3, & \rho m_{mn} > 1. \end{array}\right\}\end{aligned} \tag{4.24}$$

for heavy vector fermions. As a result, a new Dirac fermion, *i.e.* two left-handed $SU(2)$ Weyl fermions, in the fundamental representation of $SU(2)$, one obtains

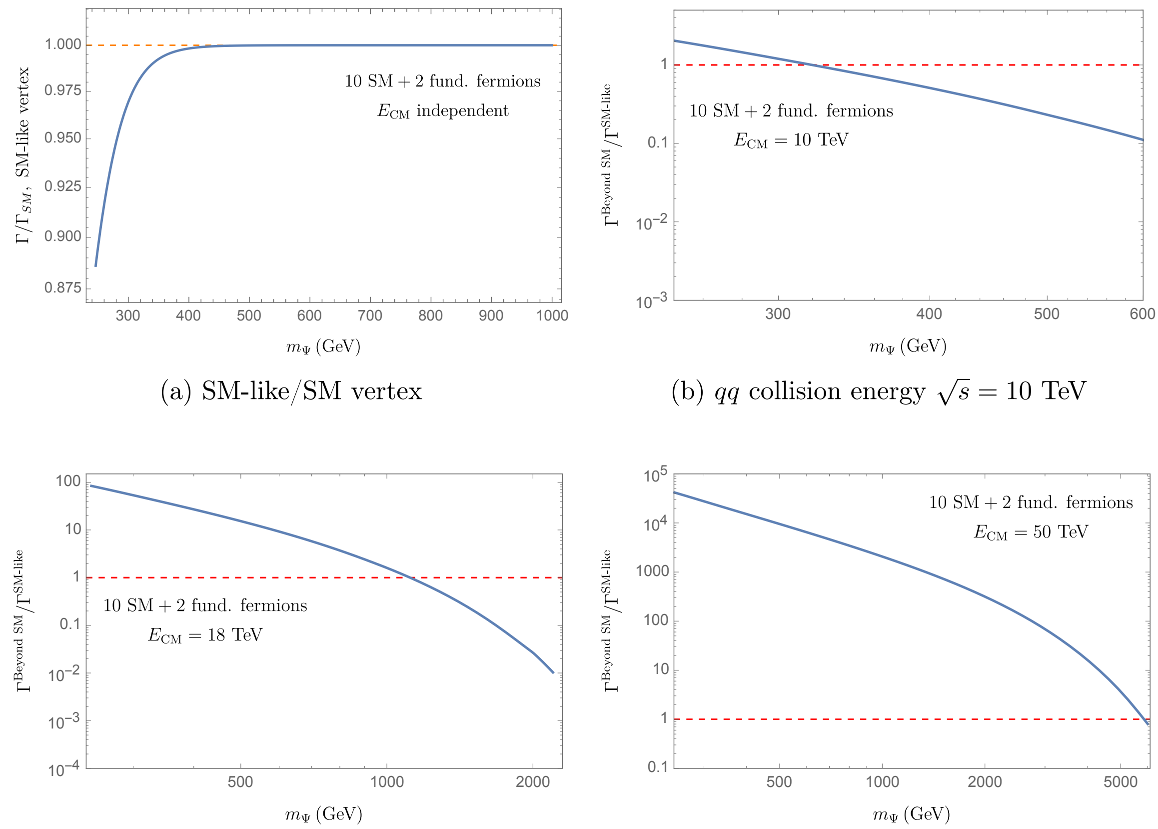
$$\Delta L^{\text{SM+vector}} = \Delta L_{\text{SM-like}} + \Delta L_{\text{new}}, \tag{4.25}$$

$$\Delta L_{\text{SM-like}} \sim \int_{1/m_\psi}^{1/v}\frac{d\rho}{\rho^5}\,d(\rho)\left[(2\pi^2\rho^3)^6(m_\psi\rho)^{2/3} \times \Pi^{12}_{m=1}\psi^{SM}_m\exp(-\pi^2\rho^2v^2)\right] \tag{4.26}$$

$$+\int_0^{1/m_\psi}\frac{d\rho}{\rho^5}\,d(\rho)\left[(2\pi^2\rho^3)^6\,(\rho m_\psi) \times \Pi^{12}_{m=1}\psi^{SM}_m\exp(-\pi^2\rho^2v^2)\right], \tag{4.27}$$

$$\Delta L_{\text{new}} \sim \int_{1/m_\psi}^{1/v}\frac{d\rho}{\rho^5}\,d(\rho)\left[(2\pi^2\rho^3)^7(\rho m_\psi)^{-1/3}\,(\Pi^{12}_{m=1}\psi^{SM}_m)\tilde{\psi}\psi\exp(-\pi^2\rho^2v^2)\right] \tag{4.28}$$

$$+\int_0^{1/m_\psi}\frac{d\rho}{\rho^5}\,d(\rho)\left[(2\pi^2\rho^3)^7\,(\Pi^{12}_{m=1}\psi^{SM}_m)\tilde{\psi}\psi\exp(-\pi^2\rho^2v^2)\right] \tag{4.29}$$



(a) SM-like/SM vertex

(b) qq collision energy $\sqrt{s} = 10$ TeV

(c) qq collision energy $\sqrt{s} = 18$ TeV

(d) qq collision energy $\sqrt{s} = 50$ TeV

Figure 4.2: SM-like/SM (upper left) and BSM/SM-like rate ratios for two additional heavy mass fermions in the fundamental representation of $SU(2)$

with

$$\begin{aligned} d(\rho) = (&-b_{\rm SM}\log(\Lambda\rho) + 2/3\log(m_\psi\rho))^4 \\ &\exp(-(8\pi^2/g_0^2 - b_{\rm SM}\log(m_\psi\rho) + 2/3\log(m_\psi\rho))) \end{aligned} \tag{4.30}$$

where we have simplified $8\pi^2/g^2_{\rm SM}(\rho)$ by expressing it in terms of the scale Λ at which the one-loop SU(2) coupling diverges:

$$\begin{aligned} \frac{8\pi^2}{g^2_{\rm SM}(\rho)} &= \frac{8\pi^2}{g(M_0)^2} - b_{\rm SM}\ln(M_0\rho) \equiv -b\log\Lambda\rho, \\ \frac{8\pi^2}{g^2_{\rm SM}(\rho = 1/\Lambda)} &\equiv 0 \Rightarrow \Lambda = \exp\left[-\frac{8\pi^2}{b_{\rm SM}g(M_0)^2}\right] M_0. \end{aligned} \tag{4.31}$$

Taking $M_0 = m_Z$, $g(m_Z) = 0.6519$, and $b_{\text{SM}} = 19/6$ for $SU(2)$ (see (3.46)), we have:

$$\begin{aligned}\Lambda &= 3.015 \times 10^{-24}\,\text{GeV},\\ d_{0,\text{SM}}(\rho) &= \text{const}'\,(\ln[\Lambda\rho])^4\,(M_0\rho)^{19/6}.\end{aligned} \tag{4.32}$$

The results can be found in Fig. 4.2. One can clearly, that decoupling is at work here! For masses $m_\psi \gg v$, the SM vertex is fully recovered with equal rates, see Fig. 4.2a. Besides, if we send the masses to infinity, the BSM process is suppressed by powers of $1/m_\psi$ as seen by the fall of in Fig. 4.2b to 4.2d. Notable is that rates are not suppressed in the full mass range. Wheras in the 10 TeV case in Fig. 4.2b, only a slight enhancement occurs below 300 GeV, the enhancement is much larger and up to higher masses for processes with qq collisions at 18 TeV and 50 TeV as seen in Fig. 4.2c and Fig. 4.2d. One can conclude that the BSM/SM-like rate ratios scale with some powers of E_{CM}^a/m_ψ^b with $a, b > 1$. In sphaleron processes, we observe that the BSM vertex actually exceeds the SM-like vertex in a certain mass and energy range. This could be interesting in collider physics experiments searching for the $B + L$ violating sphaleron process, not only because new physics enhances the rate but also to find these new particles for example in searching for displaced vertices signatures in multijet environments.

Fermions in the Adjoint Representation

A second case we study are adjoint fermions of $SU(2)$. In the adjoint case, the chiral charge Q_5 is violated by

$$\Delta Q_5 = 2\sum_r n(r)C(r) = 16 \tag{4.33}$$

for one additional Weyl fermion with $C(r) = 2$ including the 12 SM Weyl fermions. Hence, the full vertex in the adjoint theory consists of 16 Weyl fermions

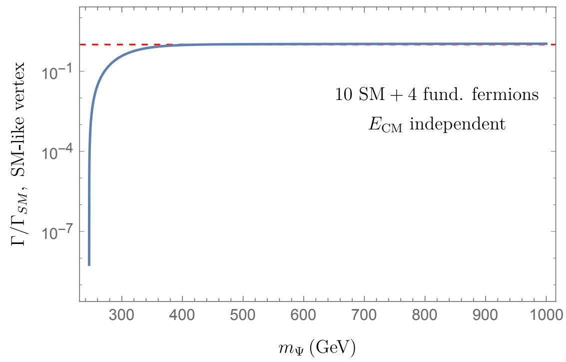
$$\mathcal{O}_{\text{SM+adj. vector}} = \prod_{i=1,2,3} (u_L d_L d_L v_L)_i (\tilde{\psi}\psi)^2\ . \tag{4.34}$$

In constructing the effective Lagrangian, we proceed in an analogous way like in the fundamental fermion case with the same power counting of ρ's and masses m_ψ. One difference to the results obtained in the case with fundamental fermions is the instanton density expression $d(\rho)$ including a group representation sensitive coefficient $C(r)$. Now, we have $C(\text{adj.}) = 2$, and hence, the difference in the beta coefficient is

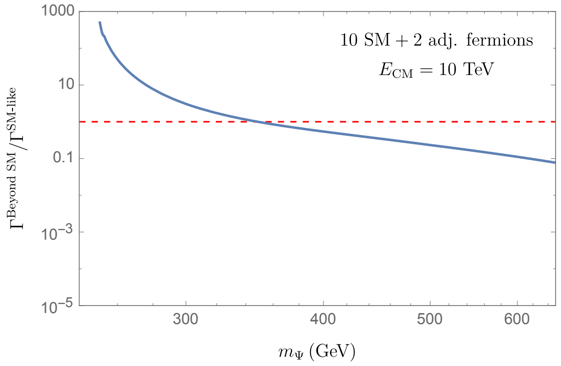
$$b_{\text{UV}} - b_{\text{IR}} = -\frac{4}{3}\ . \tag{4.35}$$

This results in a different gauge coupling behaviour

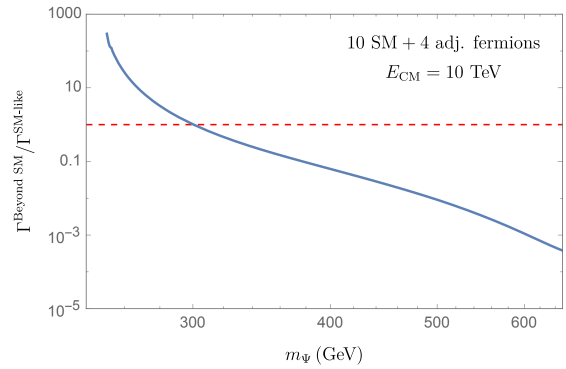
$$\frac{8\pi^2}{g_{\text{UV}}^2(\rho)} = \frac{8\pi^2}{g_{\text{IR}}^2(\rho)} + \frac{4}{3}\ln(m_\psi\rho), \tag{4.36}$$



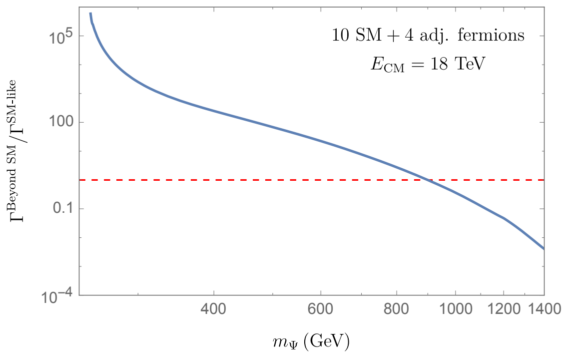
(a) SM-like/SM vertex



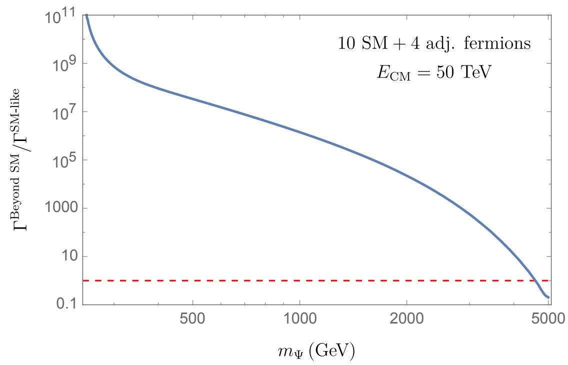
(b) qq collision energy $\sqrt{s} = 10$ TeV, 2 additional adjoint fermions



(c) qq collision energy $\sqrt{s} = 10$ TeV, 4 additional adjoint fermions



(d) qq collision energy $\sqrt{s} = 18$ TeV



(e) qq collision energy $\sqrt{s} = 50$ TeV

Figure 4.3: SM-like/SM (upper left) and BSM/SM-like rate ratios for up to four additional heavy mass fermions in the adjoint representation of $SU(2)$

and a mass scaling of $(m_\psi \rho)^{4/3}$. This is a modification of $(m_\psi \rho)^{2/3}$ compared to the case of two mass insertions by four Weyl fermions in the light fermion mass theory. For the effective

Lagrangian, we get

$$\Delta L^{\text{SM+adj.vector}} = \Delta L_{\text{SM-like}} + \Delta L_{\text{new}}, \tag{4.37}$$

$$\Delta L_{\text{SM-like}} \sim \int_{1/m_\psi}^{1/v} \frac{d\rho}{\rho^5}\, d(\rho) \left[(2\pi^2\rho^3)^6 (m_\psi \rho)^{4/3} \times \Pi_{m=1}^{12} \psi_m^{SM} \exp(-\pi^2\rho^2 v^2)\right] \tag{4.38}$$

$$+ \int_{0}^{1/m_\psi} \frac{d\rho}{\rho^5}\, d(\rho) \left[(2\pi^2\rho^3)^6 \, (\rho m_\psi)^2 \times \Pi_{m=1}^{12} \psi_m^{SM} \exp(-\pi^2\rho^2 v^2)\right], \tag{4.39}$$

$$\Delta L_{\text{new1}} \sim \int_{1/m_\psi}^{1/v} \frac{d\rho}{\rho^5}\, d(\rho) \left[(2\pi^2\rho^3)^7 (\rho m_\psi)^{1/3} \, (\Pi_{m=1}^{12} \psi_m^{SM}) \tilde{\psi}\psi \exp(-\pi^2\rho^2 v^2)\right] \tag{4.40}$$

$$+ \int_{0}^{1/m_\psi} \frac{d\rho}{\rho^5}\, d(\rho) \left[(2\pi^2\rho^3)^7 (\rho m_\psi)^1 \, (\Pi_{m=1}^{12} \psi_m^{SM}) \tilde{\psi}\psi \exp(-\pi^2\rho^2 v^2)\right] \tag{4.41}$$

$$\Delta L_{\text{new2}} \sim \int_{1/m_\psi}^{1/v} \frac{d\rho}{\rho^5}\, d(\rho) \left[(2\pi^2\rho^3)^8 (\rho m_\psi)^{-2/3} \, (\Pi_{m=1}^{12} \psi_m^{SM}) (\tilde{\psi}\psi)^2 \exp(-\pi^2\rho^2 v^2)\right] \tag{4.42}$$

$$+ \int_{0}^{1/m_\psi} \frac{d\rho}{\rho^5}\, d(\rho) \left[(2\pi^2\rho^3)^8 \, (\Pi_{m=1}^{12} \psi_m^{SM}) (\tilde{\psi}\psi)^2 \exp(-\pi^2\rho^2 v^2)\right] \tag{4.43}$$

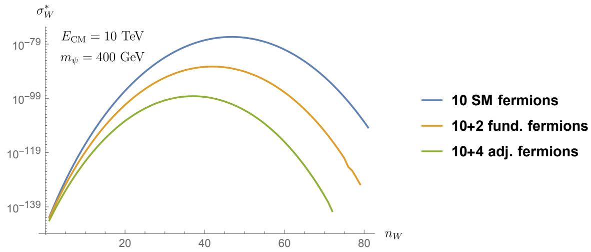
with SM-like vertices as well as BSM vertices with 2 and 4 additional fermions. In this case, the instanton density is given by

$$\begin{aligned} d(\rho) = (&-b_{\text{SM}} \log(\Lambda\rho) + 4/3 \log(m_\psi \rho))^4 \\ &\exp(-(8\pi^2/g_0^2 - b_{\text{SM}} \log(m_\psi \rho) + 4/3 \log(m_\psi \rho)))\ . \end{aligned} \tag{4.44}$$

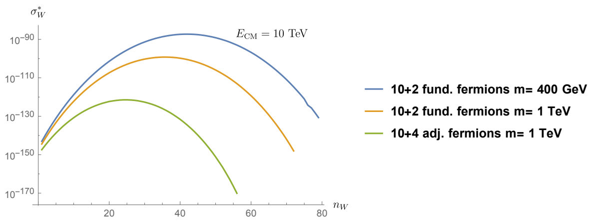
The scaling behaviour in the adjoint case is similiar to the one in the fundamental representation. Nevertheless, on the one hand, one can see in Fig. 4.3 that the SM-like/SM rate ratio approaches the SM case earlier in the mass range than in the fundamental case. On the other hand, it drops down to zero more rapidly in the lower mass range due to the higher mass suppression power in the SM-like vertex. This results in a fastly increasing BSM/SM-like vertex to lower masses for several considered collision energies. For one, the BSM vertex is more enhanced, for another it's much faster suppressed if we go to higher masses of the adjoint fermions.

4.2.3 Processes With Fermions and Bosons

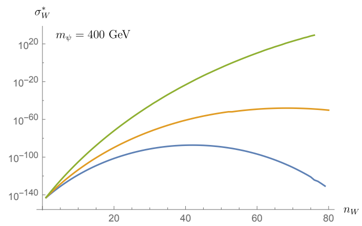
The last case we regard is to include W and Higgs bosons in our vertex. What we expect is that those bosons carry a sizable contribution of the total energy. For the W bosons, this implies that the phase space integral contribution to the rate gets larger due to the extra energy factor coming from the matrix element. As a result, we expect the total rate to be higher if we include more W bosons. Furthermore, we also expect that there is a maximum enhancement since at some point the large number of final state particles are produced at



(a) Dependence of maximum n_W on number of fermions included in the vertex.



(b) Dependence of of maximum n_W on number and masses of fermions included in the vertex.



(c) Maximum shift due to the available energy of the process.

Figure 4.4: n_W distributions for different n_f with $m_\psi = 400$ GeV (upper left) and different m_ψ (upper right), and 2 add. fermions with $m_\psi = 400$ GeV with different E_{CM}

rest due to the limited available energy of the process. For the same reason, we expect this maximum to be shifted towards smaller number of additional bosons if we include new heavy fermions in our vertex distinguishing the SM case from BSM scenarios. For the matrix element squared we use the following formulas

1. SM only vertex:

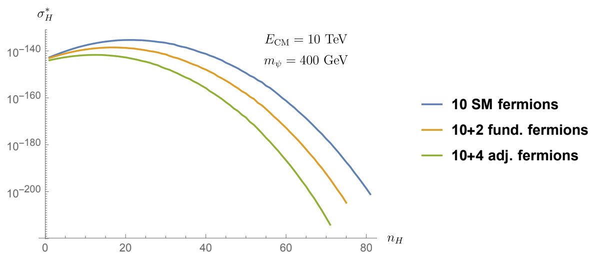
$$|\tilde{\mathcal{M}}_{SM}^{n_h,n_W}|^2 = F(n_h, n_W) \left[\int_0^{1/v} \frac{d\rho}{\rho^5} d(\rho) (2\pi^2\rho^3)^6 \, \rho^{2(n_h+n_W)} \exp(-\pi^2\rho^2 v^2) \right]^2 , \tag{4.45}$$

with

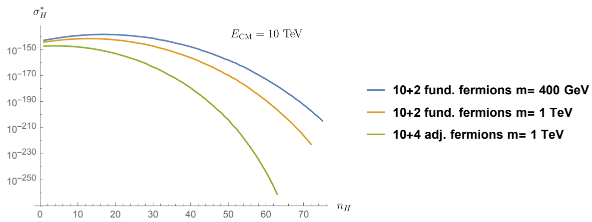
$$F(n_h, n_W) \equiv 2^{n_h+2n_W} \pi^{4(n_h+n_W)} v^{2(n_h+n_W)} . \tag{4.46}$$

2. SM-like vertex in SM+vector theory, with $m_\psi > v$:

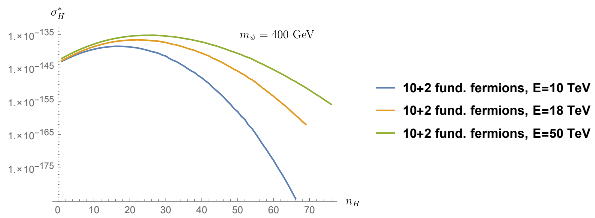
$$\begin{aligned} |\tilde{\mathcal{M}}_{SM+}^{0,n_h,n_W}|^2 = & F(n_h, n_W) \left[\int_{1/m_\psi}^{1/v} \frac{d\rho}{\rho^5} d(\rho) (2\pi^2\rho^3)^6 (m_\psi \rho)^{2/3} \, \rho^{2(n_h+n_W)} \exp(-\pi^2\rho^2 v^2) \right. \\ & \left. + \int_0^{1/m_\psi} \frac{d\rho}{\rho^5} d(\rho) (2\pi^2\rho^3)^6 \, (\rho m_\psi) \rho^{2(n_h+n_W)} \exp(-\pi^2\rho^2 v^2) \right]^2 . \end{aligned} \tag{4.47}$$



(a) Dependence of maximum n_H on number of fermions included in the vertex.



(b) Dependence of of maximum n_H on number and masses of fermions included in the vertex.



(c) Maximum shift due to the available energy of the process.

Figure 4.5: n_H distributions for different n_f with $m_\psi = 400$ GeV (upper left) and different m_ψ (upper right), and 2 add. fermions with $m_\psi = 400$ GeV with different E_{CM}

3. Beyond-SM vertex in SM+vector theory:

$$\begin{aligned} |\tilde{\mathcal{M}}_{SM+}^{2,n_h,n_W}|^2 =& F(n_h, n_W) \Bigg[\int_{1/m_\psi}^{1/v} \frac{d\rho}{\rho^5}\, d(\rho)(2\pi^2\rho^3)^7 (\rho m_\psi)^{-1/3} \rho^{2(n_h+n_W)} \exp(-\pi^2\rho^2 v^2) \\ &+ \int_0^{1/m_\psi} \frac{d\rho}{\rho^5}\, d(\rho)(2\pi^2\rho^3)^7 \rho^{2(n_h+n_W)} \exp(-\pi^2\rho^2 v^2) \Bigg]^2 . \end{aligned} \tag{4.48}$$

with the phase-space integration from eq. 4.7 and $d(\rho)$ from eq. 4.30. The adjoint case is constructed in the same way.

We obtain what we already expected as depicted in Fig. 4.4. For more fermions included in the vertex, the maximum enhancement with respect to the number of W bosons included is shifted towards lower values. In addition, for fermions with higher masses it's further shifted for the same reason. For more energy available in the process, the sharing of energies between all n final particles is saturated for larger n_W. If we include Higgs bosons in our vertex, we still get an enhancement as seen in Fig. 4.5 where we consider the same scenarios as in the W boson case. The basic behaviour is similar to the ones we obtained for the W bosons. Nevertheless, the enhancement is much weaker in the Higgs boson case. This can also be seen by focussing on the enhancement due to bosons as depicted in the $n_H - n_W$-plane of Fig. 4.6. Here, we focused on the case of a SM+2 additional fermions in the fundamental representation with mass $m_\psi = 400$ GeV. It's clearly visible that the regions of larger enhancements are shifted towards the W bosons.

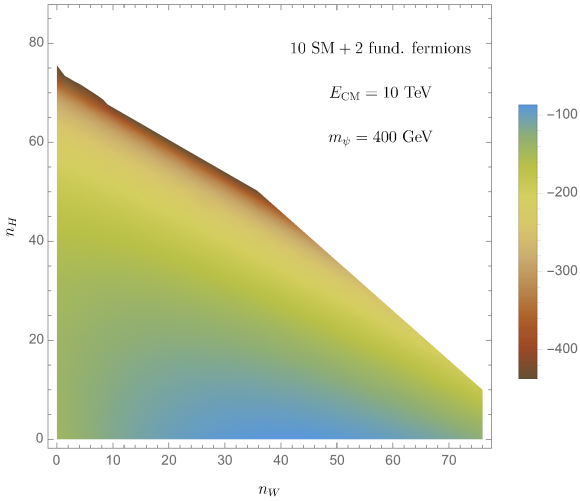


Figure 4.6: Both n_W and n_H dependency of the vertex with 10 SM + 2 add. fermions with $m_\psi = 400$ GeV. The colour shows the order of magnitude

5 Conclusion and Outlook

In this thesis, we have studied $B+L$-violating sphaleron processes and how they can be influenced by physics beyond the Standard Model, in particular in theories with additional fermions charged under $SU(2)$. In principle, every $SU(2)$ left-handed Weyl fermion contributes to the chiral anomaly, which determines the maximum number of fields entering the sphaleron vertex. We use instanton techniques to construct approximate effective Lagrangian for arbitrary BSM scenarios. First, we conclude that in case of additional chiral fermions in the $SU(2)$ gauge group, we only get vertices with BSM fermions, and no SM-like vertices. The decoupling theorem does not hold in that case. So if we detect only a BSM vertex of sphaleron process with new massive $SU(2)$ particles and no SM-like vertex, we can infer that we have at least one additional chiral fermion in the $SU(2)$ sector of the SM. In case of non-chiral extensions of the SM, we look at new particles in the fundamental and in the adjoint representation of $SU(2)$ at certain energies of the process. Both, in the case of 2 additional Weyl fermions in the fundamental and antifundamental representations, respectively, one gets two types of possible vertices. First, a SM-like vertex arises from mass insertions of the new fermions. Additionally, a BSM vertex occurs with two new $SU(2)$ Weyl fermions. With an additional Weyl fermion in the adjoint representation of $SU(2)$, we obtain three possible vertices, the SM-like vertex, a vertex with 2 additional adjoint fermions and the maximal vertex consisting of the SM+4 additional adjoint fermions. Both for additional fermions in the fundamental and adjoint representation, we see that the interaction rates of the sphaleron process are enhanced in a certain mass range and for high enough collision energies. In both cases, the SM-like vertex is reproduced in agreement with the decoupling theorem. We also observe that the maximum enhancement of the sphaleron vertex with respect to the number of W bosons included in the vertex is shifted towards lower numbers of bosons if we include more fermions in the vertex. The energy of the process is distributed amongst all particles and since the new particles carry a sizable contribution of the energy the sharing of the energy of the process is saturated for a lower number of particles included. To sum up, we have worked out that sphalerons are not only $B+L$ violating processes but also could be linked to BSM physics searches at hadron colliders. It would be interesting to study searches for BSM physics in such processes by investigating signatures of these new particles, *e.g.* by looking for disappearing tracks of the BSM fermions.

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 21.09.2017 ....