New Physics at the TeV Scale

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New physics relevant to the LHC has over the last decade become a quite baroque field. From the simple alternative of either (perturbative) supersymmetry of (strongly interacting) technicolor we have evolved into a world of many different models attempting to solve the hierarchy problem, provide a dark matter candidate and not get killed by electroweak precision data or flavor measurements. This is an incoherent set of notes on all kinds of topics I have been thinking about in the run-up to the LHC, and they have not been updated since...

Contents

User manual

These notes are based on a set of SUPA lectures given in Edinburgh between 2005 and 2008. Each of them lasted around eight hours and was meant to broaden everybody's perspective on physics beyond the Standard Model, explicitly including the lecturer's. The text is divided into different topics:

1. supersymmetry is still the leading candidate for new physics at the LHC, at least when we look at the number of search channels planned. This part is the only one which got updated more recently for a set of talks at TASI-East in Mainz...
2. extra dimensions came to life in the late 1990s, when phenomenologists noticed that they could hijack US faculty jobs from the string community claiming to work on similarly fundamental and beautiful concepts. Since then, flat extra dimensions have kind of stalled in their development while warped extra dimensions have become more popular as the weakly interacting description of strongly interacting models....
3. little Higgs models are a realization of an old idea: can we write down some kind of symmetry which protects the Higgs mass as a Goldstone via its breaking mechanism.
4. models without a Higgs boson have been around longer than the Higgs boson, or at least our theoretical insight that there should be one. They simply do not go away, so we should keep in mind that they might be able to explain some very unfortunate experimental outcomes at the LHC.

Historically, two more SUPA lectures on QCD and on Higgs physics are not part of this writeup, but have served as the basis for their own quite extensive lecture notes (arXiv:0910.4182). At a more advanced level, this writeup has also lead to a review articles on new physics at the LHC together with David Morrissey and Tim Tait (arXiv:0912.3259).
Last, but not least, the literature listed for each of these sections is not meant to cite the relevant research papers. Instead, I collected a set of review papers or advanced lecture notes supplementing this set of lecture notes in different directions of interest.

As the author of these notes I am confident that they are far from bug free. So if you think you read them very carefully and you did not email me at least a handful typos and complaints about poor explanations, you did not read them carefully enough.

I. SUPERSYMMETRY

Supersymmetry is a well-established model to solve to problems of the Standard Model: first there exists a theoretical problem, called the hierarchy problem. It describes the fact that our observed Higgs mass is light, while in a quantum theory all scalar masses are expected to range around the upper cutoff scale of the theory. Or in other words, the loop corrections to the Higgs mass and the counter term have to cancel to a very large degree. The second problem is experimental: we know that dark matter exists and that is has a particle nature. A TeV-scale extension of the Standard Model with a weakly interacting stable particle serves as an excellent candidate to solve this problem. Finally, supersymmetry allows us to build a grand unified theory as well as possible links to gravity and super-strings.

When giving a brief introduction into supersymmetry at this level we have to make a choice at the very beginning. We could, for example, begin with the supersymmetry as the last missing piece in the set of symmetries of our Standard Model. However, from there we will never make it to a semi-detailed account of the minimal supersymmetric Standard Model (MSSM). Similarly, we could generalize our four-dimensional space-time to a superspace with additional supersymmetric directions, and again we would not make it to the MSSM.
The way supersymmetry is presented here is therefore mathematically hand-waving. We start by writing down the Lagrangian for example of QED and then step by step supersymmetrize it. We guess the SUSY transformations of the fields, then check that our SUSY-QED Lagrangian is indeed supersymmetric, and then derive an algebra of the SUSY transformation. This approach closely follows the lecture notes by Ian Aitchison (hep-ph/0505105), slightly re-ordered and shortened in many places.
One aspect of supersymmetry which we will focus on a little closer are the different sources of scalar self couplings. Mathematically they might not be the most important structures in supersymmetry, but when constructing the MSSM and its Higgs sector they play a crucial role we will discuss in some length.

A. Supersymmetry as an extended symmetry

Before we can construct a supersymmetric gauge theory we need to briefly think about all symmetry operators we can apply to such a theory. These operators we can classify according to their Lorentz structure, following the original approach of Haag, Lopuszanski, and Sohnius (1975):

– Lorentz scalar, *e.g.* charge, isospin

– four-vector, *e.g.* space-time translations

– antisymmetric tensor, *e.g.* angular momentum

– symmetric tensor, but not allowed due to Coleman-Mandula theorem

The one structure not listed above, but which is nevertheless allowed is an operator with a spinor charge

$$Q|J\rangle = \left| J \pm \frac{1}{2} \right\rangle \; , \tag{1}$$

where J is the spin of the field or particle involved. In our usual gauge theories such a symmetry operator used to be unthinkable because spin fixes (Fermi- or Bose-Einstein) statistics and distinguishes matter from interaction particles.

An obvious observation is that under a supersymmetry transformation

$$\begin{aligned} Q|\phi\rangle \to & \quad |\tilde{Q}\rangle \\ \text{spin } 0 \to & \quad \text{spin } \frac{1}{2} \end{aligned} \tag{2}$$

degrees of freedom should not vanish. So the question becomes what the fermionic partner of a complex scalar (2 dof) might be. A Dirac spinor obeying the Dirac equation $(i\partial\mu\gamma^\mu - m)\Psi = 0$ has four entries, which in general means it represents four degrees of freedom on its mass shell. One way to reduce the degrees of freedom of a Dirac fermion is via chirality projections

$$P_{L,R} = \frac{1 \pm \gamma_5}{2} = \frac{1}{2}\left(1 \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{cases} .$$

This means we can use it to dissect a Dirac spinor,

$$\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix} \qquad \Rightarrow \qquad P_L \Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix} \qquad \text{and} \qquad P_R \Psi \begin{pmatrix} 0 \\ \chi \end{pmatrix} . \tag{3}$$

The two chiral fermion fields ϕ and χ have two degrees of freedom on-shell or four degrees of freedom off-shell. The obey the two-component Dirac equation

$$\begin{aligned} \sigma^\mu p_\mu \psi &= m\chi \\ \bar{\sigma}^\mu p_\mu \chi &= m\psi , \end{aligned} \tag{4}$$

in terms of the 2×2 Pauli matrices

$$\begin{aligned} \sigma^\mu &= \left(1 \, , \, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \\ \bar{\sigma}^\mu &= \left(1 \, , \, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} , \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right) . \end{aligned} \tag{5}$$

They are hermitian $(\sigma^\mu)^\dagger = \sigma^\mu$, $(\bar{\sigma}^\mu)^\dagger = \bar{\sigma}^\mu$ and normalized $\sigma_i \sigma_i = 1$. Note that Eq.(4) indicates that the two chiral fields are mixed by their mass, *i.e.* in the presence of a finite fermion mass m we have to automatically consider both components and hence the complete Dirac spinor. Now, we can write a Lagrangian for a massive fermion in Weyl spinors

$$\mathcal{L} \supset \overline{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi = \psi^\dagger i\sigma^\mu \partial_\mu \psi + \chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi - m(\psi^\dagger \chi + \chi^\dagger \psi) . \tag{6}$$

After this short discussion of notation, let us construct a supersymmetric, massless QED.

1. *Scalar electron partner*

Following the discussion above, massless QED means that in the full Lagrangian we can pick either χ or ψ for the fermion field and then add a complex scalar field. The kinetic term for the Weyl fermion follows from Eq.(6),

$$\boxed{\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi} . \tag{7}$$

We postpone a look at the degrees of freedom at this stage. The scalar field fulfills the a Klein-Gordon equation of motion,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \qquad \Rightarrow \qquad \partial_\mu \partial^\mu \phi \equiv \Box \phi = 0 . \tag{8}$$

The question is if this Lagrangian can ever be supersymmetric under a transformation of the kind

$$\left\{ \begin{matrix} \phi \\ \chi \end{matrix} \right\} \xrightarrow{\text{SUSY}} \left\{ \begin{matrix} \chi \\ \phi \end{matrix} \right\} + \delta_\xi \left\{ \begin{matrix} \phi \\ \chi \end{matrix} \right\} . \tag{9}$$

Generally, we can define a SUSY transformation proportional to a spinorial generator ξ. We can to some degree guess its form step by step: obviously, the variation of the scalar field $\delta_\xi \phi$ has to be a Lorentz invariant proportional to the fermion field χ. The corresponding transposed spinor is going to be the spinorial shift ξ^T. Because of Lorentz invariants the two Weyl spinors need to get contracted through the antisymmetric tensor $\varepsilon_{ij} = -i\sigma_2$. This gives us

$$\delta_\xi \phi = \xi^T(-i\sigma_2)\chi = (\xi_1 \xi_2) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} . \tag{10}$$

The mass dimensions are $[\delta_\xi \phi] = 1$, $[\chi] = M^{3/2}$, and correspondingly $[\xi] = M^{-1/2}$.

Following similar arguments we can try to construct the SUSY transformation of the scalar field. First, the mass dimensions of all fields are $[\delta_\xi \chi] = M^{3/2}$, $[\xi] = M^{-1/2}$, and $[\phi] = M$. That suggests that $\delta_\xi \chi \sim \phi \xi$ lacks a partial derivative $\partial_\mu \phi$ with $[\partial_\mu] = M$ to increase the mass dimensions on the right hand side. Next, Lorentz invariance requires us to saturate the open index in $\delta_\xi \chi \sim (\partial_\mu \phi)\xi$ with the only structure we have at hand, the Pauli matrices. As mentioned above, this is not Lorentz invariant unless we replace $\xi \to i\sigma_2 \xi^*$. Including a free normalization factor A this gives

$$\delta_\xi \chi = -A i \sigma^\mu (i\sigma_2 \xi^*)(\partial_\mu \phi) . \tag{11}$$

Note that compared to Aitchison's review we are choosing a different sign of A.

Before we tackle the actual Lagrangian we also need the transformations of the hermitian conjugate fields:

$$\begin{aligned} \delta_\xi \phi^\dagger =& \chi^\dagger (i\sigma_2)\xi^* \qquad\qquad \text{in analogy to} \quad (A \cdot B)^\dagger = B^\dagger A^\dagger \\ \delta_\xi \chi^\dagger =& - A(\partial_\mu \phi^\dagger)(\xi^T i\sigma_2) i\sigma^\mu . \end{aligned} \tag{12}$$

By brute force we then find

$$\begin{aligned} \delta_\xi \mathcal{L} =& \delta_\xi \left[\partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi \right] \\ =& \partial_\mu (\delta_\xi \phi^\dagger) \partial^\mu \phi + \partial_\mu \phi^\dagger \partial^\mu (\delta_\xi \phi) + (\delta_\xi \chi^\dagger) i\bar{\sigma}^\mu \partial_\mu \chi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu (\delta_\xi \chi) \\ =& \partial_\mu (\chi^\dagger \lambda \sigma_2 \xi^*) \partial^\mu \phi - \partial_\mu \phi^\dagger \partial^\mu (\xi^T i\sigma_2 \chi) - A(\partial_\mu \phi^\dagger)(\xi^T i\sigma_2 i\sigma^\mu)(i\bar{\sigma}^\nu \partial_\nu \chi) - A\chi^\dagger i\bar{\sigma}^\nu \partial_\nu i\sigma^\mu i\sigma_2 \xi^* \partial_\mu \phi \stackrel{?}{=} 0 . \end{aligned} \tag{13}$$

For illustration purposes we limit ourselves to only terms including ξ^*,

$$\begin{aligned} \delta_\xi \mathcal{L}\Big|_{\xi^*} &= \partial_\mu \chi^\dagger i\sigma_2 \xi^* \partial^\mu \phi + iA\chi^\dagger \bar{\sigma}^\nu \partial_\nu \sigma^\mu \sigma_2 \xi^* \partial_\mu \phi \\ &= i(\partial_\mu \chi^\dagger)\sigma_2 \xi^* \partial^\mu \phi + iA\chi^\dagger \sigma_2 \xi^* (\partial_\mu \partial^\mu \phi) \qquad \text{using} \;\; \bar{\sigma}^\nu \partial_\nu \sigma^\mu \partial_\mu = (\partial_0 + \bar{\sigma}\vec{\nabla})(\partial_0 - \bar{\sigma}\vec{\nabla}) = \partial_0^2 - \vec{\nabla}^2 = \partial_\mu \partial^\mu \\ &\stackrel{A=1}{=} i\partial_\mu \big(\chi^\dagger \sigma_2 \xi^* (\partial^\mu \phi) \big) \end{aligned} \tag{14}$$

The supersymmetric variation of the Lagrangian, shown here only for the term proportional to ξ^*, becomes a total derivative exactly for $A = 1$. Because total derivatives in the Lagrangian do not have any physical effect this uniquely fixes A and with it the supersymmetry transformations of ϕ and χ. Under those assumptions $\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi$ is indeed supersymmetric under the two transformations

$$\boxed{\delta_\xi \phi = -i\xi^T \sigma_2 \chi} \qquad\qquad \boxed{\delta_\xi \chi = \sigma^\mu (\sigma_2 \xi^*)(\partial_\mu \phi)} . \tag{15}$$

2. Wess-Zumino model

If we assume that ϕ and χ transform into each other under SUSY transformations they will form some kind of supersymmetric multiplet. The question arises if successive SUSY transformations simply transform the two fields into each other or under what circumstances this happens. In particular, we might hope that such a condition leads us to a specific form of the algebra of SUSY transformations which *per se* do not commute. Again, we simply compute

$$
\begin{aligned}
[\delta_\eta, \delta_\xi]\ \phi = (\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)\ \phi &= -\, i\delta_\eta(\xi^T\sigma_2\chi) - (\xi \leftrightarrow \eta) \\
&= -\, i\xi^T\sigma_2\delta_\eta\chi - (\xi \leftrightarrow \eta) && \delta_\eta \text{ only acting on SUSY fields} \\
&= -\, i\xi^T\sigma_2(-i\sigma^\mu i\sigma_2\eta^*)\partial_\mu\phi - (\xi \rightarrow \eta) \\
&= -\, i\xi^T\sigma_2\sigma^\mu\sigma_2\eta^*\partial_\mu\phi - (\xi \leftrightarrow \eta) \\
&= -\, i\xi^T(\bar\sigma^\mu)^T\eta^*\partial_\mu\phi - (\xi \leftrightarrow \eta) && \text{using} \quad \sigma_2\sigma^\mu\sigma_2 = (\bar\sigma^\mu)^T \\
&= -\, i(\xi^T\bar\sigma^{\mu T}\eta^*)^T\partial_\mu\phi - (\xi \leftrightarrow \eta) && \text{just a number } c^T \equiv c \\
&= +\, i\eta^\dagger\bar\sigma^\mu\xi^*\partial_\mu\phi - (\xi \leftrightarrow \eta) && \text{sign from Grassmann } \xi\ ,
\end{aligned}
\tag{16}
$$

giving us

$$
[\delta_\eta, \delta_\xi]\ \phi = i(\eta^\dagger\bar\sigma^\mu\xi^* - \xi^\dagger\bar\sigma^\mu\eta^*)\ \partial_\mu\phi\ . \tag{17}
$$

The mass units $[\phi] = M$, $[\eta^\dagger] = M^{-1/2} = [\xi]$, and $[\partial_\mu\phi] = [\partial_\mu][\phi] = M^2$ match. This relation suggests an operator identity $[\delta_\eta, \delta_\xi] = (\eta^\dagger\bar\sigma^\mu\xi - \xi^\dagger\bar\sigma^\mu\eta)\partial_\mu$ defining the algebra of the SUSY transformations. Before we can postulate such an operator relation we should compute the commutator acting on the Weyl fermion χ, still on foot and bare of any elegance:

$$
\begin{aligned}
\delta_\eta\delta_\xi\chi &= \delta_\eta\sigma^\mu\sigma_2\xi^*\partial_\mu\phi \\
&= \sigma^\mu\sigma_2\xi^*\partial_\mu(\delta_\eta\phi) \\
&= \sigma^\mu\sigma_2\xi^*\partial_\mu(-\eta^T i\sigma_2\chi) \\
&= -\, i\sigma^\mu\sigma_2\xi^*\eta^T\sigma_2\partial_\mu\chi \\
&= \dots && \text{skipping Aitchison Eqs.(241-247)} \\
&= -\, i\eta(\xi^\dagger\bar\sigma^\mu\partial_\mu\chi) + i\eta^T\sigma_2\sigma^\mu\sigma_2\xi^*\partial_\mu\chi \\
&= -\, i\eta(\xi^\dagger\bar\sigma^\mu\partial_\mu\chi) - i(\xi^\dagger\bar\sigma^\mu\eta^*)\partial_\mu\chi\ .
\end{aligned}
\tag{18}
$$

If we want to prove Eq.(17) on the operator level the second term is exactly what we want, while with first term is not at all appreciated.
To get rid of the first term we now have a look at the degrees of freedom for the off-shell fields, *i.e.* without requiring them to obey their respective equation of motion.

$$
\phi:\ \text{complex scalar field, 2 entries or d.o.f.} \quad ⚡ \quad \psi/\chi:\ \text{complex Weyl spinors, 4 entries or d.o.f.} \tag{19}
$$

The degrees of freedom of our two fields do not match, which means that the SUSY transformations cannot simply transform one of them into the other and back. What we need is an additional field F with two degrees of freedom. This will then give us the freedom to choose its SUSY transformation $\delta_\xi F$ as well as its contributions to $\delta_\xi\phi$ and $\delta_\xi\chi$ proportional to F such that

1. the commutator $[\delta_\eta, \delta_\xi]$ acts identically on all three fields and closes for the complete multiplet $\{\phi, \chi, F\}$
2. $\mathcal{L}(\phi, \chi, F)$ is SUSY-invariant, modulo total derivatives

We might hope to limit F to an auxiliary field, which means it should neither propagate not couple to the two physical fermion and scalar fields. If the Lagrangian is of the form $\mathcal{L}_F \sim F^\dagger F$, with the corresponding mass dimension $[F] = M^2$, the equation of motion for F becomes

$$
\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu F^\dagger)}\right) - \frac{\partial\mathcal{L}}{\partial F} = -F^\dagger = 0\ . \tag{20}
$$

This way the auxiliary field F indeed does not propagate. Next, we construct the SUSY transformations, starting with $\delta_\xi F \propto \xi^T \chi$. The mass dimensions suggest $\delta_\xi F \sim \xi^T \partial_\mu \chi$. We need to saturate the Lorentz index, and it turns out that

$$\boxed{\delta_\xi F = -i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi} \qquad \text{and} \qquad \delta_\xi F^\dagger = +i(\partial_\mu \chi^\dagger)\bar{\sigma}^\mu \xi \tag{21}$$

do the job.

To cure the problematic additional contribution to $[\delta_\eta, \delta_\xi]\chi$ we postulate

$$\boxed{\delta_\xi \phi = -i\xi^T \sigma_2 \chi} \; , \tag{22}$$

which gives us an additional contribution to Eq.(18)

$$\begin{aligned} \delta_\xi \chi\Big|_F &= \xi F \\ \delta_\eta \delta_\xi \chi\Big|_F &= \delta_\eta \xi F = -i\xi \eta^\dagger \bar{\sigma}^\mu \partial_\mu \chi \; , \end{aligned} \tag{23}$$

cancelling the unwanted of the two terms and ensuring $[\delta_\eta, \delta_\xi]\chi \sim \partial_\mu \phi$. Finally, the transformation of the scalar field does not need to be changed,

$$\boxed{\delta_\xi \chi = \sigma^\mu \sigma_2 \xi^* \partial_\mu \phi + \xi F} \; . \tag{24}$$

With that replacement the commutator $[\delta_\eta, \delta_\xi]$ acts the same way on our fermion and our scalar. The remaining task is to check this operator identity acting on F,

$$\begin{aligned} [\delta_\eta, \delta_\xi]\; F =& \delta_\eta \delta_\xi F - (\eta \leftrightarrow \xi) \\ =& \delta_\eta(-i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi) - (\eta \leftrightarrow \xi) \\ =& -i\xi^\dagger \bar{\sigma}^\mu \partial_\mu (\delta_\eta \chi) - (\eta \leftrightarrow \xi) \\ =& -i\xi^\dagger \bar{\sigma}^\mu \partial_\mu (\sigma^\nu \sigma_2 \eta^* \partial_\nu \phi + \eta F) - (\eta \leftrightarrow \xi) \\ =& -i\xi^\dagger \bar{\sigma}^\mu \sigma^\nu \sigma_2 \eta^* \partial_\mu \partial_\nu \phi - i\xi^\dagger \bar{\sigma}^\mu \eta \partial_\mu F - (\eta \leftrightarrow \xi) \\ =& i(-\xi^\dagger \bar{\sigma}^\mu + \eta^\dagger \bar{\sigma}^\mu \xi)\; \partial_\mu F \; , \end{aligned} \tag{25}$$

where in the last step the first term vanishes after adding the permutation of η and ξ. This means that for all three fields in the supersymmetric multiplet $\{\phi, \chi, F\}$ the SUSY algebra becomes

$$\boxed{[\delta_\eta, \delta_\xi] = i(\eta^\dagger \bar{\sigma}^\mu \xi^* - \xi^\dagger \bar{\sigma}^\mu \eta^*)\partial_\mu} \; . \tag{26}$$

To test the second requirement on our wish list we need to compute the SUSY transformation of the terms including F in the Lagrangian

$$\begin{aligned} \delta_\xi F^\dagger F =& (\delta_\xi F^\dagger)F + F^\dagger \delta_\xi F \\ =& i(\partial_\mu \chi^\dagger)\bar{\sigma}^\mu \xi F - iF^\dagger \xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi \\ \delta_\xi(\chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi)\Big|_F =& (\delta_\xi \chi^\dagger)i\bar{\sigma}^\mu \partial_\mu \chi + i\chi^\dagger \bar{\sigma}^\mu \partial_\mu (\delta_\xi \chi)\Big|_F \\ =& iF^\dagger \xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + i\chi^\dagger \bar{\sigma}^\mu \xi \partial_\mu F \; . \end{aligned} \tag{27}$$

Adding both contributions indeed leaves us with the total derivative $\delta_\xi \mathcal{L}_F \sim i\partial_\mu(\chi^\dagger \bar{\sigma}^\mu \xi F)$.

What we derived is the Lagrangian of the Wess-Zumino model

$$\boxed{\mathcal{L}_{\rm WZ} = \partial_\mu \phi^\dagger \partial^\mu \phi + i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + F^\dagger F} \; , \tag{28}$$

with a massless Weyl fermion χ, a massless charged scalar ϕ, and an auxiliary field F. The latter can be replaced by its equation of motion,

$$\frac{\partial \mathcal{L}}{\partial F} = F^\dagger = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu F^\dagger)} \right) = 0 \qquad \Leftrightarrow \qquad F = 0 \tag{29}$$

This means that in the Wess–Zumino Lagrangian the auxiliary field makes little sense. This will change when eventually we introduce additional terms, which give it a on-trivial equation of motion.

3. *Fermionic photon partner*

After super-symmetrizing the matter content we next describe the photon in a supersymmetric Lagrangian. Starting with the photon with two on-shell degrees of freedom we expect another Weyl fermion to appear, the photino with two degrees of freedom. This is at least true if we pretend to not have learned anything from the appearance of the F field; we really expect the photino and another auxiliary field to match all off-shell degrees of freedom.

A good guess for the Lagrangian of the supersymmetric abelian $U(1)$ gauge sector is

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \lambda^\dagger \bar{\sigma}^\mu \partial_\mu \lambda} \qquad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \; . \tag{30}$$

The fermionic part completely corresponds to the Wess-Zumino Lagrangian Eq.(28). The difference between the electron and the photino is the charge: for QED the electron has a fundamental charge and the photino is neutral, while for QCD the quarks will carry fundamental color charge and the gluino adjoint color charge.

Next, we guess the SUSY transformations of the photon and the photino fields. The mass unit of the transformed photon field is $[\delta_\xi A^\mu] = M$, while on the left hand side we expect $[\xi] = M^{-1/2}$ and $[\lambda] = M^{3/2}$ to appear. This would allow us to write $\delta_\xi A^\mu \sim \xi \bar{\sigma}^\mu \lambda$, including the Lorentz index on the right hand side while avoiding ∂^μ and its mass unit. In addition, we need to ensure that A^μ is a real field, so we postulate

$$\boxed{\delta_\xi A^\mu = \xi^\dagger \bar{\sigma}^\mu \lambda + \text{h.c.} = \xi^\dagger \bar{\sigma}^\mu \lambda + \lambda^\dagger \bar{\sigma}^\mu \xi} \; . \tag{31}$$

We construct the SUSY transformation for the photino field in terms of the gauge-invariant field strength tensor appearing in the Lagrangian. Its mass dimension is $[F_{\mu\nu}] = M^2$. If the basic structure is $\delta_\xi \lambda \sim \xi F_{\mu\nu}$ we need to contract two Lorentz indices. A good guess, modulo a normalization constant, is

$$\delta_\xi \lambda = C \frac{i}{2} \sigma^\mu \bar{\sigma}^\nu \xi F_{\mu\nu} \qquad \text{and} \qquad \delta_\xi \lambda^\dagger = -C^* \frac{i}{2} \xi^\dagger \bar{\sigma}^\nu \sigma^\mu \xi F_{\mu\nu} \; . \tag{32}$$

As before, we can compute the SUSY transformation of the Lagrangian to adjust the normalization factor C

$$\begin{aligned}
\delta_\xi \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) &= -\frac{1}{4} \left((\delta_\xi F_{\mu\nu}) F^{\mu\nu} + F_{\mu\nu} (\delta_\xi F^{\mu\nu}) \right) \\
&= -\frac{1}{4} \left((\partial_\mu \delta_\xi A_\nu - \partial_\nu \delta_\xi A_\mu) F^{\mu\nu} + F_{\mu\nu} (\partial^\mu \delta_\xi A^\nu - \partial^\nu \delta_\xi A^\mu) \right) \\
&= -\frac{1}{2} (\partial_\mu \delta_\xi A_\nu - \partial_\nu \delta_\xi A_\mu) F^{\mu\nu} \\
&= -\frac{1}{2} \partial_\mu \delta_\xi A_\nu \; F^{\mu\nu} + \frac{1}{2} \partial_\nu \delta_\xi A_\mu \; F^{\mu\nu} \\
&= -\partial_\mu (\delta_\xi A_\nu) \; F^{\mu\nu} \qquad \qquad \text{with } F^{\mu\nu} = -F^{\nu\mu} \\
&= -F^{\mu\nu} \partial^\mu (\xi^\dagger \bar{\sigma}_\nu \lambda + \lambda^\dagger \bar{\sigma}_\nu \xi) \; .
\end{aligned} \tag{33}$$

The photino contribution gives us an additional contribution, only considering the part proportional to $\xi^\dagger$,

$$\begin{aligned}
\delta_\xi(i\lambda^\dagger\bar\sigma^\mu\partial_\mu\lambda)\Big|_{\xi^\dagger} =& i(\delta_\xi\lambda^\dagger)\bar\sigma^\mu\partial_\mu\lambda\Big|_{\xi^\dagger} + i\lambda^\dagger\bar\sigma^\mu\partial_\mu(\delta_\xi\lambda)\Big|_{\xi^\dagger} \\
=& \frac{C^*}{2}\xi^\dagger\bar\sigma^\nu\sigma^\mu F_{\mu\nu}\bar\sigma^\rho\partial_\rho\lambda \\
=& \frac{C^*}{2}\xi^\dagger(\bar\sigma^\nu\sigma^\mu\bar\sigma^\rho)F_{\mu\nu}\partial_\rho\lambda \\
=& \frac{C^*}{2}\xi^\dagger[g^{\nu\mu}\bar\sigma^\rho - g^{\nu\rho}\bar\sigma^\mu + g^{\mu\rho}\bar\sigma^\nu - i\epsilon^{\mu\nu\rho\sigma}\bar\sigma_\sigma]\,F_{\mu\nu}\partial_\rho\lambda \\
=& \frac{C^*}{2}\xi^\dagger[-g^{\nu\rho}\bar\sigma^\mu + g^{\mu\rho}\bar\sigma^\nu - i\epsilon^{\mu\nu\rho\sigma}\bar\sigma_\sigma]\,F_{\mu\nu}\partial_\rho\lambda \qquad \text{integration by parts } F_{\mu\nu}\partial_\rho\lambda \sim \lambda\partial_\rho F_{\mu\nu} \\
=& \frac{C^*}{2}\xi^\dagger[-g^{\nu\rho}\bar\sigma^\mu + g^{\mu\rho}\bar\sigma^\nu]\,F_{\mu\nu}\partial_\rho\lambda \qquad \text{using } \epsilon^{\mu\nu\rho\sigma}(\partial_\rho\partial_\mu A_\nu - \partial_\rho\partial_\nu A_\mu) = 0 \\
=& \frac{C^*}{2}\xi^\dagger[-\bar\sigma^\mu\partial^\nu\lambda + \bar\sigma^\nu\partial^\mu\lambda]F_{\mu\nu} \\
=& \frac{C^*}{2}\xi^\dagger[+\bar\sigma^\nu\partial^\mu\lambda + \bar\sigma^\nu\partial^\mu\lambda]F_{\mu\nu} \qquad \text{using again } F_{\mu\nu} = -F_{\nu\mu} \\
=& C^*\xi^\dagger\bar\sigma^\nu(\partial^\mu\lambda)\,F_{\mu\nu}\;,
\end{aligned} \tag{34}$$

which precisely cancels the $\xi^\dagger$ contribution from the photon's kinetic term in Eq.(33), provided $C^* = 1$,

$$\boxed{\delta_\xi\lambda = \frac{i}{2}\sigma^\mu\bar\sigma^\nu\xi F_{\mu\nu}}\;. \tag{35}$$

This way the non-interacting photon-photino Lagrangian is, just like the electron-selectron Lagrangian, supersymmetric in itself.

To confirm the SUSY algebra Eq.(59) we need to also compute $[\delta_\eta, \delta_\xi]$ acting on the photon and the photino fields. Without going into the details we quote the result that again the algebra does not hold when we apply it to the photino field λ. Remembering the off-shell degrees of freedom the photon as a general gauge boson indeed has three degrees of freedom while the Weyl spinor has four. We can solve this by introducing one additional bosonic degree of freedom through a real scalar field D. What should the SUSY transformation of this field look like? Based on the mass dimension $[D] = M^2$ we can first try the same transformation as for F shown in Eq.(21). The only additional modification needed is that the D field is real, so we have to form a hermitian SUSY variation of D,

$$\boxed{\delta_\xi D = -i\left(\xi^\dagger\bar\sigma^\mu(\partial_\mu\lambda) - (\partial_\mu\lambda)^\dagger\bar\sigma^\mu\xi\right)}\;. \tag{36}$$

The transformation of the photino field follows the example of the electron, where the auxiliary field F is replaced by D in Eq.(24),

$$\boxed{\delta_\xi\lambda = \frac{1}{2}\bar\sigma^\mu\bar\sigma^\nu\xi F_{\mu\nu} + \xi D}\;. \tag{37}$$

To the original photon–photino Lagrangian of Eq.(30) gets the expected D^2 correction for the real auxiliary field,

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda^\dagger\bar\sigma^\mu\partial_\mu\lambda + \frac{D^2}{2}}\;. \tag{38}$$

If we assume, as before, that the SUSY transformation of the photon field $F_{\mu\nu}$ does not change, the two new contributions to the SUSY transformation of the Lagrangian are

$$\begin{aligned}
\delta_\xi \left(\frac{1}{2}D^2\right) =& D\ \delta_\xi D \\
=& - iD\xi^\dagger\bar\sigma^\mu\partial_\mu\lambda + iD(\partial_\mu\lambda)^\dagger\bar\sigma^\mu\xi \\
\delta_\xi(\lambda^\dagger i\bar\sigma^\mu\partial_\mu\lambda)\Big|_D =& (\delta_\xi\lambda^\dagger)i\bar\sigma^\mu\partial_\mu\lambda\Big|_D + \lambda^\dagger i\bar\sigma^\mu\partial_\mu(\delta_\xi\lambda)\Big|_D \\
=& iD\xi^\dagger\bar\sigma^\mu\partial_\mu\lambda + i\lambda^\dagger\bar\sigma^\mu\partial_\mu(\xi D) \\
=& iD\xi^\dagger\bar\sigma^\mu\partial_\mu\lambda + i\lambda^\dagger\bar\sigma^\mu\xi\partial_\mu D \\
=& iD\xi^\dagger\bar\sigma^\mu\partial_\mu\lambda + i\partial_\mu(\lambda^\dagger\bar\sigma^\mu\xi D) - i(\partial_\mu\lambda^\dagger)\bar\sigma^\mu\xi D
\end{aligned} \tag{39}$$

The two contribution together form a total derivative. This means that The complete non-interacting photon-photino Lagrangian is supersymmetric and the three fields $\{\lambda, A_\mu, D\}$ form a closed multiplet under the SUSY algebra Eq.(59). As for the Wess–Zumino Lagrangian the auxiliary field at this stage would simply be removed through its equation of motion $D = 0$. However, we will see that this is not the final word.

4. *Supersymmetric QED*

In the last two sections we have derived individual Lagrangians for the electron-selectron multiplet and for the photon-photino multiplet. Combining the two we can build the Lagrangian of supersymmetric QED. The only thing we need to remember is how interactions appear through the covariant derivative

$$\partial^\mu \to D^\mu := \partial^\mu + iqA^\mu \ . \tag{40}$$

This construction ensures that the Lagrangian is gauge invariant. This has two aspects: first, replacing ∂_μ in the actual Lagrangian ensures gauge invariance of the Lagrangian defined as a combination of Eq.(28) and Eq.(38),

$$\boxed{\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) + i\chi^\dagger\bar\sigma^\mu D_\mu\chi + F^\dagger F^\dagger - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda^\dagger\bar\sigma^\mu\partial_\mu\lambda + \frac{D^2}{2}} \ . \tag{41}$$

Note the annoying double use of D, D^μ and $F, F^{\mu\nu}$ which we nevertheless keep because they are standard in the literature.

The covariant derivative appears in the kinetic term of the electrons just as in the usual QED, and it should therefore appear in the supersymmetric version for the selectrons. This way both matter states will couple to photons the same way. Because in QED the covariant derivative does not appear in the photon kinetic term, it should also not appear for the photino. Otherwise the photino would couple to the photon, which should only occur for a non-abelian gauge theory like supersymmetric QCD.
In addition, replacing ∂_μ in the SUSY transformations keeps the SUSY transformations to break gauge invariance, *i.e.* SUSY transformations and gauge transformations commute,

$$\begin{aligned}
&\delta_\xi\phi = -i\xi^T\sigma_2\chi && \delta_\xi A^\mu = \alpha\xi^\dagger\bar\sigma^\mu\lambda + \text{h.c.} \\
&\delta_\xi\chi = \sigma^\mu\sigma_2\xi^* D_\mu\phi + \xi F && \delta_\xi\lambda = \alpha\frac{i}{2}\sigma^\mu\bar\sigma^\nu\xi F_{\mu\nu} + \alpha\xi D \\
&\delta_\xi F = -i\xi^\dagger\bar\sigma^\mu D_\mu\chi && \delta_\xi D = -i\alpha\xi^\dagger\bar\sigma^\mu D_\mu\lambda + \text{h.c.}
\end{aligned} \tag{42}$$

The only change as compared to the original transformations is that we have shifted the SUSY transformation in the photon–photino Lagrangian by $\xi \to \alpha\xi$.

The Lagrangian Eq.(41) describes several interactions: the photon coupling to the selectron and the electron related by a SUSY transformation of the matter fields; a non-abelian gluon would also couple to the gluino, which is not the case for the photino. What we are missing in an electron-selectron-photino coupling for

which there is no reason to be absent. So the more general question becomes: what terms can we add to the Lagrangian in Eq.(41) while keeping the theory renormalizable and SUSY invariant?
When we think about ways to link the two Lagrangians consistent with our symmetries, two interaction terms with a combined mass dimension four come to mind: first, the electron–selectron–photon vertex mentioned above, with a massless charge Aq; and second, a selectron coupling to the auxiliary field in the photon multiplet

$$\mathcal{L} \to \mathcal{L} + Aq\left((\phi^\dagger\chi)\cdot\lambda + \text{h.c.}\right) + Bq\;\phi^\dagger\phi D\;. \tag{43}$$

Obviously, these interactions mix the two multiplets and hence mix the two transformations ξ and $\alpha\xi$. What we need to compute it the SUSY transformation of the additional terms in the Lagrangian Eq.(43). Terms appearing include

– $A\;\delta_\xi\lambda \supset A\;\xi D$ giving a B-type contribution;

– $B\;\delta_\xi D \supset \alpha B\;\xi^\dagger\bar{\sigma}^\mu D_\mu\lambda$ giving an A-type contribution;

– $B\;\delta_\xi\phi \supset B\;\xi^T\chi$ giving an A-type contribution.

Skipping the details we quote that we would find $A\alpha = -B$ from all B-type contributions involving D. In addition, $A = -2\alpha$ follows from the A-type contributions plus the $\chi\chi A^\mu$ term in original Lagrangian. Last but not least, we find $B = -1$ once we induce $\phi\phi A^\mu$ in original Lagrangian. All constraints combined give us for the remaining free parameters

$$A = -\sqrt{2} \qquad\qquad B = -1 \qquad\qquad \alpha = -\frac{1}{\sqrt{2}}\;. \tag{44}$$

The charge q is fixed by $D^\mu = \partial^\mu + iqA^\mu$ and hence the electric charge of ϕ, χ, λ. The complete supersymmetric QED Lagrangian reads

$$\mathcal{L}_{\text{SQED}} = (D_\mu\phi)^\dagger(D^\mu\phi) + i\chi^\dagger\bar{\sigma}^\mu D_\mu\chi + F^\dagger F - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda^\dagger\bar{\sigma}^\mu D_\mu\lambda + \frac{D^2}{2} - \sqrt{2}q\left[(\phi^\dagger\chi)\cdot\lambda + \text{h.c.}\right] - q\phi^\dagger\phi D\;, \tag{45}$$

with the scalar electron ϕ, the Weyl-fermion electron χ, the photon A^μ, and the photino λ. One thing to remember at this stage: the electron-selectron-photino coupling is formulated in terms of the left-handed Weyl spinor and the corresponding scalar field. This structure will not change once we include another, right-handed, Weyl spinor to construct the Dirac spinor of a Standard Model fermion. Therefore, this coupling defines something like 'left-handed' or 'right-handed' scalar partners as those scalars which couple to the two chiralities of the Standard Model fermion.

5. *Scalar interactions*

The Lagrangian in Eq.(45) is complete and has all the symmetries we require, but it still includes two auxiliary fields. Each of them can be removed by its equation of motion,

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial F} &= F^\dagger = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu F^\dagger)}\right) = 0 \qquad\qquad \Leftrightarrow \qquad F = 0\\
\frac{\partial\mathcal{L}}{\partial D} &= D - q\phi^\dagger\phi = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu D^\dagger)}\right) = 0 \qquad \Leftrightarrow \qquad D = q\phi^\dagger\phi
\end{aligned} \tag{46}$$

First, we replace D, inducing four-scalar interactions proportional to the gauge coupling q. They are called <u>D-terms</u> and read

$$\mathcal{L}_{\text{SQED}} \supset \frac{D^2}{2} - q\phi^\dagger\phi D = \frac{q^2}{2}(\phi^\dagger\phi)(\phi^\dagger\phi) = q^2(\phi^\dagger\phi)(\phi^\dagger\phi) = -\frac{1}{2}q^2(\phi^\dagger\phi)^2 \tag{47}$$

$$\mathcal{L}_{\text{SQED}} = (D_\mu\phi)^\dagger(D^\mu\phi) + i\chi^\dagger\bar{\sigma}^\mu D_\mu\chi + F^\dagger F - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda^\dagger\bar{\sigma}^\mu D_\mu\lambda - \sqrt{2}q\left[(\phi^\dagger\chi)\cdot\lambda + \text{h.c.}\right] - \frac{1}{2}q^2(\phi^\dagger\phi)^2\;.$$

Second, neither the supersymmetric QED Lagrangian nor the Wess-Zumino model of Eq.(28) make sense of an auxiliary F field with a constant value zero. What we need is additional occurrences of F in the Lagrangian, which will lead to a non-trivial equation of motion. In our search for supersymmetric and renormalizable terms we can add a general function of the scalar field $W^{(1)}(\phi)$ to the Wess-Zumino Lagrangian, in complete analogy to Eq.(43),

$$\mathcal{L}_{\rm WZ} \to \mathcal{L}_{\rm WZ} + \left(W^{(1)} F + \text{h.c.} \right) \tag{48}$$

This is the same spirit as adding a scalar Higgs potential to the Standard Model Lagrangian. The only condition from renormalizability is that the mass dimension of $W^{(1)}(\phi)$ has to be two. This gives rise to a non-trivial equation of motion for F

$$\frac{\partial \mathcal{L}}{\partial F} = 0 \qquad \Leftrightarrow \qquad F^\dagger = -W^{(1)} \quad \text{and} \quad F = -W^\dagger \; . \tag{49}$$

The Wess–Zumino Lagrangian now reads

$$\mathcal{L}_{\rm WZ} \to \; \mathcal{L}_{\rm WZ} + |W^{(1)}|^2 \; . \tag{50}$$

If we only allow for one set of matter fields, that really is the end of the story, provided that the general potential $W^{(1)}(\phi)$ does not spoil the supersymmetry.
Finally (really!), introducing a general scalar potential should remind us of electroweak symmetry breaking in the Standard Model. If we want to consider massive fermions, Eq.(6) tells us that we need to allow for two Weyl spinors for the left-handed or right-handed electrons. Giving the corresponding Weyl fermions an index, the Wess–Zumino Lagrangian with a generalized electron mass term becomes

$$\mathcal{L}_{\rm WZ} \to \mathcal{L}_{\rm WZ} + |W_i^{(1)}|^2 - \frac{1}{2} \left(W_{ij}^{(2)} \chi_i^\dagger \chi_j + \text{h.c.} \right) \tag{51}$$

To be consistent, we also give the potential $W^{(1)}(\phi)$ and index. With both of these new terms we again have to ensure that the Wess–Zumino Lagrangian is supersymmetric, $\delta_\xi \mathcal{L}_{\rm WZ} = 0$. Skipping the derivation we quote that this translates into the following requirements

1. $\partial W_{ij}^{(2)} / \partial \phi_k$ has to be symmetric in the three possible indices (i, j, k);
2. $\partial W_{ij}^{(2)} / \partial \phi_k^\dagger = 0$, which means we cannot use the conjugate scalar field in $W_{ij}^{(2)}(\phi)$;
3. the symmetry of the Lagrangian under SUSY transformations can be restored by requiring a single source $W(\phi)$ of the two functions of the scalar fields, namely

$$W_i^{(1)} = \frac{\partial W}{\partial \phi_i} \qquad \text{and} \qquad W_{ij}^{(2)} = \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \; . \tag{52}$$

Skipping many of the intermediate steps this gives us

$$\begin{aligned}
\delta_\xi \mathcal{L} =& - i W_{ij}^{(2)} \, \xi^\dagger \bar{\sigma}^\mu \chi_i \partial_\mu \phi_j - i W_i^{(1)} \, \xi^\dagger \bar{\sigma}^\mu (\partial_\mu \chi_i) \\
=& - i \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \xi^\dagger \bar{\sigma}^\mu \chi_i \partial_\mu \phi_j - i \frac{\partial W}{\partial \phi_i} \xi^\dagger \bar{\sigma}^\mu (\partial_\mu \chi_i) \\
=& - i \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \partial_\mu \phi_j \, \xi^\dagger \bar{\sigma}^\mu \chi_i + i \partial_\mu \left(\frac{\partial W}{\partial \phi_i} \right) \xi^\dagger \bar{\sigma}^\mu \chi_i \qquad \text{after integrating by parts} \\
=& i \left[- \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \partial_\mu \phi_j + \partial_\mu \frac{\partial W}{\partial \phi_i} \right] \xi^\dagger \bar{\sigma}^\mu \chi_i \\
=& i \left[- \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \partial_\mu \phi_j + \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \partial_\mu \phi_j \right] \xi^\dagger \bar{\sigma}^\mu \chi_i = 0 \; .
\end{aligned} \tag{53}$$

If the scalar potentials should be renormalizable, a power series in the fields ϕ_j, and fulfill the above conditions, we can translate the general form of Eq.(52) into

$$\begin{aligned} W &= \frac{m_{ij}}{2}\phi_1\phi_j + \frac{y_{ijk}}{6}\phi_i\phi_j\phi_k \\ \Rightarrow \qquad W_i^{(1)} &= m_{ij}\phi_j + \frac{1}{2}y_{ijk}\phi_j\phi_k \\ \Rightarrow \qquad W_{ij}^{(2)} &= m_{ij} + y_{ijk}\phi_k \end{aligned} \tag{54}$$

These scalar potentials $W^{(1)}$ and $W_{ij}^{(2)}$ give rise to a multitude of effects linked to the mass/Yukawa coupling of the fermion χ,

$|W_i^{(1)}|^2$: ϕ mass term m_{ij}^2

$|W_i^{(1)}|^2$: ϕ^3 coupling $m_{ij}y_{ijk}$

$|W_i^{(1)}|^2$: ϕ^4 coupling $y_{ijk}y_{ijk}$

$W_{ij}^{(2)}\chi_i\chi_j$: χ mass term m_{ij}

$W_{ij}^{(2)}\chi_i\chi_j$: $\phi\chi^2$ coupling y_{ijk}

These F-term contributions ensure that the scalar ϕ and fermion χ masses match as well as the self couplings ϕ^3 and ϕ^4 and the $\phi\chi\chi$ Yukawa couplings match. All couplings are proportional to the particle masses and come in addition to the gauge-sector D-terms. We can safely assume that the many scalar interactions will make the corresponding supersymmetry calculations lengthy. On the other hand, we will see below that the F-terms and D-terms are crucial to understand the structure of the MSSM Higgs sector.**to here**

B. Superfields

One problem with this derivation of the supersymmetric Lagrangian is that the algebra of SUSY generators drops on our heads out of nowhere. What we found is that the matter (chiral) and photon (vector) multiplets closed under

$$\boxed{[\delta_\eta, \delta_\xi] = i\eta^\dagger\bar{\sigma}^\mu\xi^*\partial_\mu + \text{h.c.}} \; . \tag{55}$$

From quantum mechanics we know that in position space the derivative on the right hand side corresponds to the momentum operator. We can try to write the known SUSY transformation $\phi \to \phi + \delta_\xi\phi$ in terms of SUSY generators Q,

$$(1 + i\xi\cdot Q)\,\phi\,(1 - i\xi\cdot Q) = \phi + i\xi\cdot Q\phi - i\phi\xi\cdot Q = \phi + i[\xi\cdot Q, \phi] \qquad \text{with} \quad \xi\cdot Q \equiv -i\xi^T\sigma_2 Q \; . \tag{56}$$

If we identify this with a Lorentz-invariant infinitesimal transformation defined by the SUSY generator Q and the spinor-valued shift parameter ξ we find

$$\delta_\xi\phi = -i\xi^T\sigma_2\chi \stackrel{?}{=} i[\xi\cdot Q, \phi] \; . \tag{57}$$

This is not yet the complete form. From the calculations we did until now we remember that there are two independent SUSY shifts, ξ and ξ^*, which means we should include a second commutator $\bar{\xi}\cdot\bar{Q} \equiv \xi^\dagger i\sigma_2 Q^*$ and arrive at

$$\delta_\xi\phi \stackrel{?}{=} i[\xi\cdot Q + \bar{\xi}\cdot\bar{Q}, \phi] \; . \tag{58}$$

In terms of these SUSY generators we can let the SUSY algebra fall on our heads and at least get a rough idea what it means

$$\begin{aligned}\{Q_a, Q_b\} =&0\\ \{Q_a^\dagger, Q_b^\dagger\} =&0\\ \{Q_a, Q_b^\dagger\} =&(\sigma^\mu)_{ab} P_\mu \; .\end{aligned} \tag{59}$$

The SUSY generators just like fermion field operators anti-commute, and we replace $i\partial_\mu \leftrightarrow P_\mu$.

Beyond infinitesimal transformations we can interpret ξ as a coordinate θ of a superspace (x, θ, θ^*), made out of space-time and two additional Grassmann directions. A superspace transformation of a superfield $\Phi(x, \theta, \theta^*)$, which we still have to define, reads

$$\Phi(x,\theta,\theta^*) = U(x,\theta,\theta^*) \; \Phi(0,0,0) \; U^{-1}(x,\theta,\theta^*) \qquad\qquad \text{with} \quad U(x,\theta,\theta^*) = e^{ix\cdot P} e^{i\theta\cdot Q} e^{i\bar\theta\cdot\bar Q} \; . \tag{60}$$

The generalized finite unitary transformations includes the proper definitions of $x \cdot P$, $\theta \cdot Q$, and $\bar\theta \cdot \bar Q$. Two successive superspace transformations are defined as $U(a,\xi,\xi^*) U(x,\theta,\theta^*) \; \Phi(0) \; U^{-1}(x,\theta,\theta^*) U^{-1}(a,\xi,\xi^*)$ and can be calculated using the Baker-Campbell-Hausdorff formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{6}[[A,B]B]+\cdots} \; . \tag{61}$$

Our SUSY algebra from Eq.(59) gives for this repeated transformation

$$U(a,\xi,\xi^*) U(x,\theta,\theta^*) \; \Phi(0) \; U^{-1}(x,\xi,\xi^*) U^{-1}(a,\theta,\theta^*) = \Phi(x + a - i\theta\sigma^\mu\xi^*, \theta+\xi, \theta^*+\xi^*) \tag{62}$$

with an additional superspace shift $0 \to x^\mu \to x^\mu + a^\mu - i\theta\sigma^\mu\xi^*$.

Defining superspace and superfields living in this space is certainly a compelling concept. On the other hand, the question is if we can make use of them. Because θ, θ^* are Grassmann variables, *i.e.* for their components there is $(\theta_1)^2 = 0 = (\theta_2)^2$, we can expand any superfield as a finite Taylor series. For illustration purposes we again focus on one of the two Grassmann directions,

$$\Phi(x,\theta) = \Phi_0(x) + \theta\cdot\Phi_1(x) + \frac{\theta\cdot\theta}{2}\,\Phi_2(x) \tag{63}$$

In that approximation the superspace shift reads

$$\begin{aligned}\delta_\xi\Phi(x,\theta) =& - i\theta\sigma^\mu\xi^*\partial_\mu\Phi(x,\theta) + \xi\cdot\frac{\partial}{\partial\theta}\Phi(x,\theta)\\ =& - i\theta\sigma^\mu\xi^* \; \partial_\mu\left(\Phi_0 + \theta\cdot\Phi_1 + \frac{\theta\cdot\theta}{2}\,\Phi_2\right) + \xi^a\frac{\partial}{\partial\theta_a}\left(\Phi_0 + \theta\cdot\Phi_1 + \frac{\theta\cdot\theta}{2}\,\Phi_2\right)\\ =& - i\theta\sigma^\mu\xi^* \; \left(\partial_\mu\Phi_0 + \theta\cdot\partial_\mu\Phi_1 + \frac{\theta\cdot\theta}{2}\partial_\mu\Phi_2\right) + \xi\cdot\Phi_1 + (\xi\cdot\theta)\,\Phi_2\\ =&\xi\cdot\Phi_1 - i\theta\sigma^\mu\xi^*\partial_\mu\Phi_0 + (\xi\cdot\theta)\,\Phi_2 - i\theta\sigma^\mu\xi^*(\theta\cdot\partial_\mu\Phi_1) \qquad\qquad \text{with Grassmann components}\,\theta_a\\ \stackrel{!}{=}&(\delta_\xi\Phi)_0 + \theta\cdot(\delta_\xi\Phi)_1 + \frac{\theta\cdot\theta}{2}\,(\delta_\xi\Phi)_2 \; .\end{aligned} \tag{64}$$

Term by term in powers of θ we can identify these two results, giving us

$$\begin{aligned}\delta_\xi\Phi_0 &= \xi\cdot\Phi_1\\ \theta\cdot\delta_\xi\Phi_1 &= -i\theta\sigma^\mu\xi^*\partial_\mu\Phi_0 + (\xi\cdot\theta)\Phi_2 = \theta\cdot\sigma^\mu\sigma_2\xi^*\partial_\mu\Phi_0 + (\theta\cdot\xi)\Phi_2\\ (\theta\cdot\theta)\,\delta_\xi\Phi_2 &= -i\theta\sigma^\mu\xi^*(\theta\cdot\partial_\mu\Phi_1) = -(\theta\cdot\theta)\,i\xi^\dagger\bar\sigma^\mu\partial_\mu\Phi_1 \qquad \text{with}\;\sigma_2\sigma^\mu\sigma_2 = \bar\sigma^{\mu T}\end{aligned} \tag{65}$$

We we define a chiral, matter superfield as

$$\Phi(x,\theta) = \phi(x) + \theta\cdot\chi(x) + \frac{\theta\cdot\theta}{2}F(x) \tag{66}$$

This returns our individual transformations of Eq.(22), Eq.(24), and Eq.(21),

$$\begin{aligned}\delta_\xi \phi &= \xi \cdot \chi \\ \delta_\xi \chi &= \sigma^\mu \sigma_2 \xi^* \partial_\mu \phi + \xi F \\ \delta_\xi F &= -i \xi^\dagger \bar{\sigma}^\mu \partial_\mu \chi \ ,\end{aligned} \tag{67}$$

but in a much more elegant manner.

C. The MSSM

One aspect of supersymmetric QED we have neglected until now is fermion masses. Our current Lagrangian only includes a right-handed electron field, according to Eq.(6) not enough to give mass to the electron. Note that this does not mean that our scalar electron is massless. Even though this is really a trivial change we still need to include a left-handed electron ψ and its scalar partner ϕ_L. Our theory then contains a Dirac fermion $e^\pm \equiv \Psi = (\psi, \chi)$ and a set of left and right scalar electrons $\tilde{e}_{L,R}$. The vector field A^μ is the photon, so the only question left is what to do with the one remaining Weyl spinor λ. We can construct a four spinor out of λ and $-i\sigma_2 \lambda^*$

$$\Psi_M = \begin{pmatrix} \lambda \\ -i\sigma_2 \lambda^* \end{pmatrix} \ . \tag{68}$$

It has the correct Lorentz transformation properties and under charge conjugation gets mapped on itself. This means that the photino, a Majorana fermion, is its own anti-particle. Just a side remark concerning charges of Majorana spinors: for example in the kinetic term in a hermitian Lagrangian we see that they have to be real. This is possible for example for the adjoint representation of $SU(N)$.

As discussed already, there are two multiplets which form the Lagrangian of supersymmetric QCD. The set of (chiral) multiplets containing all matter particles in the MSSM are, with their charges

scalar	fermion	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$
$(\tilde{u}_L, \tilde{d}_L)$	(u_L, d_L)	$\underline{3}$	$\underline{2}$	$1/3$
$\tilde{u}_R$	u_R^C	$\underline{\bar{3}}$	$\underline{1}$	$-4/3$
$\tilde{d}_R$	d_R^C	$\underline{\bar{3}}$	$\underline{1}$	$2/3$
(H_u^+, H_u^0)	$(\tilde{H}_u^+, \tilde{H}_u^0)$	$\underline{1}$	$\underline{2}$	1
(H_d^0, H_d^-)	$(\tilde{H}_d^0, \tilde{H}_d^-)$	$\underline{1}$	$\underline{2}$	-1

Remember that the Yukawa couplings arise from $W_{ij}^{(2)}$ and supersymmetry explicitly requires $\partial W_{ij}^{(2)} / \partial \phi_2^\dagger = 0$. This is once reason why we need two Higgs doublets to give mass to up-type and down-type fermions, instead of H and $H^\dagger$ in the Standard Model. Because of the supersymmetry the masses of the fermion and the two scalar fermions are identical.

The second kind of multiplet are gauge multiplets which include interaction fields and their fermionic partners

fermion	boson	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$
$\tilde{g}$	g	$\underline{8}$	$\underline{1}$	0
$\tilde{W}^\pm, \tilde{W}^0$	$\tilde{W}^\pm W^0$	$\underline{1}$	$\underline{3}$	0
$\tilde{B}$	B	$\underline{1}$	$\underline{1}$	0

The fact that the Goldstone modes in the Higgs doublets are absorbed by the massive weak gauge fields $W^\pm$ and Z^0 indicates that there is no reason why the weak gauginos and the Higgsinos should not mix. The two mass matrices for the neutralinos and charginos then read

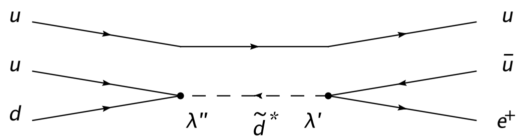
$$\begin{pmatrix} m_{\tilde{B}} & 0 & -c_\beta s_w m_Z & +s_\beta s_w m_Z \\ 0 & m_{\tilde{W}} & +c_\beta c_w m_Z & -s_\beta c_w m_Z \\ & & 0 & -\mu \\ \text{symmetric} & & -\mu & 0 \end{pmatrix} \qquad \begin{pmatrix} m_{\tilde{W}^\pm} & \sqrt{2} s_\beta m_W \\ \sqrt{2} c_\beta m_W & \mu \end{pmatrix} \tag{69}$$

with $s_\beta = \sin\beta$ and $c_\beta = \cos\beta$. The neutralino mass matrix is symmetric because if the Majorana nature of the states.

A key ingredient of the MSSM, which we have never discussed in the construction of the Lagrangian, is R parity. It implies that ever term in the Lagrangian includes an even power of supersymmetric partner fields. We need it because if we *really* carefully search for gauge invariant, renormalizable and perfectly allowed terms in the Lagrangian we will find superpotential terms of the kind

$$W = \lambda_{ijk} L_i L_j \bar{e}_K + \lambda'_{ijk} L_i Q_j \bar{d}_k + \lambda''_{ijk} \bar{u}_i \bar{d}_j \bar{d}_k \tag{70}$$

involving quark and lepton fields. Proton decay mediated through these renormalizable interactions



is about the last thing we want, because the proton life time is experimentally constrained to exceed $\sim\ 10^{16}$ GeV What we need is a suppression of well over 10 orders of magnitude, so the decay operator should include at least one factor $1/M_{\text{GUT}}$ or $1/M_{\text{Planck}}$. This is why we upgrade R parity to a global symmetry, based on the charges

$$R = \begin{cases} +1 & \text{SM particles and 2HDM} \\ -1 & \text{SUSY partners} \end{cases} \qquad \text{or} \qquad \boxed{R = (-1)^{3B+L+2S}} \tag{71}$$

in terms of the baryon number, the lepton number, and the spin of the particle. It forbids all interactions shown in Eq.(70). This parity has another extremely useful effect: it does not allow the lightest supersymmetric particle (LSP) to decay into two Standard Model states. If this LSP is stable and weakly interacting, it can serve as a candidate for dark matter. The R parity can be upgraded to an R symmetry, but at the expense of adding a significant number of degrees of freedom.

Last, but not least we need to talk about SUSY breaking. Unless we do something about it, SUSY predicts new scalar particles with a mass of $m_{\tilde{e}} = m_e = 511$ keV, which again is experimentally excluded. Going back to the original motivation of supersymmetry we can precisely analyze what type of SUSY-breaking terms in the Lagrangian are allowed without ruining the cancellation of quadratic divergences in the Higgs mass. These terms are called soft breaking terms, and while they lead to logarithmic divergences, they by definition do not re-introduce quadratic divergences. They include *e.g.* scalar masses, fermion masses, and trilinear scalar terms. Instead of discussing SUSY breaking here, we refer to a great review on the topic by Steve Martin, the famous supersymmetry primer hep-ph/9709356.
There, we can find careful discussion on different SUSY breaking mechanisms, or better different ways to communicate SUSY breaking from a hidden sector into the visible sector. They include

– gravity mediation (mSUGRA)

– gauge mediation (GMSB)

– anomaly mediation (AMSB) in extra dimensions

– gaugino mediation ($\tilde{\text{G}}$MSB) in extra dimensions

...

D. MSSM Higgs Sector

A major supersymmetric modification of the Standard Model particles and interactions happens in the Higgs sector, without involving actual supersymmetric partners. Here, the MSSM introduced a set of new states with

SM-like R parity. The reason is that supersymmetry requires $\partial W^{(2)}_{ij}/\partial\phi^\dagger_k = 0$. Our scalar potential

$$W^{(2)}_i = \partial W/\partial\phi_i = m_{ij}\phi_j + y_{ijk}\phi_j\phi_k/2 \tag{72}$$

then forces us to introduce two Higgs doublets to introduce masses for up-type and down-type fermions. From the Standard Model Higgs sector we know that the quartic Higgs self coupling is crucial to form the Higgs potential. In supersymmetry we can make use of three different sources of scalar self interactions in the Lagrangian. First, there are the F terms from the SUSY-conserving scalar potential introduced in Section I A 5

$$\begin{aligned} W \supset \mu\, H_u D_d \qquad \text{for}\ H_u &= \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix} \qquad H_d = \begin{pmatrix} H_d^0 \\ H_{\bar{d}} \end{pmatrix} \\ \mathcal{L}_W &= |\mu|^2 \left(|H_u^+|^2 + |H_d^-|^2 + |H_u^0|^2 + |H_d^0|^2 \right) \end{aligned} \tag{73}$$

In addition, we can use the gauge-coupling mediated SUSY-conserving D terms introduced at the end of Section I A 3. For the Higgs sector they involve the abelian $U(1)$ terms $D = g\phi^+\phi$ as well as the non-abelian $SU(2)$ terms $D^\alpha = g' \sum_i \phi_i^+ T^\alpha \phi_i$ with the Pauli matrices as generators T^α

$$\begin{aligned} \mathcal{L}_D =& \frac{g^2}{8}\left[\left((|H_u^+ + |H_u^0|^2) - \left(|H_d^-|^2 + |H_d^0|^2 \right) \right)^2 + 4|H_u^+ H_d^0 + H_u^0 H_d^-|^2 \right] \\ &+ \frac{g'^2}{8}\left[\left(|H_u^+|^2 + |H_u^0|^2 \right) - \left(|H_d^-|^2 + |H_d^0|^2 \right) \right]^2 . \end{aligned} \tag{74}$$

Last, but not least we can use scalar masses and self couplings as part of the set of soft SUSY breaking terms

$$\mathcal{L}_{\text{soft}} = -m^2_{H_u}\left(|H_u^+|^2 + |H_u^0|^2 \right) - m^2_{H_d}\left(|H_d^-|^2 + |H_d^0|^2 \right) + b\left(H_u^+ H_d^- - H_u^0 H_d^0 + \text{h.c.} \right) . \tag{75}$$

All these terms we can collect into the Higgs potential for a two Higgs doublet model

$$\begin{aligned} V =& \left(|\mu|^2 + m^2_{H_u} \right)\left(|H_u^+|^2 + |H_u^0|^2 \right) + \left(|\mu|^2 + m^2_{H_d} \right)\left(|H_d^0|^2 + |H_d^-|^2 \right) + b\left(H_u^+ H_d^- - H_u^0 H_d^0 + \text{h.c.} \right) \\ &+ \frac{g^2 + g'^2}{8}\left(|H_u^+|^2 + |H_u^0|^2 - |H_d^-|^2 - |H_d^0|^2 \right)^2 + \frac{g^2}{2}|H_u^+ H_d^0 + H_u^0 H_d^-|^2 \end{aligned} \tag{76}$$

Because now we have two Higgs doublets to play with we can first rotate them independently. First, we choose $H_u^+ = 0$ at the minimum of V, *i.e.* at the point given by $\partial V/\partial H_u^+ = 0$. This translates into the condition

$$bH_d^- + g^2 H_d^0 H_u^0 H_d^- = \left(b + g^2 H_d^0 H_u^0 \right)\ H_d^- = 0\ , \tag{77}$$

where the terms quadratic in H_u^+ do not contribute. One way to fulfill this condition is by requiring the term in parentheses to vanish. For the term proportional to b in V this means

$$V \supset b\left(H_u^+ H_d^- - H_u^0 H_d^0 + \text{h.c.} \right) \stackrel{\text{min}}{=} -2bH_u^0 H_d^0 = 2g^2 |H_u^0|^2 |H_d^0|^2 > 0\ , \tag{78}$$

independent of the sign of b. This does not help to find a minimum for the potential. The second way to fulfill Eq.(77) is to require $H_d^- = 0$, which gives us a minimum value of

$$\begin{aligned} V &= \left(|\mu|^2 + m^2_{H_u} \right)|H_u^0|^2 + \left(|\mu|^2 + m^2_{H_d} \right)|H_d^0|^2 - b\left(H_u^0 H_d^0 + \text{h.c.} \right) + \frac{g^2 + g'^2}{8}\left(|H_u^0|^2 - |H_d^0|^2 \right)^2 \\ &= \left(|\mu|^2 + m^2_{H_u} \right)|H_u^0|^2 + \left(|\mu|^2 + m^2_{H_d} \right)|H_d^0|^2 - 2b|H_u^0||H_d^0| + \frac{g^2 + g'^2}{8}\left(|H_u^0|^2 - |H_d^0|^2 \right)^2 , \end{aligned} \tag{79}$$

where we absorb the phase of b into a rotation of $H_d^0 H_u^0$. At the minimum, the entire b term then becomes real. One special direction of the potential is given by $|H_u^0| = |H_d^0| \equiv |H^0|$ which reduces the potential value at the minimum to

$$V = \left(2|\mu|^2 + m^2_{H_u} + m^2_{H_d} - 2b \right)\ |H^0|^2\ . \tag{80}$$

The condition that the potential be bounded from below for electroweak symmetry breaking then gives

$$\boxed{2|\mu|^2 + m_{H_u}^2 + m_{H_u}^2 > 2b} \; . \tag{81}$$

Aside from the pure existence of a minimum in the potential we need to ensure that this minimum is stationary in both directions

$$\begin{aligned} 0 \stackrel{!}{=} \left. \frac{\partial V}{\partial |H_{u,d}^0|} \right|_{|H_i^0|=v_i} &= 2\left(|\mu|^2 + m_{H_{u,d}}^2\right)|H_{u,d}^0| - 2b|H_d^0| + \left. \frac{g^2+g'^2}{4}\left(|H_{u,d}^0|^2 - |H_d^0|^2\right) 2|H_{u,d}^0| \right|_{|H_i^0|=v_i} \\ \Leftrightarrow \qquad |\mu|^2 + m_{H_{u,d}}^2 &= b + \frac{g^2+g'^2}{4}\left(v_{d,u}^2 - v_{u,d}^2\right) \end{aligned} \tag{82}$$

From the Standard Model Higgs sector we know that we can replace the gauge couplings squared by the gauge boson masses

$$m_Z^2 = \frac{g^2+g'^2}{2}\left(v_u^2+v_d^2\right) \qquad\qquad m_W^2 = \frac{g^2}{2}\left(v_u^2+v_d^2\right) \tag{83}$$

and define

$$\boxed{\tan\beta = \frac{v_u}{v_d}} \qquad \text{or} \quad v_u = v\sin\beta \qquad \text{and} \quad v_d = v\cos\beta \tag{84}$$

with the usual $v = 246$ GeV. The stability conditions Eq.(82) then read

$$|\mu|^2 + m_{H_u}^2 = b\ \cot\beta + \frac{m_Z^2}{2}\cot 2\beta \qquad\qquad |\mu|^2 + m_{H_d}^2 = b\ \tan\beta - \frac{m_Z^2}{2}\cos 2\beta \; . \tag{85}$$

This relation fixes for example b, but we will for now keep it to shorten our calculations and results.

Going back to the physics Higgs fields we start by counting the degrees of freedom in two Higgs doublets and their functions in breaking electroweak symmetry

$$\begin{array}{cc} \begin{array}{ccc} & \text{physical } H^+ \quad\searrow & \swarrow\quad \text{long. } W^+ \\ \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix} = & \begin{pmatrix} \text{Re}H_u^+ + i\,\text{Im}H_u^+ \\ v_u + \text{Re}H_u^0 + i\,\text{Im}H_u^0 \end{pmatrix} & \\ & \text{physical } h^0, H^0 \quad\nearrow & \nwarrow\quad \text{long. } W_3^0 \&\text{physical } A^0 \end{array} & \begin{array}{ccc} & \text{physical } h^0, H^0 \quad\searrow & \swarrow\quad \text{long. } W_3^0 \&\text{physical } A^0 \\ \begin{pmatrix} H_d^+ \\ H_d^- \end{pmatrix} = & \begin{pmatrix} v_d + \text{Re}H_d^0 + i\,\text{Im}H_d^0 \\ \text{Re}H_d^- + i\,\text{Im}H_d^- \end{pmatrix} & \\ & \text{physical } H^- \quad\nearrow & \nwarrow\quad \text{long. } W^- \end{array} \end{array} \tag{86}$$

As expected, we have three Goldstone modes, $\text{Im}H_u^+$, $\text{Im}H_d^-$, and one linear combination of $\text{Im}H_u^0$ and $\text{Im}H_d^0$, giving mass to the $W^\pm$ and the Z^0 boson. The remaining five degrees of freedom form scalar particles, one charged Higgs boson $H^\pm$, two neutral CP-even Higgs bosons H_u^0, H_d^0 mixing into the mass eigenstates h^0 and H^0, and a pseudo-scalar Higgs boson A^0 from the remaining imaginary part. The masses of the physical modes are given by

$$2m_i^2 = \left. \frac{\partial^2 V}{\partial |H_i|^2} \right|_{|H_j^0|=v_j} \; . \tag{87}$$

As an illustration, let us first compute pseudoscalar mass m_{A^0}

$$
\begin{aligned}
V = & \left(|\mu|^2 + m_{H_u}^2\right) (\mathrm{Im}H_u^0)^2 + \left(|\mu|^2 + m_{H_d}^2\right) (\mathrm{Im}H_d^0)^2 + 2b \;\mathrm{Im}H_u^0 \;\mathrm{Im}H_d^0 \\
& + \frac{g^2+g'^2}{8} \left[(\mathrm{Re}H_u^0)^2 + (\mathrm{Im}H_u^0)^2 - (\mathrm{Re}H_d^0)^2 - (\mathrm{Im}H_d^0)^2\right]^2 \\
\frac{\partial V}{\partial(\mathrm{Im}H_u^0)} = & 2\left(|\mu|^2 + m_{H_u}^2\right) \mathrm{Im}H_u^0 + 2b \;\mathrm{Im}H_d^0 + \frac{g^2+g'^2}{8} 2 \left[\dots\right] 2 \;\mathrm{Im}H_u^0 \\
\frac{\partial^2 V}{\partial(\mathrm{Im}H_u^0)^2} = & 2\left(|\mu|^2 + m_{H_u}^2\right) + \frac{g^2+g'^2}{2}\left[\dots\right] + \frac{g^2+g'^2}{2} \mathrm{Im}H_u^0 \; 2 \;\mathrm{Im}H_u^0 \\
\left.\frac{\partial^2 V}{\partial(\mathrm{Im}H_u^0)^2}\right|_{\mathrm{vev}} = & 2\left(|\mu|^2 + m_{H_u}^2\right) + \frac{g^2+g'^2}{2}\left(v_u^2 - v_d^2\right) \\
= & 2b\cot\beta \qquad \text{using Eq.(85)}
\end{aligned}
\tag{88}
$$

In complete analogy, we find $\partial^2 V/\partial(\mathrm{Im}H_d^0)^2 = 2b\tan\beta$ in the minimum of the potential. The mixed double derivative we can immediately read off as $\partial^2 V/\partial(\mathrm{Im}H_d^0)/\partial(\mathrm{Im}H_u^0) = b$. This means that there will be a mass matrix for the two CP-odd Higgs and Goldstone modes

$$
\mathcal{M}_A^2 = b \begin{pmatrix} \cot\beta & 1 \\ 1 & \tan\beta \end{pmatrix} \qquad \text{with the eigenvalues} \qquad \begin{cases} m_{G^0}^2 = 0 & \text{Goldstone in } Z^0 \\ m_{A^0}^2 = \dfrac{2b}{\sin 2\beta} & \text{physical } A^0 \end{cases}
$$

The mixing angle between these two Goldstone/Higgs modes is given by β, the ratio of the two vacuum expectation values contributing *e.g.* to m_Z.

The same thing we can do the two CP-even scalar Higgs bosons, starting with $\mathrm{Re}H_u^0$

$$
\begin{aligned}
V \supset & \left(|\mu|^2 + m_{H_u}^2\right) (\mathrm{Re}H_u^0)^2 - 2b \;\mathrm{Re}H_u^0 \;\mathrm{Re}H_d^0 + \frac{g^2+g'^2}{8} \left[(\mathrm{Re}H_u^0)^2 - (\mathrm{Re}H_d^0)^2 + \dots\right]^2 \\
\frac{\partial V}{\partial(\mathrm{Re}H_u^0)} = & 2\left(|\mu|^2 + m_{H_u}^2\right) \mathrm{Re}H_u^0 - 2b \;\mathrm{Re}H_d^0 + \frac{g^2+g'^2}{8} 2 \left[\dots\right] 2 \;\mathrm{Re}H_u^0 \\
\frac{\partial^2 V}{\partial(\mathrm{Re}H_u^0)^2} = & 2\left(|\mu|^2 + m_{H_u}^2\right) + \frac{g^2+g'^2}{2}\left[\dots\right] + \frac{g^2+g'^2}{2} \;\mathrm{Re}H_u^0 \; 2 \;\mathrm{Re}H_u^0 \\
= & 2\left(|\mu|^2 + m_{H_u}^2\right) + \frac{g^2+g'^2}{2}\left[\dots\right] + \left(g^2+g'^2\right) (\mathrm{Re}H_u^0)^2 \\
\left.\frac{\partial^2 V}{\partial(\mathrm{Re}H_u^0)^2}\right|_{\mathrm{vev}} = & 2\left(|\mu|^2 + m_{H_u}^2\right) + \frac{g^2+g'^2}{2}\left(v_u^2 - v_d^2 + 2v_u^2\right) \\
= & 2\left(|\mu|^2 + m_{H_u}^2\right) + \frac{g^2+g'^2}{2}\left(3v_u^2 - v_d^2\right) \\
= & 2b\cot\beta + \frac{g^2+g'^2}{2}\left(v_d^2 - v_u^2\right) + \frac{g^2+g'^2}{2}\left(3v_u^2 - v_d^2\right) \qquad \text{using Eq.(85)} \\
= & 2b\cot\beta + \left(g^2+g'^2\right) v_u^2 \\
= & 2\left(b\cot\beta + m_Z^2 \sin^2\beta\right)
\end{aligned}
\tag{89}
$$

As before, we also need the mixed derivative, which now includes two terms

$$
\begin{aligned}
\left.\frac{\partial^2 V}{\partial(\mathrm{Re}H_u^0)\partial(\mathrm{Re}H_d^0)}\right|_{\mathrm{vev}} = & -2b + \frac{g^2+g'^2}{2} \mathrm{Re}H_u^0 \; (-2) \;\mathrm{Re}H_d^0 \\
= & -2b - (g^2+g'^2) v^2 \sin\beta\cos\beta \\
= & -2b - m_Z^2 \sin 2\beta
\end{aligned}
\tag{90}
$$

From those results we obtain the mass matrix for the two CP-even Higgs bosons $\text{Re}H_u^0$ and $\text{Re}H_d^0$

$$\begin{aligned}
\mathcal{M}^2_{h,H} &= \begin{pmatrix} b\cot\beta + m_Z^2\sin^2\beta & -b - \dfrac{m_Z^2}{2}\sin 2\beta \\ -b - \dfrac{m_Z^2}{2}\sin 2\beta & b\tan\beta + m_Z^2\cos^2\beta \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{m_A^2}{2}\sin 2\beta\cot\beta + m_Z^2\sin^2\beta & -\dfrac{m_A^2+m_Z^2}{2}\sin 2\beta \\ -\dfrac{m_A^2+m_Z^2}{2}\sin 2\beta & \dfrac{m_A^2}{2}\sin 2\beta\tan\beta + m_Z^2\cos^2\beta \end{pmatrix} \\
&= \begin{pmatrix} m_A^2\cos^2\beta + m_Z^2\sin^2\beta & -\dfrac{m_A^2+m_Z^2}{2}\sin 2\beta \\ -\dfrac{m_A^2+m_Z^2}{2}\sin 2\beta & m_A^2\sin^2\beta + m_Z^2\cos^2\beta \end{pmatrix}
\end{aligned} \tag{91}$$

with an additional mixing angle α and the mass eigenvalues

$$\begin{aligned}
2m^2_{h^0,H^0} &= m_A^2 + m_Z^2 \mp \left(\left(m_A^2+m_Z^2\right)^2 - 4m_A^2 m_Z^2\cos^2 2\beta \right)^{1/2} \\
&\simeq m_A^2 \mp \left(m_A^4 - 4m_A^2 m_z^2\cos^2 2\beta \right)^{1/2} \qquad\qquad \text{for } m_A \gg m_Z \\
&\simeq m_A^2 \mp m_A^2 \left(1 - \frac{4m_Z^2}{m_A^2}\cos^2 2\beta \right)^{1/2} \\
m^2_{h^0,H^0} &= \frac{m_A^2}{2} \mp \frac{m_A^2}{2}\left(1 - \frac{2m_Z^2}{m_A^2}\cos^2 2\beta \right) = \begin{cases} m_Z^2\cos^2 2\beta & \text{bound from above} \\ m_A^2 & \text{high mass scale} \end{cases}
\end{aligned} \tag{92}$$

In the same limit $m_A^2 \sim b \gg m_Z^2$ we can immediately read off the the mixing angle α off the mass matrix

$$\mathcal{M}^2_{h,H} \simeq b \begin{pmatrix} \cot\beta & -1 \\ -1 & \tan\beta \end{pmatrix} \qquad \Rightarrow \quad \alpha = \beta \tag{93}$$

This summarizes the situation of the two-Higgs doublet model (type-II) which is part of the MSSM: instead of one physical Higgs boson we have five of them. In the limit of large m_A and large $m_{H^\pm}$ the two CP-even Higgs bosons split into a light state with a tree-level upper mass limit smaller than m_Z and a heavy state at the same mass scale as m_A. This means that in terms of Higgs searches the MSSM is actually more predictive than the Standard Model, where the Higgs mass is a free parameter. The reason is that instead of introducing a quartic Higgs couplings λ we can use the set of scalar self interactions defined by the MSSM. Loop corrections mostly from the top and stop self energies in the propagator of the light Higgs boson raise the upper limit on the Higgs mass from the tree-level value below m_Z to roughly 135 GeV, for a large mass splitting between the top quark and its supersymmetric partners.

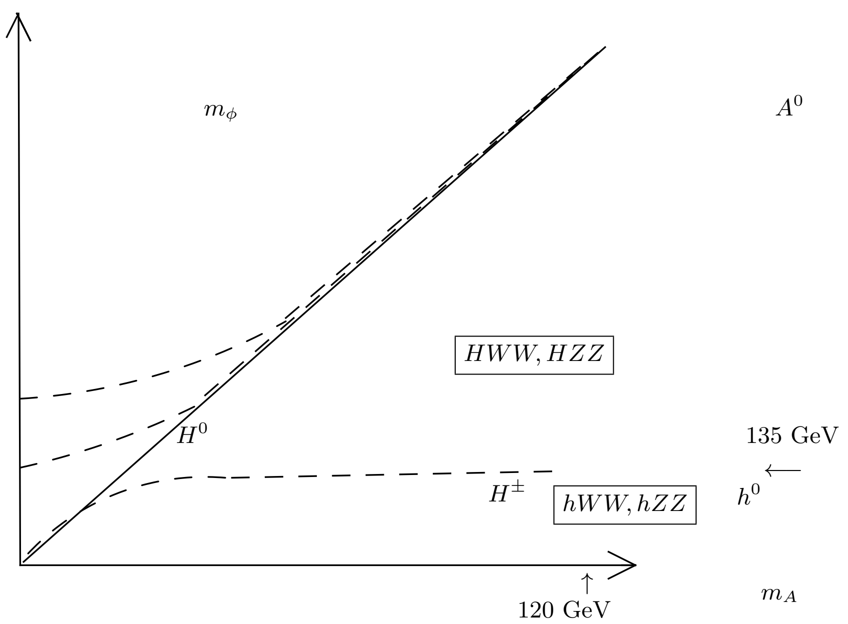
E. Literature

There are by now a great many text books and review articles on supersymmetry. Some which I really like is

– the very comprehensive review/book by Ian Aitchison (hep-ph/0505105) from where I took essentially all formulas

– the standard review on slightly more advanced SUSY and SUSY breaking by Steven Martin (hep-ph/9709356), continuously updated to lead our way to SUSY discovery at the LHC.

Acknowledgments: Supersymmetry is the model for new physics I grew up with as a graduate students. I would like to thank all the people who taught me this model together with the firm conviction that there has to be interesting physics out there. First and foremost this was Peter Zerwas, but also Michael Spira, Wim Beenakker, and on the experimental side Giacomo Polesello and Dirk Zerwas. Finally, Uli Baur served as a great role model when he criticized the MSSM believers and still never stopped believing that there is interesting new physics waiting to be discovered.



II. EXTRA DIMENSIONS

A. The Standard Model as an effective theory

Before we can even start talking about physics beyond the Standard Model, we have to define what we mean by the Standard Model. For those of you who attended Graham Kribs' journal club — remember the discussion between him and Thomas Binoth. There are different ways we can look at the Standard Model, regardless of it great success over by now many decades of measurements. Both approaches have basic structures in common:

– a gauge theory with the group structure $SU(3) \otimes SU(2) \otimes U(1)$

– massive electroweak gauge bosons masses through spontaneous symmetry breaking (Higgs mechanism with $v = 246$ GeV and m_H unknown)

– Dirac fermions in the usual doublets and with masses equal to their Yukawa couplings

There are two philosophies behind writing down a Lagrangian for this model:

1. write a renormalizable Lagrangian with all dimension–4 operators consistent with the particle content and all symmetries

2. write a general effective–theory Lagrangian with these particles and all symmetries. Higher–dimension operators will appear and have to be suppressed by some scale Λ

The difference between these two approaches are higher–dimensional operators, operators which have an explicit suppression by a large mass. If this scale Λ is large enough, we might never see the difference between the two approaches in high–energy experiments. If Λ is smaller, we might see the Standard Model break down as an effective theory, for example at the LHC, and we should be able to determine its ultraviolet completion.

1. *Experimental hints*

LEP and Tevatron

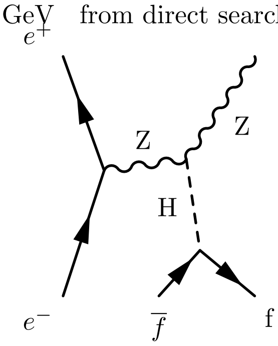
LEP(2) and Tevatron experiments have for many years tested the Standard Model to energy scales of $100 \cdots 500$ GeV. All their results are in perfect agreement with the Standard Model, apart from that fact that we could have seen direct evidence for the Higgs boson:

– electroweak gauge bosons discovered with masses $m_W \sim 80$ GeV, $m_Z \sim 91$ GeV
no anomalous W, Z decays

– 6 quarks found, $m_t \sim 172$ GeV
typical decay $t \to b\,W^+$ observed, no anomalous decays

– leptons, including τ as expected.

– electroweak precision data with global fit:
$m_H \sim 110$ GeV best value
$m_H \lessapprox 250$ GeV $1\,\sigma$ bound
$m_H > 114$ GeV from direct search at LEP2



where $f\overline{f}$ are mostly $H \to b\bar{b}$ and $Z \to l^+l^-, \nu\overline{\nu}$.

The possible problem with the electroweak precision data is the quality of the global fit. Its best χ^2 value is poor. A reason might be that some for example b observables might be inconsistent, but we do not know $\Rightarrow$ not conclusive

Muon anomalous magnetic moment (g-2)

The anomalous magnetic moment of the muon is one of the best–measured parameters in high–energy physics, even though most of the physics which goes into its determination we would call low–energy physics nowadays. Unfortunately, the Brookhaven experiment, which recently delivered the best available measurement, has been shut down. If you want to know more about this observable — Dominik Stöckinger recently wrote a great review on it.
The measured value of the anomalous magnetic moment of μ is

$$a_\mu^{\text{exp}} = \frac{1}{2}(g-2)_\mu^{\text{exp}} = (11659208 \pm 6) \cdot 10^{-10} \tag{94}$$

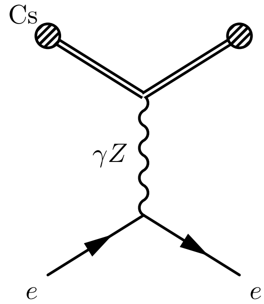
while the Standard Model predictions range between two different approaches

$$\begin{aligned} a_\mu^{\text{exp}} - a_\mu^{\text{SM}} &= (31.7 \pm 9.5) \cdot 10^{-10} \qquad : \ 3.3\sigma \\ a_\mu^{\text{exp}} - a_\mu^{\text{SM}} &= (20.2 \pm 9.0) \cdot 10^{-10} \qquad : \ 2.1\sigma \end{aligned} \tag{95}$$

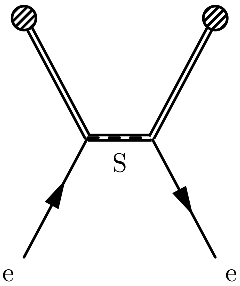
The general agreement in high–energy physics is: a 5 σ deviation from the background is called a discovery, everything else is either a rumor or a hint or a matter of taste $\Rightarrow$ not conclusive

Atomic parity violation

We know that electroweak gauge bosons have couplings which distinguish between the chirality of fermions in the $ff'W$ and ffZ vertices. In the interaction between the nucleus and the electrons in an atom, this interaction leads to parity violation, which can be measured:



Beyond the Standard Model we can look for so–called leptoquarks, scalars or vectors which carry baryon and lepton number and occur in the crossed channel compared to the usual γ, Z exchange:



Again, this experiment in Boulder was terminated with around 2 σ discrepancy between the Standard Model prediction and the final measurement $\Rightarrow$ not conclusive

Cosmology

The last Nobel prize went to studies of the cosmic microwave background. The most recent WMAP data (combined with large–scale structure measurements) confirms conclusively the existence of cold dark matter in the universe:

$$\Omega_{\rm DM} h^2 = 0.094 \ldots 0.129 \tag{96}$$

where $\Omega = 1$ is critical density for flat universe and $h = H_0/100$ km/s/Mpc $\sim$ 0.7 is just a c–number connected to the Hubble constant $H_0 \sim 73$ km/s/Mpc. Different measurements determine the matter content of the universe (averaged over all distances) to:

– baryon density $\Omega_b\, h^2 = 0.024 \pm 0.001$

– matter density $\Omega_m\, h^2 = 0.14 \pm 0.02$

Error estimates are mixture of serious studies, chemistry and miracles. Such a dark matter particle is not part of the Standard Model. Most generally, it could be a stable particle with electroweak interactions and a mass around 200 GeV. Unfortunately, we have not observed such a particle in direct of indirect searches for dark matter $\Rightarrow$ conclusive!

Flavor Physics

Flavor physics has been a major effort over the last decade — in particular once we put together B and K physics, neutrino physics, and low–energy searches for example for proton decay or neutrinoless double–beta decay. Unfortunately, all these measurements have revealed little but the existence of a finite neutrino mass of which we still do not know the over–all scale:

– proton decay not observed

– flavor changing neutral current not observed

– neutrinoless double–β decay not observed

– no unexplained effects in B physics

– no unexplained effects in K physics

– ....

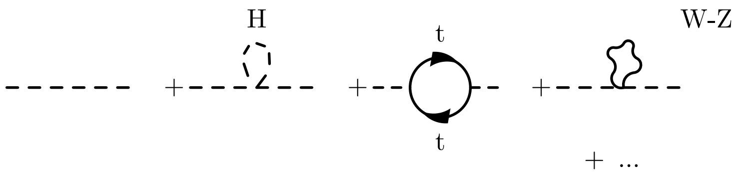
If we want to phrase it positively, we conclude that we have not found anything and not gained any clues for physics beyond the Standard Model $\Rightarrow$ $\boxed{\text{unfortunately conclusive}}$.

2. Theoretical hints

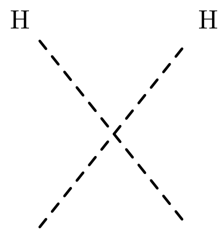
Let us start from assuming that the Standard Model is a renormalizable theory (it has no inverse powers of mass in the Lagrangian). This means that it does not have a built–in energy scale where it breaks down. An exception is gravity, because we know that at energies above the Planck scale 10^{19} GeV gravitational interactions become strong and our world should be described by some combination of the Standard Model and quantum gravity. All current observables probe scales $E \lessapprox 100$ TeV $= 10^5$ GeV so we can ignore Planck–scale or quantum gravity effects for now.

<u>Standard Model beyond tree-level</u>

At next–to–leading order, the (bare) leading order Higgs mass gets corrected by loops involving Standard Model particles:



We can for example compute the 4–point Higgs loop with the coupling:



$$= -\frac{3}{4} i g^2 \frac{m_H^2}{m_W^2}$$

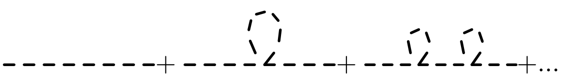
The amplitude for this diagram is given in terms of the 4–momentum q in the loop and in terms of the cutoff scale Λ. Note that the Standard Model with only dimension–4 operators does not offer an interpretation for such a scale, so at the end of the argument we have to perform the limit $\Lambda \to \infty$. Introducing a cutoff scale and sending it to infinity is certainly a physical regularization scheme:

$$
\begin{aligned}
\mathcal{M}_H^2 &= \int^\Lambda \frac{d^4q}{(2\pi)^4} \left(-\frac{3}{4} i ... \right) \frac{1}{q^2 - m_H^2} \\
&= \int \frac{d^4q}{(2\pi)^4} \left(\frac{1}{q^2 - m_H^2} - \frac{1}{q^2 - \Lambda^2} \right) \qquad \text{Pauli–Villars regularization} \\
&= (m - H^2 - \Lambda^2) \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 - m_H^2)(q^2 - \Lambda^2)} \\
&= (m_H^2 - \Lambda^2) \int \frac{d^4q}{(2\pi)^4} \int_0^1 dx \int_0^1 dy \frac{2\delta(1 - x - y)}{[(q^2 - m_H^2)x + (q^2 - \Lambda^2)y]^2} \\
&= 2(m_H^2 - \Lambda^2) \int_0^1 dx \int_0^1 dy \, \delta(1 - x - y) \int \frac{d^4q}{(2\pi)^4} \frac{1}{[q^2 - x m_H^2 - y\Lambda^2]^2} \\
&\sim 2(m_H^2 - \Lambda^2) \int_0^1 dx \int_0^1 dy \, \delta(1 - x - y) \frac{i}{16\pi^2} \\
&= -\frac{2i\Lambda^2}{16\pi^2} \left(-\frac{3}{4} i g^2 \frac{m_H^2}{m_W^2} \right) \qquad \text{now with couplings} \\
&= -\frac{3}{32\pi^2} g^2 \frac{m_H^2}{m_W^2} \Lambda^2
\end{aligned} \tag{97}
$$

where in the first line we have set $(-3/4i...) = 1$ for simplicity. The Pauli–Villars regularization using a cutoff scale Λ works as:

$$
\frac{1}{q^2 - m_H^2} - \frac{1}{q^2 - \Lambda^2} = \begin{cases} \dfrac{1}{q^2 - m^2} & q^2 L_L \Lambda^2 \\ \dfrac{1}{q^2 - q^2} & q^2 \gg \Lambda^2 \end{cases} \tag{98}
$$

How does this affect the mass?



$$
\begin{aligned}
&= \frac{1}{q^2 - m_H^2} + \frac{1}{q^2 - m_H^2} \mathcal{M}_H^2 \frac{1}{q^2 - m_H^2} + \frac{1}{q^2 - m_H^2} \mathcal{M}_H^2 \frac{1}{q^2 - m_H^2} \mathcal{M}_H^2 \frac{1}{q^2 - m_H^2} + ... \\
&= \frac{1}{q^2 - m_H^2} \sum_{n=0}^{\infty} \left(\mathcal{M}_H^2 \frac{1}{q^2 - m_H^2} \right)^n = \frac{1}{q^2 - m_H^2} \frac{1}{1 - \mathcal{M}_H^2 \dfrac{1}{m^2 - m_H^2}} = \frac{1}{q^2 - m_H^2 - \mathcal{M}_H^2}
\end{aligned} \tag{99}
$$

This means, the next–to–order contributions shift the leading–order unrenormalized mass $m_{H,b}^2$ to the unrenormalized next–to–leading order value $m_{H,b}^2 + \mathcal{M}_H^2$. We could calculate all Standard–Model corrections proportional to Λ^2 and obtain:

$$
m_H^2 = m_{H,b}^2 + \frac{3g^2}{32\pi^2} \frac{\Lambda^2}{m_W^2} \left[m_H^2 + 2m_W^2 + m_Z^2 - \frac{4n_f}{3} m_t^2 \right] \tag{100}
$$

This form is dictated by the fact that the Higgs couples to every Standard–Model particle proportional to its mass. This formula means that the unrenormalized Higgs mass will always be driven to the cutoff Λ of the Standard Model, unless we do something about it.
The naive solution $m_H^2 + 2m_W^2 + m_Z^2 - 4n_f m_t^2/3 = 0$ is called Veltman's condition, but it is of course only 1–loop solution. Moreover, talk to Martin Schmaltz about it and watch his (correct) rant about different particles in the loop behaving differently in the Pauli–Villars regularization.

Dimensional regularization

In modern calculations we usually use dimensional regularization, $d^4q \to d^Nq$. In his QCD book Rick Field gives the formula for the relevant Feynman integral:

$$\int \frac{d^Nq}{(2\pi)^N} \frac{(q^2)^R}{(q^2-m^2)^M} = \frac{i(-)^{R-M}}{(16\pi^2)^{N/4}} (m^2)^{R-M+N/2} \frac{\Gamma(R+N/2)\Gamma(M-R-N/2)}{\Gamma(N/2)\Gamma(M)} \tag{101}$$

With $R = 0, M = 1, N = 4-2\varepsilon$ we find

$$\begin{aligned}\int \frac{d^Nq}{(2\pi)^N} \frac{(q^2)^R}{(q^2-m^2)^M} &= -\frac{i}{(16\pi^2)^{N/4}} (m_H^2)^{-1+N/2} \frac{\Gamma(N/2)\Gamma(1-N/2)}{\Gamma(N/2)\Gamma(1)} \\ &= -\frac{i}{(16\pi^2)^{1-\varepsilon/2}} m_H^{2-2\varepsilon} \frac{\Gamma(\varepsilon)}{\varepsilon-1} \\ &= -\frac{i}{(16\pi^2)^{1-\varepsilon/2}} m_H^{2-2\varepsilon} \frac{e^{-\gamma_E\varepsilon}}{\varepsilon-1} \left(\frac{1}{\varepsilon}+\frac{\zeta_2}{2}\varepsilon+...\right)\end{aligned}$$

We can use a simple trick $x^\varepsilon = \exp(\log x^\varepsilon) = \exp(\varepsilon \log x) = 1 + \varepsilon \log x + \varepsilon^2/2 \log^2 x + ...$ to compute the limit $\varepsilon \to 0$

$$\int \frac{d^Nq}{(2\pi)^N} \frac{(q^2)^R}{(q^2-m^2)^M} = \frac{i}{16\pi^2} m_H^2 \left(\frac{1}{\varepsilon} + \mathcal{O}(\varepsilon^0)\right) \tag{102}$$

The next–to–leading order contribution to the Higgs mass now looks like $m_H^2/\varepsilon + \mathcal{O}(\varepsilon^0)$ and will be removed by for example on–shell or $\overline{\text{MS}}$ renormalization. The problem with this magical vanishing of the quadratic divergence is that dimensional regularization $(4-2\varepsilon)$ is not a physical regularization scheme, as far as we can see...

Numerical results

We can quantify the level of fine tuning, which would be required to remove the huge next–to–leading order contributions using a counter term.

$$m_H^2 = m_{H,b}^2 + \mathcal{M}_H^2 - \delta m_H^2 \tag{103}$$

which using for example $\Lambda = 10$ TeV implies

$$\delta m_H^2 \sim \mathcal{M}_H^2 = \begin{cases} -\frac{3}{8\pi^2}\lambda_t^2\Lambda^2 \sim -(2 \text{ TeV})^2 & t \text{ loop} \\ \frac{1}{16\pi^2} g^2\Lambda^2 \sim (100 \text{ TeV})^2 & W \text{ loop} \\ \frac{1}{16\pi^2}\lambda^2\Lambda^2 \sim (500 \text{ TeV})^2 & H \text{ loop}\end{cases} \tag{104}$$

For a varying cutoff scale Λ we find:

$$m_H = m_{H,b}^2 - \delta m_H^2 + \begin{cases} (-100+10+5)\ (200 \text{ GeV})^2 & \text{for } \Lambda = 10 \text{ TeV} \\ (-10000+1000+500)\ (200 \text{ GeV})^2 & \text{for } \Lambda = 100 \text{ TeV} \\ ... \end{cases} \tag{105}$$

To summarize our arguments for physics beyond the Standard Model, before we discuss possible scenarios, here is the short list:

1. the experimental reason to believe in BSM physics is dark matter (or the experience that until now every increase in energy has brought us in new physics). Any new 100 GeV WIMP can do the job
2. the theoretical reason to believe in BSM physics is the lack of stability of fundamental scalar masses in perturbative field theory

⇒ So what could this new physics at the TeV scale be?

supersymmetry	cancel Λ^2 terms
little Higgs (bosonic supersymmetry)	cancel Λ^2 terms
composite-Higgs models: technicolor, topcolor, ...	cut off integral
extra dimensions	$\Lambda_{\text{Planck}} \to \text{TeV}$
...	...

B. Fundamental Planck scale in (4+n) dimensions

1. *Einstein–Hilbert action and proper time*

To understand the trick of extra dimensions, we have to generalize the Einstein–Hilbert action to $(4+n) > 4$ dimensions.
Let's first remember/learn what this action means (following *e.g.* Peacock's book):

$$S = -\frac{1}{2} \int d^4x \sqrt{-g} M_\star^2 \, R \tag{106}$$

The root $\sqrt{-g}$ can also be written as $\sqrt{|g|}$, remembering that we are using the metric $\eta_{\mu\nu} = (+,-,-,-)$.
Start with the relativistic distance between two space–time points

$$ds^2 = \eta_{\mu\nu} x^\mu x^\nu \tag{107}$$

where we call

$$\begin{aligned} &ds^2 < 0, \;\; \text{spacelike} \\ &ds^2 = 0, \;\; \text{lightlike} \\ &ds^2 > 0, \;\; \text{timelike(the only allowed for massive particles)} \end{aligned} \tag{108}$$

To get a feeling for what ds^2 means let's integrate it along a path in space–time:

– (trivial case of) object at rest: $\Delta x^\mu = (\Delta t, 0, 0, 0)$

$$\int d^4x \sqrt{ds^2} = \int \sqrt{\eta_{\mu\nu} x^\mu x^\nu} = \int \sqrt{(dt)^2} = \int dt = \Delta t \tag{109}$$

which is just the time felt by this observer at rest

– moving along $\hat{x}$ direction

$$\begin{aligned} \int d^4x \sqrt{ds^2} &= \int \sqrt{(dt)^2 - (dx)^2} && \text{invariant under Lorentz trafos, go check...} \\ &= \int \sqrt{(dt)^2} && \text{in proper Lorentz frame} \\ &= \Delta t && \text{again time felt by the resting observer} \end{aligned} \tag{110}$$

⇒ definition of proper time:

$$d\tau = \sqrt{\eta_{\mu\nu} x^\mu x^\nu} \tag{111}$$

Start with the equations describing a freely falling object

$$\begin{aligned}
\frac{d^2x^\mu}{d\tau^2} &= 0 && \text{equation of motion for trajectory } x^\mu \\
d\tau^2 &= \eta_{\mu\nu} dx^\mu dx^\nu && \text{just definition of } d\tau^2 \\
dx^\mu &= \frac{\partial x^\mu}{\partial y^\nu} dy^\nu && \text{coordinate transformation}
\end{aligned} \tag{112}$$

This means for the proper time

$$\begin{aligned}
d\tau^2 &= \eta_{\mu\nu} dx^\mu dx^\nu \\
&= \eta_{\mu\nu} \frac{\partial x^\mu}{\partial y^\rho} dy^\rho \frac{\partial x^\nu}{\partial y^\sigma} dy^\sigma \\
&= \left(\eta_{\mu\nu} \frac{\partial x^\mu}{\partial y^\rho} \frac{\partial x^\nu}{\partial y^\sigma} \right) dy^\rho dy^\sigma \equiv g_{\rho\sigma} dy^\rho dy^\sigma
\end{aligned} \tag{113}$$

→ definition of general metric tensor and its transformation law

$$\boxed{g_{\rho\sigma} = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial y^\rho} \frac{\partial x^\nu}{\partial y^\sigma}} \tag{114}$$

Similarly, we can transform the equation of motion (just sketched here)

$$\begin{aligned}
\frac{dx^\mu}{d\tau} &= \frac{\partial x^\mu}{\partial y^\nu} \frac{\partial y^\nu}{d\tau} \\
0 \equiv \frac{d^2x^\mu}{d\tau^2} &= \frac{d}{d\tau} \frac{\partial x^\mu}{\partial y^\nu} \cdot \frac{\partial y^\nu}{d\tau} + \frac{\partial x^\mu}{\partial y^\nu} \frac{\partial^2 y^\nu}{d\tau^2} \\
&= \frac{dy^\nu}{d\tau} \frac{\partial x^\mu}{\partial y^\nu \partial y^\rho} \frac{\partial y^\rho}{d\tau} + \frac{\partial x^\mu}{\partial y^\nu} \frac{\partial^2 y^\nu}{d\tau^2} \\
\Leftrightarrow 0 \equiv \frac{d^2 y^\nu}{d\tau^2} &+ \Gamma^\nu_{\rho\sigma} \frac{dy^\rho}{d\tau} \frac{dy^\sigma}{d\tau}
\end{aligned} \tag{115}$$

→ definition of Christoffel symbol

$$\boxed{\Gamma^\nu_{\rho\sigma} = \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial^2 x^\mu}{\partial y^\rho \partial y^\sigma}} \tag{116}$$

which together with the metric tensor completely describes the kinematics in general relativity. The next question would be — can we express for example the Christoffel symbol in terms of the metric?
Compute

$$\frac{\partial g_{\rho\sigma}}{\partial y^\lambda} = \frac{\partial}{\partial y^\lambda} \left(\eta_{\mu\nu} \frac{\partial x^\mu}{\partial y^\rho} \frac{\partial x^\nu}{\partial y^\sigma} \right) = \Gamma^\mu_{\rho\lambda} g_{\sigma\mu} + \Gamma^\mu_{\sigma\lambda} g_{\rho\mu} \tag{117}$$

which by exchanging indices can be combined to

$$\frac{\partial g_{\mu\nu}}{\partial y^\lambda} + \frac{\partial g_{\mu\lambda}}{\partial y^\nu} - \frac{\partial g_{\nu\lambda}}{\partial y^\mu} = 2\Gamma^\rho_{\nu\lambda} g_{\rho\mu} \tag{118}$$

using $g_{\rho\mu} g^{\mu\rho} = 1$.
In other words, we can express the Christoffel symbols in terms of the metric tensors (and its derivatives):

$$\boxed{\Gamma^\rho_{\nu\lambda} = \frac{1}{2} g^{\rho\mu} \left(\frac{\partial g_{\mu\nu}}{\partial y^\lambda} + \frac{\partial g_{\mu\lambda}}{\partial y^\nu} - \frac{\partial g_{\nu\lambda}}{\partial y^\mu} \right)} \tag{119}$$

If we now have a guess, we would think that we can write the action for general relativity in terms of the metric tensor.

3. Simple invariant Lagrangian

Again, I will just sketch how we can write down a simple Lagrangian for general relativity. The transformation of the metric tensor from one coordinate system x into another one x' reads:

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial^2 x^\sigma}{\partial x'^\nu} \, g_{\rho\sigma} \tag{120}$$

which fixes the Jacobian

$$\det g'_{\mu\nu} = \left| \frac{\partial x}{\partial x'} \right|^2 \det g_{\mu\nu} \tag{121}$$

which we can use to compensate the Jacobian from the integration measure

$$d^4 x'^\mu = \left| \frac{\partial x'}{\partial x} \right| \, d^4 x^\mu \tag{122}$$

to build a very simple Lorentz–invariant Lagrangian

$$\boxed{\int d^4 x \, \sqrt{-g} \; \rho = \text{const}} \tag{123}$$

for any Lorentz–scalar density ρ.
What kind of scalar — made out of $g_{\mu\nu}$ and possibly $\Gamma^\mu_{\rho\sigma}$ — can we use to for example include sources (particles) in this action? A guess would be to find a tensor made out of second derivatives to be useful in a field equation and gives the right special relativistic limit. Luckily, there is a unique tensor which serves this purpose (following Peacock's book): the Riemann tensor

$$R^\mu_{\alpha\beta\gamma} \equiv \frac{d\Gamma^\mu_{\alpha\gamma}}{dx^\beta} - \frac{d\Gamma^\mu_{\alpha\beta}}{dx^\gamma} + \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\gamma\alpha} - \Gamma^\mu_{\sigma\gamma}\Gamma^\sigma_{\beta\alpha} \tag{124}$$

which can be contracted to give the Ricci tensor

$$R_{\alpha\beta} \equiv R^\mu_{\alpha\beta\mu} \tag{125}$$

and the Ricci scalar

$$R \equiv g^{\alpha\beta} R_{\alpha\beta} \equiv R^\mu_\mu \tag{126}$$

which is precisely the scalar we are looking for to put into our action!
Before constructing the action we should check the mass units:

$$\begin{aligned} [d^4 x] &= m^{-4} \\ [g] &= m^0 \\ [\Gamma] &= m \\ [R] &= m^2 \end{aligned} \tag{127}$$

which means that $[d^4 x \sqrt{-g} R] = m^{-2}$. So we are not quite there yet, the mass unit of the action is still wrong. We have to introduce a fundamental mass parameter into general relativity which we call $M_\star$. We arrive at:

$$\boxed{S = -\frac{1}{2} \int d^4 x \sqrt{-g} M_\star^2 \, R} \tag{128}$$

with a conventional numerical c-number factor in front. Since we are still talking about 4 dimensions we can identify $M_\star \equiv M_{\text{Planck}}$, with the Planck mass measured through the gravitational coupling G_N.

For the rest of the discussion of large flat extra dimensions I will largely follow Graham Kribs' hep-ph/0605325, both in logic and notation. Our first task is to write down the Einstein–Hilbert action in (n+4) dimensions, to see how extra space dimensions actually solve the hierarchy problem. Inserting the Ricci scalar — derived from the Riemann tensor as unique a building bock for Einstein's field equations — into our old action we obtain the correct $(4+n)$–dimensional action as

$$S_{\text{bulk}} = -\frac{1}{2} \int d^{4+n}x \, \sqrt{-g^{(4+n)}} \; M_\star^{n+2} \; R^{(4+n)} \tag{129}$$

where bulk means that this action governs our $(4+n)$–dimensional space, to be distinguished from 'brane', which refers to a 4–dimensional subspace where all or some of our Standard Model field live. The increased power of the Planck mass $M_\star$ is again chosen to correct the over–all mass dimension. The mass dimension of the $(4+n)$–dimensional Ricci scalar is the same as it's 4–dimensional counterpart, because it is created by the number of space–time derivatives.
This formula still has to be filled with physics content, *i.e.* we have to define $g^{(4+n)}$ and $R^{(4+n)}$. Of course, we have to distinguish the different Dirac indices we are talking about. In general, the usual space–time vector get extended to $x_A = \{x_\mu, y_j\}$ where the usual Greek indices run from $\mu = 0 \cdots 3$ and the additional Roman indices run fill the remaining n components. I will try to stick to running the Roman indices as $j = 4 \cdots (4+n)$. The capital Roman index describes the bulk and runs from $A = 0 \cdots (4+n)$. The bulk metric can be written as

$$\begin{aligned} ds^2 &= g^{(4+n)}_{MN} dx^M dx^N && M, N = 0, ..., n+4-1 \\ &= g^{(4)}_{\mu\nu} dx^\mu dx^\nu - dx^j dx_j && j = 4, ..., n+4-1 \\ &= (\eta_{\mu\nu} + h_{\mu\nu})\, dx^\mu dx^\nu - dx^M dx_M && \text{allowing for a 4–dimensional graviton} \\ &= (\eta_{\mu\nu} + h_{\mu\nu})\, dx^\mu dx^\nu - r^2 d\Omega_{(n)} && \text{after compactifying } j \text{ on a torus} \end{aligned} \tag{130}$$

Note that this simple model requires the <u>extra dimensions to be flat</u> (compactifying on a torus does not mean we bend them, it is just another way of referring to periodic boundary conditions). The power of r arises because ds^2 is bilinear in the space–time vector. At this point, I should reiterate the specific requirements we have to make on the extra dimensions to make the following argument.

– if we write the split $(4+n)$–dimensional metric tensor it looks like $g^{(4+n)} = g^{(4)} \otimes (-1)$, as long as we assume that the extra dimensions are flat. We cannot allow any sources (particles or stars or black holes) off our Standard Model brane. Sources on our brane will of course affect the bulk, but we will discuss later how a mathematically infinitely narrow brane is unrealistic. So we might imagine looking at a slightly wider brane and ignore the bulk region close to the brane, so we can assume that the extra dimensions are indeed flat. For the Einstein–Hilbert action this means $\sqrt{g^{(4+n)}} = \sqrt{g^{(4)}}$, as long as the extra dimensions are perpendicular to our (3+1)–dimensional brane.

– the special geometry of the extra dimensions allows us to rewrite and if possible integrate out all additional dimensions, as long as we assume that the extra dimensions are orthogonal to our brane, as suggested by the diagonal metric tensor:

$$\int d^{4+n}x = \int d^4x \, r^n \, d\Omega_{(n)} = (2\pi r)^n \int d^4x \equiv V_{\text{torus}} \int d^4x \tag{131}$$

– from looking at Riemann's tensor you can guess that flat, orthogonal, extra dimensions without any sources will hardly affect the Ricci scalar. More specifically, the Ricci scalars in 4 and $(4+n)$ dimensions can be linked through Einstein's field equations, which we have not talked about yet. In the absence of matter (which we are assuming for the extra dimensions) they read

$$R_{jk} - \frac{1}{n+2} g_{jk} R = 0 \tag{132}$$

which after contracting with g^{jk} requires $R = 0$. In other words, the extra–dimensional part of R is zero, or $R^{(4+n)} = R^{(4)}$.

We can combine these pieces and simplify the higher–dimensional bulk action

$$\begin{aligned}
S_{\text{bulk}} &= -\frac{1}{2} M_\star^{n+2} \int d^{4+n}x \, \sqrt{-g^{(4+n)}} \, R^{(4+n)} \\
&= -\frac{1}{2} M_\star^{n+2} \, (2\pi r)^n \int d^4x \, \sqrt{-g^{(4)}} \, R^{(4)} \\
&\equiv -\frac{1}{2} M_{\text{Planck}}^2 \int d^4x \sqrt{-g^{(4)}} \, R^{(4)}
\end{aligned} \tag{133}$$

In the last line we have matched the two theories, *i.e.* we have assumed that from a 4–dimensional point of view the actions have to be identical, as long as we do probe high enough energy scales to observe quantum–gravity effects.
This leads us to the basis of extra dimensions as a solution to the hierarchy problem: the 4–dimensional Planck scale M_{Planck} which we measure on our brane/in our world is not the fundamental scale of gravity. It is merely a derived parameter which depends on the fundamental $(4+n)$–dimensional Planck scale and on the geometry of the extra dimensions, in the simplest case the compactification radius of the n–dimensional torus. Matching the two theories gives

$$\boxed{M_{\text{Planck}} = M_\star \, (2\pi r M_\star)^{n/2}} \tag{134}$$

The derived 4–dimensional Planck scale is indeed measured to be around 10^{19} GeV. If we can assume that the correction factor $(2\pi r M_\star)^n$ is large we can postulate that the fundamental Planck scale $M_\star$ is not much larger than 1 TeV. In that case the cutoff of our field theory is not much above the expected Higgs boson mass and there is no problem with the stability of the two scales m_H and $M_\star$, which we introduced as the hierarchy problem.
Assuming $M_\star = 1$ TeV we can solve the equation above for r for a given number of extra dimensions and obtain the compactification radius:

n	r
1	10^{12} m
2	10^{-3} m
3	10^{-8} m
...	...
6	10^{-11} m

Obviously, the case $n = 1$ is dangerous, because gravity gets modified at large distances. For larger values of n we have to test Newtonian gravity at small distances, which is harder. However, small distances just means larger energies, and we might be able to find cosmological or collider observables which are sensitive to such effects. Graham in his overview discusses quite a few of them.

C. Gravitons in flat (4+n) dimensions

Note that at this point we have not talked about particles in the theory. Let's still assume that Standard Model fields do not propagates in more than 4 dimensions. All we postulate is a continuation of Newtonian gravity into $(4+n)$ dimensions.
Let's consider two masses on our Standard Model brane. For large distances $r' \gg r$ the two masses are far enough apart that the curled–up extra dimensions will not be resolved. Again, we can think about large distances as small energies, which means that our test energy $1/r'$ is too small to see effects coming in at much larger energies $1/r$. Which means that for $r' \gg r$ we observe ordinary 4–dimensional Newtonian gravity.
Probing smaller distances r' (or higher energies $1/r'$) the 4–dimensional distance will fit into the extended extra dimensions, which means that gravity propagates into all $(4+n)$ dimensions and the volume integral over the

n–dimensional torus is cut off by a $(4+n)$–dimensional sphere with radius r':

$$V(r') = \begin{cases} -G_N \dfrac{m_1 m_2}{r'} & r' \gg r \qquad \text{(4–dim theory at small energies)} \\ -G_N^{(4+n)} \dfrac{m_1 m_2}{r'} \sim -G_N \dfrac{M_{\text{Planck}}^2}{M_\star^2} \dfrac{m_1 m_2}{r'} = -G_N \, (2\pi M_\star)^n \, \dfrac{m_1 m_2}{r'^{1-n}} & r' L_L r \end{cases} \tag{135}$$

tp: this argument sound totally convincing, but I think the power of r' should actually be $1+n$, have to check, damnit! For the 4–dimensional theory Newton's constant is defined as $G_N = 1/(16\pi M_{\text{Planck}}^2)$. Modulo c-number pre-factors it is obvious that the fundamental Planck scale in the bulk is given by $G_N^{(4+n)} = 1/(16\pi M_\star^2)$. Such a modification of Newtonian gravity can be tested experimentally without even looking at the details of a model!

1. *Propagating an extra–dimensional graviton*

Now we understand why large extra space dimensions solve the hierarchy problem. We also know how Newtonian gravitation is modified. However, if $M_\star \sim 1$ TeV and we can probe these scales at colliders, we have to understand quantum gravity effects at colliders. Which takes us to Kaluza–Klein effective theories.
First, we expand the $(4+n)$–dimensional metric around the flat metric η_{MN}, treating the resulting $(n+4)$–dimensional graviton field h_{MN} as a small perturbation:

$$\begin{aligned} ds^2 &= g_{MN}^{(4+n)} dx^M dx^N \qquad\qquad\qquad M,N = 0,...,3+n \\ &= \left(\eta_{MN} + \frac{1}{M_\star^{n/2+1}} h_{MN} \right) dx^M dx^N \end{aligned} \tag{136}$$

The factor $1/M_\star^{n/2+1}$ fixes the mass unit of the graviton to $[h] = m^{1+n/2}$. In general, for a boson in $(4+n)$-dimensions we would expect to be able to write down a squared mass term (corresponding to the Klein–Gordon equation) in the Lagrangian, which means $[d^{4+n}x \, m^2 \, SS] = m^{-4-n} \, m^2 \, m^{2(1+n/2)} = m^0$ for the correct mass dimension of the bosonic field.
At this stage we would not get around computing Ricci tensor/scalar (which I will of course not do in this lecture), to express the left–hand side of Einstein's equation $R_{AB} - g_{AB} R/(2+n)$ in terms of the graviton field h_{AB} and its derivatives. Here is how it would be done. We start with the definitions in $(n+4)$ dimensions and a short-hand notation of derivative with respect to x_C:

$$\begin{aligned} R_{AB} &= R^M_{ABM} = \frac{d\Gamma^M_{AM}}{dx^B} - \frac{d\Gamma^M_{AB}}{dx^M} + \Gamma^M_{SB}\Gamma^S_{MA} - \Gamma^S_{SM}\Gamma^S_{BA} \\ R &= g^{AB} R_{AB} \\ \Gamma^M_{AB} &= \frac{1}{2} g^{MS} \left(\frac{\partial g_{SA}}{\partial x^B} + \frac{\partial g_{SB}}{\partial x^A} - \frac{\partial g_{AB}}{\partial x^S} \right) \\ &= \frac{1}{2} \left(\eta^{MS} - \frac{1}{M_\star} h^{MS} \right) (\partial_B h_{SA} + \partial_A h_{SB} - \partial_s h_{AB}) \end{aligned} \tag{137}$$

Note that the definition of R_{AB} includes total derivatives with respect to x. Just to give you an idea we can compute the partial derivative of the Christoffel symbols:

$$\begin{aligned}\partial_B \Gamma^M_{AM} &= \frac{1}{2}\partial_B \left(\eta^{MS} - \frac{1}{M_\star^{1+n/2}} h^{MS}\right)(\partial_M h_{SA} + \partial_A h_{SM} - \partial_S h_{AM}) \\ &+ \frac{1}{2}\left(\eta^{MS} - \frac{1}{M_\star^{1+n/2}} h^{MS}\right)(\partial_B\partial_M h_{SA} + \partial_B\partial_A h_{SM} - \partial_B\partial_S h_{AM}) \\ &= -\frac{1}{2M_\star^{1+n/2}}\partial_B h^{MS}(\partial_M h_{SA} + \partial_A h_{SM} - \partial_S h_{AM}) \\ &+ \frac{1}{2}\left(\partial_B\partial^S h_{SA} + \partial_B\partial_A h^S_S - \partial_B\partial_S h^S_A\right) \\ &- \frac{1}{2M_\star^{1+n/2}} h^{MS}(\partial_B\partial_M h_{SA} + \partial_B\partial_A h_{SM} - \partial_B\partial_S h_{AM})\end{aligned} \tag{138}$$

$$\begin{aligned}\partial_M \Gamma^M_{AB} &= \frac{1}{2}\partial_M \left(\eta^{MS} - \frac{1}{M_\star^{1+n/2}} h^{MS}\right)(\partial_B h_{SA} + \partial_A h_{SB} - \partial_S h_{AB}) \\ &+ \frac{1}{2}\left(\eta^{MS} - \frac{1}{M_\star^{1+n/2}} h^{MS}\right)(\partial_M\partial_B h_{SA} + \partial_M\partial_A h_{SB} - \partial_M\partial_S h_{AB}) \\ &= -\frac{1}{2M_\star^{1+n/2}}\partial_M h^{MS}(\partial_B h_{SM} + \partial_A h_{SB} - \partial_S h_{AB}) \\ &+ \frac{1}{2}\left(\partial^S\partial_B h_{SA} + \partial^S\partial_A h_{SB} - \partial^S\partial_S h_{AB}\right) \\ &- \frac{1}{2M_\star^{1+n/2}} h^{MS}(\partial_M\partial_B h_{SA} + \partial_M\partial_A h_{SB} - \partial_M\partial_S h_{AB})\end{aligned} \tag{139}$$

The other kind of terms in R_{AB} yields similar terms of the kind:

$$\Gamma^M_{SB}\Gamma^S_{MA} = \frac{1}{4}\left(\eta - \frac{1}{M_\star^{1+n/2}}...h\right)(\partial h + ...)\left(\eta - \frac{1}{M_\star^{1+n/2}}...h\right)(\partial\eta + ...) \tag{140}$$

Combining all these terms should (according to Graham, who cites another paper by Gian Giudice, Ricardo Rattazzi and James Wells) give the final result for the left–hand side of Einstein's equations:

$$\begin{aligned}&R_{AB} - \frac{1}{n+2} g_{AB} R \\ &= \frac{1}{M_\star^{1+n/2}}\left[\Box h_{AB} - \partial_A\partial^C h_{CB} - \partial_B\partial^C h_{CA} + \partial_A\partial^B h^C_C - \eta_{AB}\Box h^C_C + \eta_{AB}\partial^C\partial^D h_{CD}\right]\end{aligned} \tag{141}$$

for a general metric/graviton in $(4+n)$ dimensions. The d'Alembert operator is defined as $\Box = \partial_C\partial^C$.

2. *Brane matter and bulk gravitons*

The right–hand side of Einstein's equations is given by the energy–momentum tensor, again normalized to the proper mass dimension. The energy–momentum tensor can be computed from the Lagrangian for the respective theory using a function derivative with respect to the metric. We will give examples later, at this point all we need to know is that T_{AB} is a function which includes all particle fields which live on our brane (and possibly in the bulk):

$$\boxed{R_{AB} - \frac{1}{2+n} g_{AB} R = -\frac{T_{AB}}{M_\star^{2+n}}} \tag{142}$$

The usual 4–dimensional $T_{\mu\nu}$ we have to generalize to $(4+n)$ dimensions. Obviously, the tensor rank (size of the matrix) increases from 4×4 to $(4+n)\times(4+n)$. Moreover, each entry now has dimension m^{4+n}, which requires the proper normalization using the usual (only) mass scale $M_\star$.

Our model is still assuming that all Standard Model fields are confined to our brane. In that case we can simplify the $(4+n)$–dimensional energy–momentum tensor:

– all matter is localized to $y_j = 0$, which means all entries in the energy–momentum tensor have to be localized:

$$\boxed{T_{AB}(x;y) = T_{AB}(x)\delta^{(n)}(y)} \tag{143}$$

note that this $\delta(y)$ is dodgy from a Heisenberg uncertainty point of view, because we pretend to know exactly the location of a matter particle in the extra dimension. Which means we know nothing about its momentum into this direction. However, the momentum we also localize, so that scattering processes with exclusively Standard Model particles observe 4–momentum conservation on the brane. The solution to this problem is to postulate a small finite width of the Standard Model brane. As far as this size is larger than the inverse energy scale we are probing our system with, this approximation is not a problem. Naturally, a size $1/M_\star$ will be fine, because above this scale we will not be able to compute anything with our effective Kaluza–Klein theory anyhow.

– finite entries into the energy–momentum tensor only appear on the brane, which together with the last point gives us the form of the energy momentum tensor as it appears on the right–hand side of Einstein's equations:

$$T_{AB}(x;y) = \eta^\mu_A\,\eta^\nu_B\,T_{\mu\nu}(x)\,\delta^{(n)}(y) = \begin{pmatrix} T_{\mu\nu}(x)\,\delta^{(n)}(y) & 0 \\ 0 & 0 \end{pmatrix} \tag{144}$$

Of course, the x appearing in the argument of our energy–momentum tensor do not have to be only space coordinates. It means that all arguments of $T_{\mu\nu}$ are localized to the Standard Model brane. As our usual check we look at the mass dimensions of our different objects: if $[T_{AB}] = m^{4+n}$ and $[\delta^{(n)}(y)] = m^n$ (remember how is cancels integrations) then $[T_{\mu\nu}] = m^4$, as expected.

The set of $(n+4)^2$ Einstein equations now splits into homogeneous equations for the bulk (including the bulk–brane mixing indices) and into a regular inhomogeneous equation for the brane:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2+n} g_{\mu\nu} R &= -\frac{T_{\mu\nu}}{M_\star^{2+n}} && \text{is a 4-dimensional theory!} \\ R_{\mu k} - \frac{1}{2+n} g_{\mu k} R &= 0 \\ R_{jk} - \frac{1}{2+n} g_{jk} R &= 0 && \text{the condition for solving the hierarchy problem } R^{(4)} \equiv R^{(4+n)} \end{aligned} \tag{145}$$

Just as before, we postulate periodic boundary conditions in all extra dimensions, with a compactification radius r

$$x_M = (x_\mu; y_i) \qquad i \ge 1 \qquad y_i \equiv y_i + 2\pi r \tag{146}$$

which means we can write $h_{AB}(x_M)$ as a Fourier series in the extra dimensions:

$$h_{AB}(x;y) = \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_j=-\infty}^{\infty} \frac{h_{AB}^{(m)}(x)}{\sqrt{(2\pi r)^n}} e^{i\frac{m_j y_j}{r}} \tag{147}$$

Note that we are now evaluating the graviton in a mixed position space x_μ and (Fourier–) momentum space $(y_j \mapsto m_j)$.
We can rewrite the left–hand side of the Einstein equations in this mixed space. We already phrased it in terms of the $(4+n)$-dimensional graviton field h_{AB} and its derivatives. Take for example the first (d'Alembert) term:

$$R_{AB} - \frac{1}{2+n} g_{AB} R \sim \frac{1}{M_\star^{1+n/2}} \left(\Box h_{AB} + \cdots\right) \equiv -\frac{T_{AB}}{M_\star^{2+n}} = -\frac{\eta^\mu_A \eta^\nu_B T_{\mu\nu} \delta^{(n)}(y)}{M_\star^{2+n}} \tag{148}$$

The d'Alembert term can be written in its Fourier components

$$\begin{aligned}
\Box h_{AB} &= \sum_{m_j} \frac{1}{(2\pi r)^{n/2}} \partial_C \partial^C \left[h^{(m)}_{AB}(x)\, e^{i(m\cdot y)/r} \right] \\
&= \sum_{m_j} \frac{1}{(2\pi r)^{n/2}} \partial_C \left[\left(\delta^C_\mu h^{(m)}_{AB}(x) + h^{(m)}_{AB}(x) \frac{i m_j}{r} \delta^C_j \right)\, e^{i(m\cdot y)/r} \right] \\
&= \sum_{m_j} \frac{1}{(2\pi r)^{n/2}} \left[\left(\partial_\mu \partial^\mu h^{(m)}_{AB}(x) + 0 \right) + \left(\delta^C_\mu h^{(m)}_{AB}(x) + h^{(m)}_{AB}(x) \frac{i m_j}{r} \delta^C_j \right) \frac{i m_k}{r} \eta_{Ck} \right] e^{i(m\cdot y)/r} \\
&= \sum_{m_j} \frac{1}{(2\pi r)^{n/2}} \left[\Box h^{(m)}_{AB}(x) - h^{(m)}_{AB}(x) \frac{m^j m_j}{r^2} \right] e^{i(m\cdot y)/r} \\
&= \sum_{m_j} \frac{1}{(2\pi r)^{n/2}}\, e^{i(m\cdot y)/r} \left(\Box + \hat{k}^2 \right) h^{(m)}_{AB}(x)
\end{aligned} \tag{149}$$

with $\hat{k} \equiv m_j/r$ and $\hat{k}^2 \equiv \sum |m_j/r|^2$. The d'Alembert box operator acting on $h^{(m)}_{AB}(x)$ is just $\partial_\mu \partial^\mu$. This Fourier transform works the same way for $h_{\mu\nu}, h_{\mu j}, h_{jk}$.
The right-hand side of Einstein's equations are either zero

$$\sum_{m_j} \frac{1}{(2\pi r)^n} 0 \cdot e^{i(m\cdot y)/r} = 0 \tag{150}$$

or functions of the 4-dimensional variables

$$\sum_{m_j} \frac{1}{(2\pi r)^n}\, f(x) \cdot e^{i(m\cdot y)/r}. \tag{151}$$

Even though the graviton fields $h^{(m)}_{AB}(x)$ are not yet the physical fields we will define in a minute, we already see the structure of the equation of motion for all fields involved: they include a quadratic term $(\Box^2 + \hat{k}^2) h_{AB}(x) = ...$ which means that they have masses m_j/r, where m_j are integers. Kaluza-Klein gravitons have a massless ground state $m_j = 0$ and excited states labeled by $|\overrightarrow{m}_j|$.

3. Kaluza–Klein towers

The detailed form of the physical graviton fields is not particularly important. Their precise definition can be found in the Giudice-Rattazzi-Wells paper, their counting of degrees of freedom in Graham's lectures. The Einstein equations in the most convenient form look like:

$$\begin{aligned}
(\Box + \hat{k}^2)\, G^{(k)}_{\mu\nu} &= \frac{1}{M_{\text{Planck}}} \left[-T_{\mu\nu} + \left(\frac{\partial_\mu \partial_\nu}{\hat{m}^2} + \eta_{\mu\nu} \right) \frac{T^\lambda_\lambda}{3} \right] && \text{massive graviton} \\
(\Box + \hat{k}^2)\, H^{(\overrightarrow{k})} &= \frac{1}{2 M_{\text{Planck}}} \sqrt{\frac{3(n-1)}{n+2}}\, T^\mu_\mu && \text{scalar, includes radion} \\
(\Box + \hat{k}^2)\, V^{(k)}_{\mu j} &= 0 && \text{graviscalars} \\
(\Box + \hat{k}^2)\, S^{(k)}_{jk} &= 0 && \text{massive graviphotons}
\end{aligned} \tag{152}$$

The structure of these equation reveals a few particularities: the fields $V^{(k)}_{\mu j}$ and $S^{(k)}_{jk}$ do not couple to the Standard model, because in the presence of a general energy-momentum tensor they still behave like free massless fields. The massive gravitons $G^{(k)}_{\mu\nu}$ couple to the Standard Model. Their Fourier coordinate only appears as a mass term $\hat{k}^2$ and in the coupling to the trace of the energy-momentum tensor. This means their couplings are level–degenerate and their masses and couplings depend only on the length, but not on the orientation of the vector m_j.

I can only quote the properties of conformally invariant theories, where $T^\mu_\mu = 0$. For such massless theories we find

$$\boxed{(\Box + \hat{k}^2)\, G^{(k)}_{\mu\nu} = -\frac{T^{\mu\nu}}{M_{\rm Planck}}} \tag{153}$$

which describes physical gravitons at the LHC, produced by quark or gluon interactions and either vanishing or decaying to leptons.
The scalar mode $H^{(\vec{k})}$ plays a special role. Its massless mode is called a radion and corresponds to a fluctuation of the volume of the compactified extra-dimension. We assume that the compactification radius r is somehow stabilized, and such mechanism gives mass to the radion. More importantly, the radion only couples to a massive theory, so it is not surprising that as a scalar with no Standard Model charge it will mix with a Higgs boson without very drastic effects.
Before we discuss the coupling of gravitons to Standard Model particles we introduce a mechanism for summing over the Kaluza-Klein levels. The mass splitting between the KK states is given by $1/r$ which translates into ($M_\star = 1$ TeV as before):

$$\delta m \sim \frac{1}{r} = 2\pi M_\star \left(\frac{M_\star}{M_{\rm Planck}} \right)^{2/n} = \begin{cases} 0.003 \text{ eV} & (n=2) \\ 0.1 \text{ MeV} & (n=4) \\ 0.05 \text{ GeV} & (n=6) \end{cases} \tag{154}$$

On the scale of modern light-energy experiments, this mass splitting is tiny, 0.003 eVL_L0.1 MeVL_L0.05 GeV$L_L m_Z$. This means, at colliders we will be confronted with towers composed out of a huge number of tightly spaces massive gravitons with identical couplings to Standard–Model particles. Instead of summing for example over all gravitons radiated off an LHC process, we can integrate over a continuous graviton mass space.
For n dimensions we want to compute the numbers of gravitons with masses between $|k|$ and $|k + dk|$. k represents the number of gravitons in the compactified n dimensions. In other words, we need to integrate an n-dimensional sphere:

$$dN = S_{n-1}\, |k|^{n-1}\, d|k| \tag{155}$$

with the area of an n-sphere

$$S_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{156}$$

This density in terms of states we can translate into a mass density kernel, using

$$\frac{dm}{d|k|} = \frac{1}{r} \quad \Rightarrow \quad dN = S_{n-1}\, r^n\, m^{n-1}\, dm \quad \Rightarrow \quad dN = S_{n-1} \frac{1}{(2\pi M_\star)^n} \left(\frac{M_{\rm Planck}}{M_\star} \right)^2 m^{n-1}\, dm \tag{157}$$

We will later think of this distribution as a kernel in for example final-state phase space integrals. Two properties of this distribution dN can be easily read off. The integral is IR finite

$$\begin{aligned} \int dN = \int_0^\mu S_{n-1} \frac{1}{(2\pi M_\star)^n} \left(\frac{M_{\rm Planck}}{M_\star} \right)^2 m^{n-1}\, dm &= S_{n-1} \frac{1}{(2\pi M_\star)^n} \left(\frac{M_{\rm Planck}}{M_\star} \right)^2 \left. \frac{m^n}{n} \right|_0^\mu \\ &= S_{n-1} \frac{1}{(2\pi M_\star)^n} \left(\frac{M_{\rm Planck}}{M_\star} \right)^2 \frac{\mu^n}{n} \end{aligned} \tag{158}$$

and in the UV it is strongly peaked, the stronger the larger n. Note that the limit between the well-defined IR tail and the sharp UV peak is based on the specific of the model, in our case on the existence of only one compactification length scale.

4. Graviton Feynman rules

To compute graviton production cross sections like $pp \to$ KK+jet or $pp \to$ KK $\to \mu^+\mu^-$ we need to couple the gravitons to the Standard model, i.e. write a proper Lagrangian.
Start from the general relativity Lagrangian in $(4+n)$ dimensions

$$S = -\frac{1}{2}\int d^{4+n}x\,\sqrt{|g|}\,M_\star^{n+2}\,R = \int d^{4+n}\mathcal{L}^{(4+n)} \tag{159}$$

which we can express in term of the graviton field h_{AB} (still in (4+n) dimensions)

$$\boxed{\mathcal{L} = -\frac{1}{2}h^{AB}\Box h_{AB} + \frac{1}{2}h_A^A\Box h_B^B - h^{AB}\partial_A\partial_B h_C^C + h^{AB}\partial_A\partial_C h_B^C - \frac{1}{M_\star^{1+n/2}}h^{AB}T_{AB}} \tag{160}$$

The last term corresponds to the right–hand side of the Einstein equations. Instead of deriving this coupling term we can at least check its consistency: if the equations for pure Newtonian gravity give

$$R_{AB} - \frac{1}{n+2}\,g_{AB}\,R = 0 \tag{161}$$

then the inhomogeneous term on the right–hand side

$$R_{AB} - \frac{1}{n+2}\,g_{AB}\,R = -\frac{T_{AB}}{M_\star^{2+n}} \tag{162}$$

has to correspond to a term in in the Lagrangian, given by the usual Euler–Lagrange equations:

$$\frac{\partial\mathcal{L}_{\text{add}}}{\partial h^{AB}} = \frac{\partial}{\partial h_{AB}}\left(-\frac{1}{M_\star^{1+n/2}}h^{AB}T_{AB}\right) = -\frac{T_{AB}}{M_\star^{1+n/2}} \tag{163}$$

Note the mismatch in powers of $M_\star$. We have used a different normalization for Einstein's equation; for $\mathcal{L}$ the proper mass unit is indeed $[1/M_\star^{2+n}\,h^{AB}\,T_{AB}] = m^{-1+n/2}\,m^{1+n/2}\,m^{4+n} = m^{4+n}$.
The tensor graviton field h_{AB} we as usually Fourier transform and express in the more appropriate 4-dimensional Kaluza–Klein fields. Let's assume a massless Standard Model, or in other words we are going to study QED and QCD with KK gravitons. This is appropriate for LHC or linear–collider observables, as long as we stay away from top–quark production. In that case all we are left with is:

$$\begin{aligned}\mathcal{L} = -\sum\Big[&-\frac{1}{2}G^{\mu\nu}(\Box + m^2)G_{\mu\nu} + \frac{1}{2}G_\mu^\mu(\Box + m^2)G_\nu^\nu - G^{\mu\nu}\partial_\mu\partial_\nu G_\lambda^\lambda + G^{\mu\nu}\partial_\mu\partial_\lambda G_\nu^\lambda \\ &- \frac{1}{M_{\text{Planck}}}\,G^{\mu\nu}T_{\mu\nu}\Big]\end{aligned} \tag{164}$$

Two things we observe:

1. the graviton spin–2 propagator is a mess
2. the interaction with massless Standard Model particles is easy

Next, we have to compute the energy–momentum tensor, for example for QED:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\,\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \\ &= -\frac{1}{2}\,\partial_\mu A_\nu\,\partial^\mu A^\nu + \frac{1}{2}\,\partial_\mu A_\nu\,\partial^\nu A^\mu\end{aligned} \tag{165}$$

which means

$$\frac{\partial\mathcal{L}}{\partial A_\nu} = 0 \qquad\qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2}\partial^\mu A^\nu + \frac{1}{2}\partial^\nu A^\mu \tag{166}$$

and gives for the Euler–Lagrange equations (Maxwell equation):

$$0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = -\frac{1}{2}\partial_\mu \partial^\mu A^\nu + \partial^\nu(\partial_\mu A^\mu) = -\frac{1}{2}\left(\Box A^\nu - \partial^\nu(\partial_\mu A^\mu)\right) \tag{167}$$

Remember the link with Noether's theorem and the conserved current

$$\partial_\mu j_\mu(x) = 0 \qquad \text{with} \qquad j^\mu = \frac{\partial}{\partial(\partial_\mu A_\nu)}\delta A_\nu \tag{168}$$

Similarly, define the energy–momentum tensor:

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)}\partial^\nu A_\rho - \mathcal{L}\eta^{\mu\nu} \\ &= \left(-\frac{1}{2}\partial^\mu A^\rho + \frac{1}{2}\partial^\rho A^\mu\right)\partial^\nu A_\rho + \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\eta^{\mu\nu} \\ &= -\frac{1}{2}\left(\partial^\mu A^\rho - \partial^\rho A^\mu\right)\partial^\nu A_\rho + \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\eta^{\mu\nu} \\ &= -\frac{1}{2}F^{\mu\rho}\partial^\nu A_\rho + \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\eta^{\mu\nu} \end{aligned} \tag{169}$$

From this we can (following Richard Ball's lecture) compute the symmetric energy–momentum tensor

$$\boxed{T^{\mu\nu} = -F^{\mu\rho}F_\rho^\nu + \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\ \eta^{\mu\nu}} \tag{170}$$

Including fermions we find the energy–momentum tensor for complete massless QED:

$$\begin{aligned} \frac{-1}{M_{\text{Planck}}}T_{\mu\nu}G^{\mu\nu} = \frac{-1}{M_{\text{Planck}}}\Big[&\frac{i}{4}\overline{\Psi}\left(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu\right)\Psi - \frac{i}{4}\left(\partial_\mu\overline{\Psi}\gamma_\nu + \partial_\nu\overline{\Psi}\gamma_\mu\right)\Psi \\ &+ \frac{1}{2}e_Q\overline{\Psi}\left(\gamma_\mu A_\nu + \gamma_\nu A_\mu\right)\Psi + F_{\mu\rho}F_\nu^\rho + \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\Big]G^{\mu\nu} \end{aligned} \tag{171}$$

To obtain the Feynman rules we have to just extract the terms proportional to the relevant external fields:

$$\begin{aligned} &f(k_1) - f(k_2) - G_{\mu\nu} && -\frac{i}{4M_{\text{Planck}}}\left(W_{\mu\nu} + W_{\nu\mu}\right) \\ & && \text{with} \quad W_{\mu\nu} = (k_1 + k_2)_\mu\gamma_\nu \\ &f(k_1) - f(k_2) - A_\sigma - G_{\mu\nu} && -\frac{i}{2M_{\text{Planck}}}e_Q\left(X_{\mu\nu\sigma} + X_{\nu\mu\sigma}\right) \\ & && \text{with} \quad X_{\mu\nu\sigma} = \gamma_\mu\eta_{\nu\sigma} \\ &A_\rho(k_1) - A_\sigma(k_2) - G_{\mu\nu} && -\frac{i}{M_{\text{Planck}}}\left(W_{\mu\nu\rho\sigma} + W_{\nu\mu\rho\sigma}\right) \\ & && \text{with} \quad W_{\mu\nu\rho\sigma} = \frac{1}{2}\eta_{\mu\nu}\left(k_{1\sigma}k_{2\rho} - (k_1\cdot k_2)\eta_{\rho\sigma} + ...\right) \end{aligned} \tag{172}$$

The same thing we can do for QCD (gluon with Dirac and SU(3) indices) to be able to compute LHC cross sections:

$$\begin{aligned} &f - f - g_\sigma^a - G_{\mu\nu} && -\frac{i}{2M_{\text{Planck}}}g_S T^a\ \left(X_{\mu\nu\sigma} + X_{\nu\mu\sigma}\right) \\ &g_\rho^a - g_\sigma^b - G_{\mu\nu} && -\frac{i}{M_{\text{Planck}}}g^{ab}\left(W_{\mu\nu\rho\sigma} + W_{\nu\mu\rho\sigma}\right) \end{aligned} \tag{173}$$

plus a $gggG$ vertex due to gluon self coupling...

5. ADD gravitons at the LHC

Flat and large extra dimensions are often named ADD after the early papers by Nima Arkani-Hamed, Savas Dimopoulos and Gia Dvali. We can compute the rate for real graviton emission at the LHC $pp \to KK$+jet on the parton level, using the Feynman rules derived above. It is two–step procedure, first computing the rate for the radiation of one KK state and then adding the entire KK tower:

$$\begin{aligned} d\sigma^{\text{one graviton}} &= | < f, G|T^{\mu\nu} h_{\mu\nu}|p_1, p_2 > |^2 \, (2\pi)^4 \, \delta^4(p_i - p_f) \frac{d\Phi_f}{F(p_1, p_2)} \\ d\sigma^{\text{KK tower}} &= d\sigma^{\text{one graviton}} \frac{S_{\delta-1} \, m^{n-1} \, dm}{(2\pi M_\star)^n} \left(\frac{M_{\text{Planck}}}{M_\star} \right)^2 \end{aligned} \tag{174}$$

The (huge) factor M^2_{Planck} from the KK tower summation gets absorbed into the matrix element square, i.e the effective coupling we see after adding the tower is $1/M_\star \sim 1/$ TeV instead of $1/M_{\text{Planck}}$, because of the integration over the all states in the KK tower!
Virtual s–channel gravitons can be observed in $q\bar{q} \to \mu^+\mu^-$ and $gg \to \mu^+\mu^-$ processes. The amplitude reads

$$\begin{aligned} \mathcal{A} &= \frac{1}{M^2_{\text{Planck}}} \sum \left(T_{\mu\nu} \frac{P_{\mu\nu\alpha\beta}}{s - m^2_{\text{KK}}} T_{\alpha\beta} + \frac{n-1}{3(n+2)} \frac{T^\mu_\mu T^\nu_\nu}{s - m^2_{KK}} \right) \\ &= \frac{1}{M^2_{\text{Planck}}} \sum \left(T_{\mu\nu} \frac{P_{\mu\nu\alpha\beta}}{s - m^2_{\text{KK}}} T_{\alpha\beta} \right) \qquad \text{for massless particles} \\ &= \frac{1}{M^2_{\text{Planck}}} \sum \left(T_{\mu\nu} \frac{\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\nu}\eta_{\alpha\beta}/3}{2(s - m^2_{\text{KK}})} T_{\alpha\beta} \right) \qquad \text{leading in } 1/m_{\text{KK}} \\ &= \frac{1}{M^2_{\text{Planck}}} \sum \left(\frac{1}{s - m^2_{\text{KK}}} \frac{1}{2} \left(T_{\mu\nu} T^{\mu\nu} + T_{\mu\nu} T^{\nu\mu} - 0 \right) \right) \end{aligned} \tag{175}$$

Because the KK tower couples universally to Standard Model particles, the virtual–graviton amplitude is simply a sum over propagators, in our case in the s channel

$$\boxed{\mathcal{A} = \frac{1}{M^2_{\text{Planck}}} T_{\mu\nu} T^{\mu\nu} \sum \frac{1}{s - m^2_{\text{KK}}}} \tag{176}$$

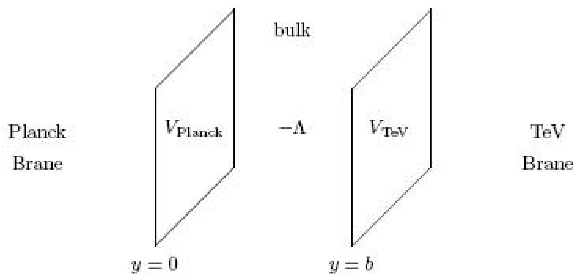
Again we integrate over the KK tower, up to a cutoff in the m_{KK} integral Λ and obtain a general dimension–8 operator

$$\mathcal{A} \sim \frac{S_{\delta-1}}{2} \frac{\Lambda^{n-2}}{M_\star^{n+2}} \tag{177}$$

This is not good because of the powers of the unknown cutoff in the numerator. A good effective theory should not give cross section predictions which basically require knowledge of the UV completion of the theory to produce sensible results. We can make such assumptions for example completing our KK theory with open or closed string resonances. Or we simply observe that gravity might be non-perturbatively UV–save and use this behavior to compute well–defined LHC cross sections. But we are still thinking about how to solve this...

D. Warped extra dimensions

Briefly after the flat (ADD) models, another way of solving the hierarchy problem was suggested by Lisa Randall and Raman Sundrum. Again it makes use of one extra dimension, but one which is specifically not flat. This finite extra dimension is bounded by two branes, on one of which we exist with all Standard Model particles (RS-I).
Strictly speaking, we compactify our 5th dimension on a S^1/Z_2 orbifold. S^1 is simply a circle (just like the torus in ADD), which is equivalent to periodic boundary conditions. S^1/Z_2 means we map one half of this circle on the other, so we really only have half a circle with no periodic boundary conditions, but two different branes at $y = 0$ and $y = b$, y being the additional space coordinate x^4.



I will skip everything that has to do with the cosmological constant and the Planck brane and focus on the hierarchy problem, *i.e.* $m_H L_L M_{\rm Planck}$ on the TeV brane. Instead, we will focus on the TeV brane with its effective 4–dimensional gravitons and their Feynman rules

1. *Newtonian gravity in a warped extra dimension*

Nobody can stop us from postulating a 5–dimensional metric:

$$\boxed{ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2} \quad \Leftrightarrow \quad g_{AB} = \begin{pmatrix} e^{-2k|y|}\eta_{\mu\nu} & 0 \\ 0 & \eta_{jk} \end{pmatrix} \tag{178}$$

The metric in 4 orthogonal directions to y depends on $|y|$. The absolute value appearing in $|y|$ corresponds to the Z_2 (orbifolding) as S^1/Z_2. When looking at our (3+1)–dimensional brane we can take into account the warp factor $e^{-2k|y|}$ in two ways (with some caveats):

1. use $g_{\mu\nu} = \eta_{\mu\nu}e^{-2k|y|}$ everywhere, which is a pain but possible
2. replace x^μ in 5 dimensions by effective coordinates $e^{-k|y|}d\tilde{x}^\mu$ and $g_{\mu\nu}$ by $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ (tilde means 4–dimensional variables)

The second vision means we shrink our effective 4–dimensional metric along y and forget about the curved space, because the warp factor does not depend on x^μ. The general–relativity action for Newtonian gravity we can write in terms of the 5–dimensional fundamental Planck scale $M_{\rm RS}$. In our hand–waving argument we have to transform the 5–dimensional Ricci scalar. Just looking at the mass dimensions we see that R has mass dimension 2, or by looking at the definition of R x dimension (-2). This suggests that the 4–dimensional Ricci scalar $\tilde{R}$ which we see in 4 dimensions should roughly scale like $x^{-2} \sim \tilde{x}^{-2}\exp(+2k|y|)$, leading us to a wild guess $R \sim \tilde{R}\exp(+2k|y|)$. The formula for the action with separated x and y integrals we start from already includes the effective 4–dimensional coordinates:

$$\begin{aligned}
S &= -\frac{1}{2}\int_0^b dy \int d^4\tilde{x}\, e^{-4k|y|} R\, M_{\rm RS}^3 \\
&\sim -\frac{M_{\rm RS}^3}{2}\int_0^b dy \int d^4\tilde{x}\, e^{-4k|y|}\; \tilde{R}\, e^{2k|y|} \\
&= -\frac{M_{\rm RS}^3}{2}\int_0^b dy\, e^{-2k|y|} \int d^4\tilde{x}\, \tilde{R} \\
&= -\frac{M_{\rm RS}^3}{2}\left(-\frac{1}{2k}e^{-2kb}+\frac{1}{2k}\right)\int d^4\tilde{x}\; \tilde{R} \qquad \text{obviously } b>0 \\
&= -\frac{M_{\rm RS}^3}{4k}\left(1-e^{-2kb}\right)\int d^4\tilde{x}\; \tilde{R} \\
&\sim -\frac{M_{\rm RS}^3}{4k}\int d^4\tilde{x}\tilde{R} \qquad \text{assume } kb \gg 1, \text{ for reasons seen later} \\
&\equiv -\frac{M_{\rm Planck}^2}{2}\int d^4\tilde{x}\; \tilde{R} \qquad \text{the usual dimensional–analysis matching} \Rightarrow
\end{aligned} \tag{179}$$

The naive matching with 4–dimensional Newtonian gravity (in this case just naive dimensional analysis) means $M_{\rm Planck}^2 \sim M_{\rm RS}^3/(2k)$. This does not solve the hierarchy problem because it looks like $M_{\rm RS} \sim k \sim M_{\rm Planck} \sim 10^{19}$ GeV is the most reasonable solution.

Fortunately, this is not the whole story. Consider the Standard Model Lagrangian on the TeV brane ($y = b$) in the $\tilde{x}^\mu$ coordinates, *i.e.* with a warp factor. If we want to solve the hierarchy problem, the scalar Higgs part is crucial:

$$\begin{aligned}
S_{\rm SM} &= \int d^4\tilde{x}\, e^{-4kb}\, \mathcal{L}_{\rm SM} \\
&= \int d^4\tilde{x}\, e^{-4kb}\; \left[(D_\mu H)^\dagger (D^\mu H) - \lambda(H^\dagger H - v^2)^2 + ...\right]
\end{aligned} \tag{180}$$

From the Higgs–mass term we see that we can rescale all Standard Model fields — in that case H as well as v — by the warp factor on the TeV brane $\exp(-kb)$. The same we have to do for the space coordinate, as described above and for gauge fields which appear in the covariant derivative. To get rid of the entire pre-factor we have to absorb four powers of the $\exp(-kb)$ in each term of the Standard Model Lagrangian.

If we only consider contributions to $\mathcal{L}_{\rm SM}$ of mass dimension 4, we can simply rescale all SM fields according to their mass dimension:

$$\begin{aligned}
\tilde{H} &= e^{-kb} H \qquad && \text{scalars} \\
\tilde{A}_\mu &= e^{-kb} A_\mu \qquad && \text{or } \tilde{D}_\mu = e^{-kb} D_\mu \\
\tilde{\Psi} &= e^{-3kb/2}\Psi \qquad && \text{fermions}
\end{aligned} \tag{181}$$

which also means for all masses

$$\begin{aligned}
\tilde{m} &= e^{-kb} m \\
\tilde{v} &= e^{-kb} v
\end{aligned} \tag{182}$$

while the Yukawa couplings as dimensionless parameters in the Lagrangian are not affected. If we assume $kb \sim 35$ we find

$$\boxed{\tilde{v} \sim 0.1\, e^{-kb}\, M_{\rm Planck} \sim 0.1 \text{ TeV}} \tag{183}$$

Note that the derived Planck scale $M_{\rm Planck}$ is still large. To solve the hierarchy problem we have shifted all dimensionful parameters, including the Higgs mass by the warp factor $e^{-k|y|} = e^{-kb}$. The fundamental Higgs mass and the fundamental Planck mass are of the same order, only the derived Higgs mass (and all mass scales on the TeV brane) appears smaller, because of the warped geometry in the 5th dimension. In contrast, on the Planck brane at, where the warp factor is $\exp(-k|y|) = 1$, nothing has happened.

Before we introduce metric fluctuations (gravitons) into our RS model, it turns out to be useful to rewrite the metric by rescaling the 5th dimension $y \to z$ to be able to write the metric as:

$$\boxed{ds^2 = e^{-A(z)}\left(g_{\mu\nu}dx^\mu dx^\nu - dz^2\right)} \tag{184}$$

To simplify things we assume for the following brief discussion $y > 0$. This is obviously justified, as long as we limit our interest to the TeV brane. First, we define $A(z) = 2ky$ and rewrite the metric using the ansatz:

$$e^{-2ky} = e^{-A(z)} = \frac{1}{(1+kz)^2} \quad \Leftrightarrow \quad A(z) = 2\log(k|z|+1) \tag{185}$$

The Planck brane at $y = 0$ sits at $z = 0$. Assuming $k > 0$ we find that $y > 0$ corresponds to $z > 0$. To check if we indeed obtain the correct metric, we start from the two variables being connected as:

$$\begin{aligned} & y = \frac{1}{k}\log(1+kz) \qquad \Leftrightarrow \qquad z = \frac{1}{k}\left(e^{ky}-1\right) \qquad \Rightarrow \qquad \frac{dz}{dy} = e^{ky} \\ \Rightarrow \qquad & dy = e^{-ky}\,dz = e^{-A(z)/2}\,dz \end{aligned} \tag{186}$$

and indeed find the correct pre-factor of dz^2.
To introduce tensor gravitons we write the relevant part of the metric:

$$ds^2 = e^{-A(z)}\left(\eta_{\mu\nu} + h_{\mu\nu}(x,z)\,dx^\mu dx^\nu - dz^2\right) \tag{187}$$

The left-hand side of Einstein's equations we know is $G_{AB} = R_{AB} - R\,g_{AB}/(n+2)$. Including a finite warp factor $A = 2ky \neq 0$ gives rise to an additional term, which in our case ($g_{AB} = e^{-A}\eta_{AB}$) reads:

$$\begin{aligned} \delta G_{AB} &= \frac{2+n}{2}\left[\frac{1}{2}\partial_A A\partial_B A + \partial_A\partial_B A + \eta_{AB}\left(\partial_C\partial^C A - \frac{1+n}{4}\partial_C A\partial^C A\right)\right] \\ &= \frac{3}{2}\left[\frac{1}{2}\partial_A A\partial_B A + \partial_A\partial_B A + \eta_{AB}\left(\partial_C\partial^C A - \frac{1}{2}\partial_C A\partial^C A\right)\right] \qquad\qquad d=5 \\ &= \begin{cases} \frac{3}{2}\left(\frac{1}{2}A'^2 + A'' - A'' + \frac{1}{2}A'^2\right) = \frac{3}{2}A'^2 & \qquad G_{55} \\ \frac{3}{2}\eta_{\mu\nu}\left(A'' - \frac{1}{2}A'^2\right) & \qquad G_{\mu\nu} \end{cases} \end{aligned} \tag{188}$$

Combined with some source–free right–hand side of Einstein's equations just proportional to the cosmological constant, this gives us the proper description of our two branes. As a matter of fact, the G_{55} equation is already solved by our ansatz for $A(z)$.
Instead of looking at the branes in 5–dimensional space, we use the formula to write down the the effects of introducing a graviton perturbation on the TeV brane. Csaba Csaki leaves calculating the additional contribution δG_{AB} introduced by $A \neq 0$ in $g_{AB} = e^{-A}(\eta_{AB} + h_{\mu\nu})$ in the gauge $h^\mu_\mu = 0 = \partial_\mu h^\mu_\nu$ as a fairly involved exercise, and I will do the same thing. The Einstein equations without sources become:

$$-\frac{1}{2}\partial_C\partial^C h_{\mu\nu} + \frac{2+n}{4}\,\partial^C A\,\partial_C h_{\mu\nu} = 0 \tag{189}$$

They have a linear term which does not look at all like an equation of motion and which we therefore do not like. We can get rid of it rescaling (as usual) $h_{\mu\nu} = e^{(2+n)/4}\widetilde{h}_{\mu\nu}$, according to the bosonic mass dimension $[h] = m^{1+n/2}$. This gives

$$\begin{aligned} -\frac{1}{2}\partial^C\partial_C\widetilde{h}_{\mu\nu} + \left(\frac{(2+n)^2}{32}\partial^C A\,\partial_C A - \frac{2+n}{8}\partial_C A\,\partial^C A\right)\widetilde{h}_{\mu\nu} = 0 & \\ -\frac{1}{2}\partial^C\partial_C\widetilde{h}_{\mu\nu} + \left(\frac{9}{32}A'^2 - \frac{3}{8}A''\right)\widetilde{h}_{\mu\nu} = 0 \qquad \text{using} A = A(z) & \end{aligned} \tag{190}$$

as the equation of the motion for the rescaled graviton field $\widetilde{h}_{\mu\nu}$. We can solve this equation of the motion for $\widetilde{h}_{\mu\nu}(x,z)$ by separating variables:

$$\widetilde{h}_{\mu\nu}(x,z) = \widehat{h}_{\mu\nu}(x)\,\Phi(z) \tag{191}$$

which yields

$$\begin{aligned}0 &\equiv -\partial^C\partial_C\left(\widehat{h}_{\mu\nu}(x)\,\Phi(z)\right) + \left(\frac{9}{32}A'^2 - \frac{3}{8}A''\right)\widehat{h}_{\mu\nu}(x)\,\Phi(z)\\ &= -\left(\partial^C\partial_C\widehat{h}_{\mu\nu}(x)\right)\Phi(z) - \widehat{h}_{\mu\nu}(x)\left(\partial_z^2\Phi(z)\right) + \left(\frac{9}{32}A'^2 - \frac{3}{8}A''\right)\widehat{h}_{\mu\nu}(x)\Phi(z)\end{aligned} \tag{192}$$

If we simply give a mass to the tensor graviton $\widehat{h}_{\mu\nu}$ using the ansatz

$$\partial_\mu\partial^\mu\widehat{h}_{\mu\nu} = m^2\widehat{h}_{\mu\nu} \tag{193}$$

we can plug this into the equation of motion and get an equation out of which $\widehat{h}_{\mu\nu}$ drops out trivially:

$$\begin{aligned}-m^2\widehat{h}_{\mu\nu}\,\Phi - \left(\partial_z^2\Phi\right)\widehat{h}_{\mu\nu} + \left(\frac{9}{16}A'^2 - \frac{3}{4}A''\right)\widehat{h}_{\mu\nu}\,\Phi = 0\\ \Leftrightarrow \qquad -\left(\partial_z^2\Phi\right) + \left(\frac{9}{16}A'^2 - \frac{3}{4}A''\right)\Phi = m^2\Phi\end{aligned} \tag{194}$$

This is a Schrödinger–type equation of Φ, with a potential term:

$$V(z) = \frac{9}{16}A'^2 - \frac{3}{4}A'' \tag{195}$$

Given the form $A(z) = 2\log(k|z|+1)$, we can compute the potential

$$\begin{aligned}z>0 \qquad & A' = \frac{2}{kz+1}k = \frac{2k}{kz+1} = \frac{2k}{k|z|+1} && \Rightarrow \qquad A'^2 = \frac{4k^2}{(k|z|+1)^2}\\ & A'' = 2k\cdot\frac{-1}{(kz+1)^2}\cdot k = -\frac{2k^2}{(k|z|+1)^2}\\ z<0 \qquad & A' = \frac{2}{-kz+1}(-k) = \frac{-2k}{k|z|+1} && \Rightarrow \qquad A'^2 = \frac{4k^2}{(k|z|+1)^2}\\ & A'' = -2k\cdot\frac{}{(-kz+1)^2}\cdot(-k) = \frac{2k^2}{(k|z|+1)^2}\end{aligned} \tag{196}$$

For the potential on our brane this means ($z>0$):

$$V(z) = \frac{9}{16}\frac{4k^2}{(k|z|+1)^2} + \frac{3}{4}\frac{2k^2}{(k|z|+1)^2} = \frac{15}{4}\frac{k^2}{(k|z|+1)^2} \tag{197}$$

First, we have a zero mode which solves the equation:

$$\begin{aligned}&-\partial_z^2\Phi^{(0)} + V(z)\Phi^{(0)} = 0\\ &\Rightarrow \qquad \Phi^{(0)}(z) = e^{-3A(z)/4}\\ &\Rightarrow \qquad h^0_{\mu\nu} = e^{+3A/4}\,\widetilde{h}^{(0)}_{\mu\nu} = e^{+3A/4}\,\hat{h}^{(0)}_{\mu\nu}\,\Phi^{(0)} = \hat{h}^{(0)}_{\mu\nu}(x)\end{aligned} \tag{198}$$

Rewriting $z\to y$, we find $\Phi^{(0)}(y) = e^{-3k|y|/4} = e^{-3kb/4}$ on our TeV brane. Indeed, gravity on the TeV brane is weak because of the exponentially suppressed wave–function overlap.

Using the form of $V(z)$ we can compute the masses of the KK gravitons on the TeV brane

$$-\partial_z^2\Phi + \frac{15}{4}\frac{k^2}{(k|z|+1)^2}\Phi = m^2\Phi \tag{199}$$

The boundary conditions on the brane are given by the orbifold identification $y \to -y$ which requires for $(z > 0)$

$$\begin{aligned} 0 \equiv -\partial_z h_{\mu\nu} = \partial_z(e^{+3A/4}\hat{h}_{\mu\nu}\Phi) &= \left(\frac{3}{4}A'\Phi + \partial_z\Phi\right) e^{3A/4}\hat{h}_{\mu\nu} \\ &= \left(\frac{3}{2}\frac{k}{kz+1}\Phi + \partial_z\Phi\right) e^{3A/4}\hat{h}_{\mu\nu} \end{aligned} \tag{200}$$

which implies

$$\partial_z^2\Phi = -\frac{3}{2}k\Phi\Big|_{\text{Planck}} \qquad \partial_z^2\Phi = -\frac{3}{2}\frac{k}{kz+1}\Phi\Big|_{\text{TeV}} \tag{201}$$

With these boundary conditions the solution of the equation of motion can be expressed in terms of Bessel functions, which are numbered by an index which corresponds to the mass introduced above:

$$\Phi_m(z) = \frac{1}{\sqrt{kz+1}}\left[a_m\, Y_2\left(m\left(z+\frac{1}{k}\right)\right) + b_m\, J_2\left(m\left(z+\frac{1}{k}\right)\right)\right] \tag{202}$$

More importantly, the masses of these modes are given in terms of the roots of the Bessel function

$$\boxed{m_j = x_j\, k\, e^{-kb}} \qquad \text{with } J_1(x_j) = 0$$
$$\text{or } x_j = 3.8, 7.0, 10.2, 16.5, .. \text{ for } j = 1, 2, 3, 4... \tag{203}$$

This means that the KK excitations in the RS I model with one warped extra dimensions are not quite equally spaced. To compute the mass values we remember that we can choose $kb \sim 35$ and $k \sim M_{\text{Planck}}$ to solve the hierarchy problem: $ke^{-kb} \sim$ TeV. In other words, the KK gravitons in the warped model have TeV–scale masses and mass differences. Obviously, this is phenomenologically very different for the large (ADD) extra dimensions. For warped extra dimensions we will not produce a tightly spaced KK tower, but for example distinct heavy s–channel excitations. One advantage of such a scenario is that we can measure things like the KK masses and spins at colliders directly.

To answer the question if we can measure these properties we have to compute the coupling strength of KK gravitons to matter, like quarks or gluons or electrons as the initial state in collider experiments. Remember that in the ADD case we had found tiny Planck–suppressed couplings for each individual KK graviton, which corresponded to an inverse–TeV–scale coupling once we integrated over the KK tower. For the warped model the relative coupling strengths on the Planck brane and on the TeV brane are approximately given by the ration of the wave function overlaps. While the zero–mode graviton has to be strongly localized on the Planck brane, to explain the weakness of Newtonian graviton the TeV brane, the KK gravitons do not have strongly peaked wave functions in the additional dimension. Hence, the ratio of wave functions becomes (assuming that the Bessel functions with their normalized arguments will not make a big difference):

$$\boxed{\frac{\Phi(z)\big|_{\text{TeV}}}{\Phi(z)\big|_{\text{Planck}}} \sim \frac{\sqrt{kz+1}\big|_{\text{Planck}}}{\sqrt{kz+1}\big|_{\text{TeV}}} \sim \frac{1}{e^{kb/2}}} \tag{204}$$

The coupling of the KK states is given by the left–hand side of Einstein's equations which enters the Lagrangian just as for the large extra dimensions. We have to distinguish between the flat zero mode with un-suppressed wave function overlap and the KK modes with the wave function normalization $\sim 1/\sqrt{kz+1}$:

$$\mathcal{L} \sim \frac{1}{M_{\text{Planck}}}T^{\mu\nu}h^{(0)}_{\mu\nu} + \frac{1}{M_{\text{Planck}}e^{-kb}}T^{\mu\nu}\sum h^{(m)}_{\mu\nu} \tag{205}$$

This means that the Randall–Sundrum–KK gravitons indeed couple with TeV scale gravitational strength and can be produced at colliders in sufficient numbers, provided they are not too heavy. Similarly to the flat extra dimensions, the couplings of the different KK excitations are (approximately) universal.

E. Ultraviolet Completions

In this addendum I will briefly describe the problem how to formulate an ultraviolet completion of extra–dimensional models. For example in ADD models the LHC can explicitely probe energy ranges above M_{Planck}, either in real graviton emission or in virtual graviton exchange. As we saw in the last sections, real graviton emission as well as virtual graviton exchange is only suppressed by powers of $M_\star$, after we integrate over the entire KK tower.
Strictly speaking, this statement is not correct. When we for example write down the higher–dimensional operator arising from s–channel graviton exchange, it will come with powers of M_{Planck} in the denominator, due to the graviton couplings. In addition, it will have powers of the ultraviolet cutoff Λ of the KK integration in the numerator, and the two of them only cancel if we assume $\Lambda = M_{\text{Planck}}$. This is motivated by the conservative estimate that for energies above M_{Planck} our KK effective theory does not describe the graviton exchange correctly and that setting all contributions arising from the ultraviolet completion of our theory to zero will be on the safe side for LHC predictions. If we knew the structure of the ultraviolet completion of the KK effective theory, which would need to be something like a quantum theory of gravity, we could compute these contributions and take them into account for the LHC cross section prediction.

1. *String theory*

One possible ultraviolet completion of gravity could be string theory. The effects of such a hypothetical UV completion are nicely computed in a classical paper by Maxim Perelstein and others (hep-ph/0001166): in general, we can compute for example the scattering $q\bar{q} \to \mu^+\mu^-$ without using Feynman rules, but will nevertheless arrive at the Standard–Model result as the leading term. In addition, string theory predicts a common form factor for all different helicity amplitudes contributing to this process. This form factor is essentially the Veneziano amplitude and includes the inverse string scale $\alpha' = 1/M_S^2$. While we do not exactly know the size of this scale, for extra–dimensional models it has to be between the well–tested weak scale $v = 246$ GeV and $M_\star$. Perelstein and collaborators compute this Veneziano form factor for the process $e^+e^- \to \gamma\gamma$, which is equivalent to $gg \to \mu^+\mu^-$, and expand it in powers of α':

$$\frac{\Gamma(1-\alpha' s)\,\Gamma(1-\alpha' t)}{\Gamma(1-\alpha'(s+t))} = \frac{\Gamma(1-s/M_S^2)\,\Gamma(1-t/M_S^2)}{\Gamma(1-(s+t)/M_S^2)} = 1 - \frac{\pi^2}{6}\,\frac{st}{M_S^4} + \mathcal{O}\left(M_S^{-6}\right) \tag{206}$$

The parameters s and t are the usual Mandelstam variables in the $(2 \to 2)$ process. This form of the string corrections corresponds to our KK effective field theory, modulo a normalization factor which relates the two mass scales M_S and M_{Planck}. Hence, this series in M_S is not what we are interested as the UV completion of our theory.
The string theory approach becomes more interesting at higher energies. The Veneziano form factor we gave above is proportional to $\Gamma(1-s/M_S^2)$, which has poles for negative integer arguments $1 - s/M_S^2 = -(n+1)$ for $n = 1, 2, ...$ These poles lie at $s = nM_S^2$, which tells us that the string resonances in the s channels have to appear as $1/(1-nM_S^2)$ in the transition amplitude. Starting from the energy threshold M_S our UV completion consists of real particles of mass $\sqrt{n}M_S$ appearing in our amplitude. This is the kind of UV completion we are looking for and which we can base cross–section calculations on.
Note that scattering partons with energies above the fundamental Planck scale probes the trans–Planckian regime of our theory of gravity without necessarily producing black–hole solutions. Black holes can occur in colliders, but they require the two partons to scatter at very high energies while at the same time getting closer than the Schwarzschild radius. The Schwarzschild radius r_h depends on the collider energy, and the production cross section of a black hole is essentially the geometric factor πr_h^2, provided the two beam collide with a small enough impact parameter. The question how these black holes can then be detected depends largely on the question is we actually produce a thermalized black hole, which would just decay to may particles via Hawking radiation, whereas otherwise the signature would look very similar to an old–fashioned contact–interaction.

According to a classical paper by Weinberg, another UV completion of gravity could be described by the possible existence of a gravitational fixed point. Such a fixed point would not be a unique feature of extra–dimensional models, but in contrast to the four–dimensional case the LHC could observe it in such models with a low fundamental Planck scale. In other words, we can simply generalize a well-established field of gravitational research.
The starting point for our argument is a renormalization group analysis of the effective action of gravity, *i.e.* the generalization of the Einstein–Hilbert action to scale–dependent parameters:

$$\frac{1}{16\pi G_k} \int \sqrt{|g|} \left(\Lambda_k + R + \mathcal{O}(R^2) \right) + S_{\text{matter},k} + S_{\text{gf},k} + S_{\text{ghosts},k} \tag{207}$$

The first term in the action is the cosmological constant, the second term is the Ricci scalar describing free gravity, and the remaining terms are the Standard–Model action without any gravity terms. Because of the Ricci scalar's mass dimension two, higher powers of R correspond to higher orders in $1/M_\star$, the only scale present in the gravitational part of the action. We will briefly discuss the limitations of perturbative gravitation later.
The index k refers to an energy scale at which we evaluate these parameters, for example the gravitational coupling $G \sim 1/M_\star^2$. Scale–dependent parameters are of course nothing new, we know for example how to evaluate the strong coupling $\alpha_s(\mu_R)$ at proper values of the renormalization scale. What we need to know to evolve our theory from one scale to another is the renormalization group equation for the gravitational coupling. Since we know the mass dimension of the gravitational coupling constant we can use a renormalization scale μ to define its dimensionless version in $(4+n)$ dimensions and add the usual renormalization constant in front:

$$g(\mu) = G\,\mu^{2+n} \quad \longrightarrow \quad Z(\mu)^{-1}\,G\,\mu^{2+n} \tag{208}$$

As the anomalous dimension of any field or Lagrangian parameter we refer to the quantum (or renormalization) contribution to the classical mass dimension of the bare field or parameter appearing in the Lagrangian. In this case the anomalous dimension of the gravitational coupling is $\eta = -d\log Z/d\log\mu = -1/Z\, dZ/d\log\mu$. In terms of this anomalous dimension, which in general will be a function of μ, we can write down a renormalization group equation for $g(\mu)$:

$$\begin{aligned} \frac{dg}{d\log\mu} &= \frac{d}{d\log\mu}\left(\frac{1}{Z}\,G\,\mu^{2+n}\right) = G\left(-\frac{1}{Z^2}\frac{dZ}{d\log\mu}\,\mu^{2+n} + \frac{1}{Z}\,\mu\,\frac{d\mu^{2+n}}{d\mu}\right) \\ &= \frac{1}{Z}\,G\,\mu^{2+n}\left(-\frac{1}{Z}\frac{dZ}{d\log\mu} + (2+n)\right) = (\eta+n+2)\;g \end{aligned} \tag{209}$$

This equation can have two fixed points. First, vanishing values of g are stable with respect to scale variations, which means the running of the gravitational coupling has a fixed point at $g = 0$. A fixed point at the trivial value $g = 0$ we call a Gaussian fixed point. This fixed point exists for any value of η and describes the usual regime of Einstein–Hilbert gravity we know.
Let's assume that $\eta > 0$, so that the change of $g(\mu)$ with μ has a positive sign. This means that for large positive and negative values of $\log\mu$ there could be another fixed point for a finite values $g = g_*$, where

$$\boxed{\eta(\mu) = -(2+n)} \qquad\qquad \boxed{G \sim \frac{g_*}{\mu^{2+n}}} \tag{210}$$

The scale factor valid around the fixed–point regime implies that for small scales the dimensionful gravitational coupling would become large, while for large scale it would be suppressed by a scale factor $1/\mu^{2+n}$. Note that in this argument we have omitted constant terms in the solution of the differential equation, so that we should not claim that the gravitational coupling vanishes at large scales.

The system we really need to solve is a coupled set of differential equations including the renormalization group equation for $\eta(\mu)$ and for the cosmological constant $\lambda(\mu) = \Lambda(\mu)/\mu^2$. However, for example in the papers by Martin Reuter or Daniel Litim we see that the general pattern of the non–Gaussian fixed point does not change, and indeed the gravitational coupling will be asymptotically free, *i.e.* become small in the ultraviolet. The expressions for the physical observables in the UV fixed point in the literature are

$$\lambda_* = \frac{D^2 - D - 4 - \sqrt{2D(D^2-D-4)}}{2(D-4)(D-1)}$$
$$g_* = \Gamma\left(\frac{D}{2}+2\right)\,(4\pi)^{D/2-1}\,\frac{\left(\sqrt{D^2-D-4}-\sqrt{2D}\right)^2}{2(D-4)^2(D+1)^2} \qquad D = 4+n \tag{211}$$

Note that this UV behavior of the gravitational couplings is exactly the opposite of what we usually think of when we are concerned with gravity becoming a strongly interacting theory at the Planck scale. Weak gravity in the ultraviolet we can think of as asymptotically free gravity.
In the usual sense we consider a theory renormalizable if in the far ultraviolet its coupling strength becomes infinitely small. Weinberg's approach, on which this study of the UV behavior of gravity builds, is to generalize the concept of renormalizability to theories with a finite UV limit of the coupling. Because of the vanishing of G in the UV, we still expect no unphysical UV divergences in such a theory. This of course does not mean that gravity will be a perturbatively renormalizable field theory — it cannot, because it has a coupling constant with an inverse mass dimension, but ultraviolet safety is a useful extension of the usual perturbative renormalizability condition which has been proven to hold for Yang–Mills theories.

Unfortunately, in the trans–Planckian energy regime we cannot write a perturbative series for example in R. However, the existence of a non–trivial fixed point for the gravitational coupling has been shown including higher–order corrections in the Einstein–Hilbert action up to $\sqrt{g}R^8$ and including a coupling to matter fields. While this ordering scheme in powers of R is of course not well defined once we are looking at energies beyond $M_\star$, there is no good reason for this fixed–point behavior to change at some arbitrary higher power of R. Moreover, it is interesting to notice that our fixed–point theory naively appears to break down if we include the R^2 term in the action. This term leads to propagators of the mass dimension $1/p^4$, which can be considered sub–leading remainders of the sum of two propagators with the leading behavior $1/p^2$, provided one of these propagators appears with a negative sign. Such particles are usually referred to as ghosts and are unphysical degrees of freedom. They should not appear in our theory! On the other hand, work by Gomez and Weinberg gives us reasons to believe that such ghost contributions vanish after taking into account all orders in R. Because one should not trust the perturbative expansion of the effective action in R this is a particularly welcome result, increasing our trust in the stable fixed–point behavior which has until now appeared order by order in R.

The next question is: how do we include the leading effects of this fixed–point behavior in our LHC calculations without having to take into account the running of all parameters with the scales for example present in the virtual–graviton propagator. The obvious way is to include a running gravitational coupling, following the paper by JoAnne Hewett and Tom Rizzo. The coupling of the integrated KK–graviton tower to the energy–momentum tensor, given by $1/M_\star^{2+n}$, is simply modified by a form factor

$$\frac{1}{M_\star^{2+n}} \longrightarrow \frac{1}{M_\star^{2+n}}\,\left[1+\left(\frac{\mu}{aM_\star}\right)^{2+n}\right]^{-1} \tag{212}$$

with a fudge factor $a \sim 1$. For large scales μ this form factor becomes smaller, (hopefully) regularizing the LHC cross section prediction in the ultraviolet.

When writing down the integral over the virtual KK graviton propagator $1/(s - m_{\text{KK}}^2)$ we see that there are two integrals, one over the KK tower m_{KK} and one over the partonic center-of-mass energy $\sqrt{s}$ or over the parton momentum fractions (x_1x_2). In the form–factor approach the authors choose $\mu = \sqrt{s}$, which regularizes this dimension of this integral, but not the other. The m_{KK} integral they still have to cut off and integrate into the form $1/M_\star^{2+n}$ instead of the integrand's single–graviton coupling $1/M_{\text{Planck}}^{2+n}$.

Separating the two integrations gives us another handle at the beneficial effects of the renormalization–group running of gravity. In the energy range $\sqrt{s} < M_\star$ we can clearly identify the IR and the UV regime of the m_{KK} integral. The transition between these two regimes should take part around $\Lambda_{\text{trans}} \equiv M_\star$, the only scale known to the theory. Because we do not know the exact matching behavior we simply assume a sudden change between the regimes at $m_{\rm KK} = \Lambda_{\rm trans}$. From QCD studies we expect the dominant difference between the IR and UV regimes to be the anomalous dimension of the gravitational coupling and of the graviton field. The scalar graviton propagator then becomes:

$$P(s, m_{\rm KK}) = \begin{cases} \dfrac{1}{s + m_{\rm KK}^2} & m < \Lambda_{\rm trans} \qquad ({\rm IR}) \\ \dfrac{M_\star^{n+2}}{(s + m_{\rm KK}^2)^{n/2+2}} & m > \Lambda_{\rm trans} \qquad ({\rm UV}) \end{cases} \tag{213}$$

In the regime $\sqrt{s} < M_\star$ this change in the anomalous dimension indeed regularizes the $m_{\rm KK}$ integration in the virtual–graviton amplitude. Compared to a simple cut off at Λ the effective dimension-8 operator describing virtual graviton exchange in the production process $gg \to \mu^+\mu^-$ shifts from

$$\begin{aligned} \mathcal{S} &= \frac{S_{n-1}}{M_\star^{2+n}} \int_0^\Lambda dm \; m^{n-1} \, P(s,m) \\ &= \frac{S_{n-1}}{M_\star^4} \, \frac{1}{n-2} \, \left(\frac{\Lambda}{M_\star} \right)^{n-2} \, \left[1 + \mathcal{O} \left(\frac{s}{\Lambda^2} \right) \right] \\ &\to \frac{S_{n-1}}{M_\star^4} \, \frac{1}{n-2} \, \left(\frac{\Lambda_{\rm match}}{M_\star} \right)^{n-2} \, \left(1 + \frac{n-2}{4} \right) \left[1 + \mathcal{O} \left(\frac{s}{\Lambda^2} \right) \right] \end{aligned} \tag{214}$$

We can refer to the cut-off result as the IR contribution of the integral, and it is indeed proportional to the cut-off Λ in the numerator. For the combined fixed–point IR and UV integral this dependence is replaced by a power dependence on the matching scale, which for good reasons we assume to be $M_\star$. The IR part of the integral is of course independent of the UV completion and a function of the number of extra dimensions n. In this specific case the UV part of the integral turns out to be independent of n, except for the geometry factor S_{n-1}. We see that for larger values of $n > 5$ the UV contribution can be numerically dominant when computing LHC signal rates.
In the remaining part of the integration region $\sqrt{s} > M_\star$ we probe gravity clearly beyond the Planck scale. This also means that in the LHC scattering amplitude the fundamental Planck scale should only appear as a coupling, but not as a dynamic mass scale. Modulo a c-number normalization we simply estimate any gravity–induced operator by factors of $\sqrt{s}$ to get the correct mass dimension. The matching around $\sqrt{s} \sim M_\star$ is unfortunately not determined: either we fix the matching scale to $M_\star$ and adjust the prefactor of the high–$\sqrt{s}$ contribution to match the well–known low–$\sqrt{s}$ solution; or we fix the normalization of both parts and compute the matching scale. Interestingly, the latter gives a matching scale more than a factor two below $M_\star$...to be continued....?

F. Literature

There is a huge number of papers available on extra dimensions, most notably a huge number of great original papers. Here, I would like to list some more pedagogical reviews which I read to prepare this lecture and which I can recommend to everybody who is interested in deepening their knowledge (ordered by appearance in the lecture):

– a very good and seriously complete review on dark matter is the one by Bertone, Hooper, Silk (hep-ph/0404175). Dan also wrote a popular book on the same topic, you can find it on Amazon

– the argument about the Higgs–mass divergence at one loop you can find in Martin Schmaltz' hep-ph/0210415

– a great collection of loop formulas and a great appendix including integrals is Rick Field's book 'Applications of Perturbative QCD'

– most of this lecture is based on Graham Kribs' TASI lecture (hep-ph/0605325)

– more very useful TASI lectures you can find by Csaba Csaki (hep-ph/0404096 and hep-ph/0510275)

– and by Raman Sundrum (hep-th/0508134)

– for the more formally interested, there is a great introduction by Gregory Gabadadze (hep-ph/0308112)

– the as far as I am concerned best paper written on extra dimensions is Gian Giudice, Riccardo Rattazzi and James Wells' hep-ph/9811291

– starting from some ideas on $n = 1$ we have tried to review the LHC prospects for ADD models in hep-ph/0408320

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III. LITTLE HIGGS MODELS

A. Electroweak Symmetry Breaking

To discuss the motivation of a new–physics model, like the Little–Higgs models, we have to sketch the Standard Model Lagrangian, including mass terms. These introductory comments are particularly nicely presented in Wolfgang Kilian's book, and I will try to follow his conventions. Fermion fields have mass dimension 3/2, so it is easy to add mass terms to the dimension-4 Lagrangian. The only thing we have to make sure is that we combine the left– and right–handed doublet and singlets properly

$$\mathcal{L}_3 \sim -\overline{Q}_L M_Q Q_R - \overline{L}_L M_L L_R + ... \tag{215}$$

Dirac mass terms simply link $SU(2)$ doublet fields for leptons and quarks with right–handed singlets and gives all fermions in the Standard Model masses. In general, these mass terms can be diagonal matrices in generation space, which implies that we might have to rotate the fermion field from an interaction basis into the mass basis where these mass matrices are diagonal. The only problem with these mass terms is that they are not gauge invariant... The interaction of fermions with gauge bosons is most easily written in terms of covariant derivatives. The terms

$$\mathcal{L}_4 \sim \overline{Q}_L i \not{D} Q_L + \overline{Q}_R i \not{D} Q_R + \overline{L}_L i \not{D} L_L + \overline{L}_R i \not{D} L_R - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} ... \tag{216}$$

describe electromagnetic interactions using such a covariant derivative $D_\mu = \partial_\mu + ieqA_\mu$ with the photon field collected in the field–strength tensor $A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The same form works for the weak interactions, except that the weak interaction knows about the chirality of the fermion fields, so we have to distinguish $\not{D} \to \not{D}_{L,R}$. The covariant derivatives in terms of the $SU(2)$ basis matrices read

$$\begin{aligned}
D_{L\mu} &= \partial_\mu + ieqA_\mu + ig_Z \left(-q s_W^2 + \frac{\tau^3}{2} \right) + i \frac{g}{\sqrt{2}} \left(\tau^+ W_\mu^+ + \tau^- W_\mu^- \right) \\
D_{R\mu} &= D_{L\mu} \Big|_{\tau \equiv 0} \\
\tau^+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \tau^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
\tau^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned} \tag{217}$$

Note that we can write the Pauli matrices as $\tau^{1,2,3}$ as well as $\tau^{+,-,3}$. The latter form of the generators corresponds to the two charged and one neutral vector bosons. While the usual basis is written in terms of complex numbers, the second set of generators reflects the fact that for $SU(2)$ as for any $SU(N)$ we can find a set of real generators in the adjoint representation. When we exchange the two bases we only have to make sure we get the factors $\sqrt{2}$ right

$$\begin{aligned}
&\sqrt{2} \left(\tau^+ W_\mu^+ + \tau^- W_\mu^- \right) = \sqrt{2} \begin{pmatrix} 0 & W_\mu^+ \\ 0 & 0 \end{pmatrix} + \sqrt{2} \begin{pmatrix} 0 & 0 \\ W_\mu^- & 0 \end{pmatrix} \equiv \tau^1 W_\mu^1 + \tau^2 W_\mu^2 = \begin{pmatrix} 0 & W_\mu^1 \\ W_\mu^1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -iW_\mu^2 \\ iW_\mu^2 & 0 \end{pmatrix} \\
&\Longleftrightarrow W_\mu^+ = \frac{1}{\sqrt{2}} \left(W_\mu^1 - iW_\mu^2 \right) \qquad W_\mu^- = \frac{1}{\sqrt{2}} \left(W_\mu^1 + iW_\mu^2 \right)
\end{aligned} \tag{218}$$

The third term in the Standard Model Lagrangian we have to have a close look at is the dimension-2 mass term for gauge bosons which we know as

$$\mathcal{L}_2 \sim M_W^2 W^{+,\mu} W_\mu^- + \frac{1}{2} M_Z^2 Z^\mu Z_\mu . \tag{219}$$

The factor 1/2 in front of the W mass corresponds to the factors $1/sqrt2$ in the $SU(2)$ generators $\tau^\pm$. Of course, in the complete Standard Model Lagrangian there are many additional terms, *e.g.* kinetic terms of all kinds, but they do not affect our discussion of $U(1)_Y$ and $SU(2)_L$ gauge invariance. We know already that the problems with gauge invariance lies in the dimension-2 and dimension-3 mass terms.

Again following Wolfgang's book we write down the local $U(1)_Y$ and $SU(2)_L$ transformations. We start with a slightly complicated–looking way of writing the abelian hypercharge $U(1)$ transformations, making it more obvious how they mix with the neutral component of $SU(2)$ to give the electric charge

$$\begin{aligned}
V^\dagger(x) = \exp\left(\frac{i}{2}\beta(x)\tau^3\right) \quad &\Leftrightarrow \quad V(x) = \exp\left(-\frac{i}{2}\beta(x)\tau^3\right) \\
\exp(-i\beta q)\exp\left(\frac{i}{2}\beta\tau^3\right) &= \exp\left(-i\beta\frac{\mathbb{1}+\tau^3}{2}\right)\exp\left(\frac{i}{2}\beta\tau^3\right) \qquad q \equiv \frac{y\mathbb{1}+\tau^3}{2} \\
&= \exp\left(-i\frac{\beta}{2}y\mathbb{1} - i\beta\frac{\tau^3}{2} + i\beta\frac{\tau^3}{2}\right) \qquad y_Q = \frac{1}{3} \quad y_L = -1 \\
&= \exp\left(-i\frac{\beta}{2}y\mathbb{1}\right)
\end{aligned} \tag{220}$$

The numbers $y_{Q,L}$ are the quark and lepton hypercharges of the $U(1)$ symmetry in the Standard Model. Properly combined with the isospin they give the correct electric charges $q_{Q,L}$. From the manipulations above we see that the combination of $\exp(-i\beta q)$ and $V(x)$ written down in the beginning is proportional to $\exp(\mathbb{1})$ and hence an abelian transformation. When combining the different exponentials a la Baker–Campbell–Hausdorff we have to remember that $\mathbb{1}$ commutes with any matrix, as does $\exp(-i\beta y_Q\mathbb{1}/2)$. Left and right–handed quark and lepton fields transform under the electric–charge $U(1)$ as

$$\begin{aligned}
L_L &\to \exp\left(-i\frac{\beta}{2}y_L\mathbb{1}\right)L_L \qquad Q_L \to \exp\left(-i\frac{\beta}{2}y_Q\mathbb{1}\right)Q_L \\
L_R &\to \exp\left(-i\frac{\beta}{2}q_L\mathbb{1}\right)L_R \qquad Q_R \to \exp\left(-i\frac{\beta}{2}q_Q\mathbb{1}\right)Q_R
\end{aligned} \tag{221}$$

Similarly, we define the local (adjoint) weak $SU(2)$ transformation

$$U(x) = \exp\left(-i\alpha^a(x)\frac{\tau^a}{2}\right) \qquad a = 1,2,3 \tag{222}$$

which only transforms the left–handed fermion fields and leaves the right–handed fields untouched

$$\begin{aligned}
L_L &\to UL_L \qquad Q_L \to UQ_L \\
L_R &\to L_R \qquad Q_R \to Q_R
\end{aligned} \tag{223}$$

It is obvious that left–right mass terms are not invariant under this left–handed $SU(2)$ gauge transformation

$$\overline{Q}_L M_Q Q_R \to_U \overline{Q}_L U^{-1} M_Q Q_R \neq \overline{Q}_L M_Q Q_R \tag{224}$$

In other words, to write a gauge–invariant Lagrangian for massive fermions (and vector bosons) we have to add something to our minimal Standard Model Lagrangian. Note that this addition does not have to be a fundamental scalar Higgs field, dependent on how picky we are with the properties of our new Lagrangian beyond its gauge invariance.

1. *Sigma Model*

One way of solving this problem which at this point almost looks like a cheap trick is to introduce an additional field $\Sigma(x)$. Properties like the quantum numbers of Σ will become obvious from it's appearance in the Lagrangian. Obviously, the equation of motion for the Σ field will also have to follow from the way we introduce it in the Lagrangian. We first use it to modify the fermionic mass term and make it gauge invariant under the weak $SU(2)$ transformation

$$\overline{Q}_L \Sigma M_Q Q_R \to_U \overline{Q}_L U^{-1} \Sigma^{(U)} M_Q Q_R \equiv \overline{Q}_L \Sigma M_Q Q_R \qquad \Longleftrightarrow \qquad \Sigma \to \Sigma^{(U)} = U\Sigma \tag{225}$$

The first thing we notice about Σ is it mass dimension $m^0 = 1$. The same we can do for the $SU(2)$ transformation V which mixes later on with the hypercharge

$$\begin{aligned}
\overline{Q}_L \Sigma M_Q Q_R \to_V & \overline{Q}_L V \exp(i\beta q) \Sigma^{(V)} M_Q \exp(-i\beta q) Q_R \\
&= \overline{Q}_L \Sigma^{(V)} V \exp(i\beta q) M_Q \exp(-i\beta q) Q_R \qquad \text{assuming } M_Q \text{ diagonal} \\
&= \overline{Q}_L \Sigma^{(V)} V M_Q Q_R \\
&\equiv \overline{Q}_L \Sigma V M_Q Q_R \\
\Sigma \to \Sigma^{(V)} = \Sigma V^\dagger & \qquad \Longleftrightarrow \qquad \boxed{\Sigma \to U \Sigma V^\dagger}
\end{aligned} \tag{226}$$

This means for any Σ with this transformation property the $\mathcal{L}_3$ part of the Lagrangian has the required $U(1) \times SU(2)$ symmetry. Note that from the way it transforms Σ is a 2×2 matrix with mass dimension zero. We have shown by construction that including a Σ field in the fermionic mass term indeed gives a $U(1)_Y$ and $SU(2)_L$-invariant Lagrangian, without saying much about possible representations of Σ for example in terms of physical fields

$$\boxed{\mathcal{L}_3 \sim -\overline{Q}_L \Sigma M_Q Q_R - \overline{L}_L \Sigma M_L L_R + \text{h.c.} + ...} \tag{227}$$

To write down a gauge–invariant gauge–boson mass we start with the left–handed covariant derivative

$$\begin{aligned}
D_{L\mu} &= \partial_\mu + ig' \left(q - \frac{\tau^3}{2} \right) B_\mu + ig W_\mu^a \frac{\tau^a}{2} \\
&= \partial_\mu + ig' \frac{y}{2} B_\mu + ig W_\mu^a \frac{\tau^a}{2}
\end{aligned} \tag{228}$$

We skip the reasoning for this, but whoever is interested can show that the covariant derivative acting on the Σ field in the gauge–symmetric Lagrangian has to be

$$D_\mu \Sigma = \partial_\mu \Sigma - ig' \Sigma B_\mu \frac{\tau^3}{2} + ig W_\mu^a \frac{\tau^a}{2} \Sigma \tag{229}$$

Instead of showing how we would have to write a gauge–invariant mass terms for the W and Z bosons we start with a promising ansatz. If we introduce $V_\mu \equiv \Sigma (D_\mu \Sigma)^\dagger$ and $T = \Sigma \tau^3 \Sigma^\dagger$ we can write the boson mass term as

$$\boxed{\mathcal{L}_2 = -\frac{v^2}{4} \, \text{Tr}[V_\mu V^\mu] - \beta' \frac{v^2}{8} \, \text{Tr}[T V_\mu] \, \text{Tr}[T V^\mu]} \tag{230}$$

The trace acts on the 2×2 $SU(2)$ matrices. We will show the specific form soon for the different gauge choices.

The problems in our Σ–field model are additional terms of mass dimension 4 we can write down using the (dimensionless) field Σ and which are gauge invariant. For such terms we have to find a selection rule or symmetry which only allows the Σ terms in the Lagrangian which we need to include massive fields. Without the trace we can construct terms which are forbidden by gauge invariance

$$\Sigma^\dagger \Sigma \to (U \Sigma V^\dagger)^\dagger (U \Sigma V^\dagger) = V \Sigma^\dagger U^\dagger U \Sigma V^\dagger = V \Sigma^\dagger \Sigma V^\dagger \neq \Sigma^\dagger \Sigma \tag{231}$$

On the other hand, $\text{Tr}(\Sigma^\dagger \Sigma) = \text{Tr}(V \Sigma^\dagger \Sigma V^\dagger) = \text{Tr}[\Sigma^\dagger \Sigma]$ is gauge invariant, which allows the additional potential terms (terms with no derivatives)

$$\boxed{\mathcal{L}_\Sigma = -\frac{\mu^2 v^2}{4} \text{Tr}(\Sigma^\dagger \Sigma) + \frac{\lambda v^4}{16} \left(\text{Tr}(\Sigma^\dagger \Sigma) \right)^2} \tag{232}$$

with properly chosen prefactors μ, v, λ. The factors μ and v have mass dimension one while λ has mass dimension zero. To give mass to the gauge bosons we have to assume that $\text{Tr}(\Sigma^\dagger\Sigma)$ assumes a finite value after we deal properly with the field Σ. The simplest way to achieve this is to generally assume

$$\Sigma(x) = \mathbb{1} \tag{233}$$

This assumption is called unitary gauge. In this gauge the covariant derivative again becomes

$$D_\mu\Sigma = igW_\mu^a \frac{\tau^a}{2} - ig' B_\mu \frac{\tau^3}{2} \tag{234}$$

Moreover, we can simply compute the auxiliary field V_μ in unitary gauge

$$\begin{aligned} V_\mu &= -igW_\mu^a \frac{\tau^a}{2} + ig' B_\mu \frac{\tau^3}{2} \\ &= -igW_\mu^+ \frac{\tau^+}{\sqrt{2}} - igW_\mu^- \frac{\tau^-}{\sqrt{2}} - igW_\mu^3 \frac{\tau^3}{2} + ig' B_\mu \frac{\tau^3}{2} \\ &= -i\frac{g}{\sqrt{2}} \left(W_\mu^+ \tau^+ - W_\mu^- \tau^- \right) - ig_Z Z_\mu \frac{\tau^3}{2} \qquad \text{with } Z_\mu = c_W W_\mu^3 - s_W B_\mu \text{ and } g_Z = \frac{g}{c_W}, g' = \frac{s_W}{c_W} g \end{aligned} \tag{235}$$

This field gives for the first of the two terms in the gauge–boson mass Lagrangian

$$\begin{aligned} \text{Tr}[V_\mu V^\mu] &= -2\,\frac{g^2}{2} W_\mu^+ W_\mu^- \text{Tr}(\tau^+\tau^-) - \frac{g_z^2}{4} Z_\mu Z_\mu \text{Tr}(\tau_3^2) \\ &= -g^2 W_\mu^+ W_\mu^- - \frac{g_z^2}{2} Z_\mu Z_\mu \end{aligned} \tag{236}$$

The second term proportional to β' better is similarly simple in unitary gauge

$$\begin{aligned} T &= \Sigma\tau^3\Sigma^\dagger = \tau^3 \\ \Rightarrow \text{Tr}(TV_\mu) &= \text{Tr}\left(-ig_Z Z_\mu \frac{\tau_3^2}{2} \right) = -ig_Z Z_\mu \frac{\text{Tr}(\mathbb{1})}{2} = -ig_Z Z_\mu \\ \Rightarrow \text{Tr}(TV_\mu)\ \text{Tr}(TV^\mu) &= -g_Z^2 Z_\mu Z^\mu \end{aligned} \tag{237}$$

Combining both terms gives the gauge boson masses

$$\begin{aligned} \mathcal{L}_2 &= -\frac{v^2}{4} \left(-g^2 W_\mu^+ W^{-\mu} - \frac{g_Z^2}{2} Z_\mu Z^\mu \right) - \beta' \frac{v^2}{8} \left(-g_Z^2 Z_\mu Z^\mu \right) \\ &= \frac{v^2 g^2}{4} W_\mu^+ W^{-\mu} + \frac{v^2 g_Z^2}{8} Z_\mu Z^\mu + \beta' \frac{v^2 g_z^2}{8} Z_\mu Z^\mu \\ &= \frac{v^2 g^2}{4} W_\mu^+ W^{-\mu} + \frac{v^2 g_Z^2}{8} (1+\beta')\, Z_\mu Z^\mu \end{aligned} \tag{238}$$

Identifying the masses and assuming the universality of neutral and charged current interactions ($\beta' = 0$) we find

$$M_W = \frac{gv}{2} \qquad\qquad M_Z = \frac{g_Z v}{2}. \tag{239}$$

This scale choice for $\Sigma(x)$ is not the only one possible. The weakest assumption to obtain finite gauge–boson masses would be $\langle \text{Tr}(\Sigma^\dagger(x)\Sigma(x)) \rangle \neq 0$ in the vacuum. In the canonical normalization we write

$$\frac{1}{2} \langle \text{Tr}(\Sigma^\dagger(x)\Sigma(x)) \rangle = 1 \qquad \forall x \tag{240}$$

which can also be fulfilled through

$$\Sigma^\dagger(x)\Sigma(x) = 1\!\!1 \qquad \forall x \tag{241}$$

This means $\Sigma(x)$ is now a unitary matrix which like any 2×2 unitary matrix can be expressed in terms of the Pauli matrices

$$\boxed{\Sigma(x) = \exp\left(\frac{-i}{v}\vec{w}(x)\right)} \qquad \text{with} \quad \vec{w}(x) = w^a(x)\tau^a. \tag{242}$$

Note that $\vec{w}(x)$ has mass dimension one, so it can be a physical scalar field. The normalization scale v is given by the energy scale of our Lagrangian. For reason which will be obvious in a few seconds, $\vec{w}(x)$ is called the non–linear representation of the symmetry related Σ field. Using the commutation properties of the Pauli matrices We can expand Σ as

$$\begin{aligned}
\Sigma &= 1\!\!1 - \frac{i}{v}\vec{w} + \frac{1}{2}\frac{(-1)}{v^2}w^a\tau^a w^b\tau^b + \frac{1}{6}\frac{i}{v^3}w^a\tau^a w^b\tau^b w^c\tau^c \\
&= 1\!\!1 - \frac{i}{v}\vec{w} - \frac{1}{2v^2}w^a w^a 1\!\!1 + \frac{i}{6v^3}w^a w^a \vec{w} \\
&= \left(1 - \frac{1}{2v^2}w^a w^a \pm ...\right) 1\!\!1 - \frac{i}{v}\left(1 - \frac{1}{6v^2}w^a w^a \pm ...\right)\vec{w}
\end{aligned} \tag{243}$$

From this expression we can for example read off the Feynman rules.

Obviously, a third way of expressing a unitary field Σ in terms of the Pauli matrices is the properly normalized linear representation

$$\boxed{\Sigma(x) = \frac{1}{\sqrt{1 + \frac{w^a w^a}{v^2}}}\left(1 - \frac{i}{v}\vec{w}(x)\right)} \tag{244}$$

The different ways of writing the Σ field in terms of the Pauli matrices cannot have any impact on the physics. However, the three forms of $\Sigma(x)$ we briefly discussed (unitary gauge Σ=1, exponential and linear representation) have different Feynman rules and Green's functions, and for a given problem one or the other might be the most efficient to use in computations or proofs. For example in electroweak calculations, the proof of renormalizability was first formulated in unitary gauge. Loop calculations might be more efficient in the Feynman gauge, because of the simplified propagator structure, while some QCD processes benefit from an explicit projection on the physical external gluons. Modern tree–level helicity amplitudes are usually computed in the unitary gauge, etc. Each of these techniques clearly have their strengths and weaknesses.

For example from the introductions to supersymmetry and extra dimensions in recent semesters we know that if we do not introduce something new, the Standard Model with gauge-bosons masses violates unitarity, most notably in $WW \to WW$ scattering. This argument can even be used to fix all the Higgs couplings, the only remaining free parameter is the Higgs mass, because unitarity arguments always affect the high–energy (*i.e.* massless) limit of the theory. In other words, our Σ model can only be viewed as an effective theory unless we give the new field a physical meaning. To extend the simple Σ model we can allow for fluctuations of $\text{Tr}(\Sigma^\dagger\Sigma)$ around the vacuum value $\Sigma^\dagger\Sigma = 1$ and parameterize the new degrees of freedom as a physical field

$$\boxed{\Sigma \to \left(1 + \frac{H}{v}\right)\Sigma} \tag{245}$$

which means for our usual trace

$$\frac{1}{2}\,\text{Tr}(\Sigma^\dagger\Sigma) = \left(1 + \frac{H}{v}\right)^2 \tag{246}$$

The non-dynamic limit is again $\Sigma^\dagger\Sigma = 1 \Longleftrightarrow H = 0$. Interpreting the fluctuations around the non-trivial vacuum as a physical Higgs field is really nothing but the usual Higgs mechanism (named after one of the University of Edinburgh's most famous sons), except that the static limit has a proper definition as an effective gauge–invariant theory, the Σ model. This way, the Higgs field does not have to be fundamental, but could just be one step in a ladder built out of effective theories. The potential terms $\mathcal{L}_\Sigma$ produce a potential for the new Higgs field H

$$\mathcal{L}_2 = -\frac{\mu^2 v^2}{2}\left(1+\frac{H}{v}\right)^2 + \frac{\lambda v^4}{2}\left(1+\frac{H}{v}\right)^4 + ... \tag{247}$$

The dots stand for higher–dimensional terms which might or might not be there, just like in the Standard Model. Some of them are not forbidden by any symmetry, but they are not realized at tree level in the Standard Model. In the static limit we have to recover the vacuum condition $\mathrm{Tr}(\Sigma^\dagger(x)\Sigma(x))/2 = 1$, so there $H = 0$ and hence $\mathcal{L}_2 = 0$ means $\mu^2 = \lambda v^2$.

Just as for the Σ field alone we can move from the simple unitary gauge to a different (linear) representation of the Σ field including a physical Higgs scalar

$$\Sigma \to \left(1+\frac{H}{v}\right)\mathbb{1} - \frac{i}{v}\vec{w} = \mathbb{1} + \frac{1}{v}\begin{pmatrix} H - iw^3 & -i\sqrt{2}w^+ \\ -i\sqrt{2}w^- & H + iw^3 \end{pmatrix} = \mathbb{1} + \frac{1}{v}(\tilde{\Phi}\Phi) \tag{248}$$

The last step is just another way to write the 2×2 matrix in terms of the two doublets

$$\tilde{\Phi} = \begin{pmatrix} H - iw^3 \\ -i\sqrt{2}w^- \end{pmatrix} \qquad \Phi = \begin{pmatrix} -i\sqrt{2}w^+ \\ H + iw^3 \end{pmatrix} \tag{249}$$

These two doublets give mass to up–type and down–type fermions.

Instead of deriving both relevant doublets from one physical Higgs doublet Φ and $\tilde{\Phi}$ we can include two sigma fields in the fermion–mass terms

$$\mathcal{L}_3 \sim -\overline{Q}_L M_{Qu}\Sigma_u \frac{1+\tau^3}{2} Q_R - \overline{Q}_L M_{Qd}\Sigma_d \frac{1-\tau^3}{2} Q_R + ... \tag{250}$$

and in the gauge–boson mass terms

$$\mathcal{L}_2 = \frac{v_u^2}{2}\,\mathrm{Tr}\left[(D_\mu\Sigma_u)^\dagger D^\mu\Sigma_u\right] + \frac{v_d^2}{2}\,\mathrm{Tr}\left[(D_\mu\Sigma_d)^\dagger D^\mu\Sigma_d\right] \tag{251}$$

Each of the two Σ fields we can express in the usual linear representation

$$\Sigma_j = \mathbb{1} + \frac{1}{v_j}\Phi_j^0 - \frac{i}{v_j}\vec{\Phi}_j \qquad\qquad i = u,d \quad \vec{\Phi}_j = \Phi_j^a\tau^a. \tag{252}$$

From the gauge–boson masses we know that

$$v_u^2 + v_d^2 = v^2 \qquad \Longleftrightarrow \qquad v_u = v\sin\beta \quad v_d = v\cos\beta \tag{253}$$

which means that the longitudinal vector bosons are

$$\vec{w} = \cos\beta\;\vec{\Phi}_u + \sin\beta\;\vec{\Phi}_d \tag{254}$$

This two–Higgs doublet model is for example the minimal choice in supersymmetric extensions of the Standard Model. But type-II two–Higgs doublet models where one Higgs doublet gives mass to up-type and another one to down–type fermions are much more general than that.

From the discussion in the last section we have seen that electroweak symmetry breaking with a simple sigma field or Higgs doublet links the couplings of neutral and charged currents firmly to the masses of the W and Z bosons. After the precision measurements at LEP this link has turned into a seriously strong constraint on all kind of new–physics models. As a matter of fact, this constraint is responsible for the almost death of (technicolor) models which describe the Higgs boson as a bound state under a new QCD-like interaction.

We remember that the Lagrangian for the gauge-boson masses involves two terms, both symmetric under $SU(2) \times U(1)$ and hence allowed in the electroweak Standard Model

$$\mathcal{L}_2 = -\frac{v^2}{4} \,\text{Tr}[V_\mu V^\mu] - \beta' \,\frac{v^2}{8}\, \text{Tr}[TV_\mu]\, \text{Tr}[TV^\mu] \tag{255}$$

In unitary gauge we actually computed the mass terms coming from $\text{Tr}[V_\mu V^\mu]$, which gave M_W and M_Z proportional to $g \equiv g_W$ and g_Z. Their relative size can be expressed in terms of the weak mixing angle θ_w, together with the assumption that G_F or g universally govern charged current ($W^\pm$) and neutral-current (W^3) interactions. This relations at tree level is simply

$$\frac{M_W^2}{M_Z^2} = c_w^2. \tag{256}$$

A free parameter ρ breaking this relation can be introduced as a shift

$$\boxed{g_Z^2 \to g_Z^2 \cdot \rho} \qquad\qquad m_Z \to m_Z \cdot \sqrt{\rho}\,, \tag{257}$$

which from measurements it is very strongly constrained to be unity. In $\mathcal{L}_2$ the Z-mass term proportional to β' precisely predicts the deviation $\rho = 1 + \beta' \neq 1$. To bring our Lagrangian into agreement with measurements we better find a reason to constrain β' to zero, and the $SU(2) \times U(1)$ gauge symmetry unfortunately does not do the job.

Looking ahead, we will find that $\rho = 1$ is violated in the Standard Model, for example by the difference in up-type and down-type quark masses $m_b \neq m_t$. Which means we are looking for an approximate symmetry of the entire Standard Model, but in particular a good symmetry in the $SU(2)$ gauge sector. There is one possibility...
We can replace the $SU(2)_L \times U(1)_Y$ symmetry with a larger symmetry $SU(2)_L \times SU(2)_R$, which obviously would have to act like

$$\begin{aligned} &\Sigma \to U\Sigma V^\dagger \qquad U \in SU(2)_L \qquad V \in SU(2)_R \\ &\text{Tr}(\Sigma^\dagger\Sigma) \to \text{Tr}\left[V\Sigma^\dagger U^\dagger U\Sigma V^\dagger\right] = \text{Tr}[\Sigma^\dagger\Sigma] \qquad \text{(because of circular trace)} \end{aligned} \tag{258}$$

From the definition of the covariant derivative $D_\mu\Sigma$ including a simple τ^3 we can already guess that the complete group $SU(2)_R$ will not allow B-field interactions which are proportional to $s_W \sim \sqrt{1/4}$. It also does not allow $\beta' \neq 0$, but it does allow all terms in the Higgs potential $\mathcal{L}_\Sigma$. Giving the Σ field a finite vacuum expectation value Σ field changes the picture: in the minimal (non–Higgs) version and in the unitary gauge the Σ field now reduces to $\mathbb{1}$, which for the combined $SU(2)$ transformations means

$$\langle\Sigma\rangle \to \langle U\Sigma V^\dagger\rangle = \langle U\mathbb{1}V^\dagger\rangle = UV^\dagger \equiv \mathbb{1} \tag{259}$$

The last step, *i.e.* the symmetry requirement for the Lagrangian can only be satisfied if we require $U = V$. In other words, the vacuum expectation value for Σ or for the Higgs field breaks $SU(2)_L \times SU(2)_R$ to the diagonal subgroup $SU(2)_{L+R}$. The technical term is precisely defined this way — the two $SU(2)$ symmetries reduce to one remaining symmetry which can be written as $U = V$. In the extended symmetry group the ρ parameter is indeed protected to be $\rho = 1$, while under only the diagonal symmetry group we can accommodate a general ρ.

Leading corrections to the ρ parameter come from Higgs loops in the case $g' \neq 0$

$$\Delta\rho \sim -\frac{11 G_F M_Z^2 s_W^2}{24\sqrt{2}\pi^2} \log \frac{m_h^2}{M_Z^2}. \tag{260}$$

Others come from virtual bottoms and tops in the W and Z self energies

$$\begin{aligned}
\Delta\rho &\sim \frac{3G_F}{8\sqrt{2}\pi^2}\left(m_t^2 + m_b^2 - 2\frac{m_t^2 m_b^2}{m_t^2 - m_b^2}\log\frac{m_t^2}{m_b^2}\right) \\
&\sim \frac{3G_F}{8\sqrt{2}\pi^2}\left(2m_b^2 + \delta - 2\frac{(m_b^2+\delta)m_b^2}{\delta}\log\left(1 + \frac{\delta}{m_b^2}\right)\right) \qquad m_t^2 = m_b^2 + \delta \\
&= \frac{3G_F}{8\sqrt{2}\pi^2}\left(2m_b^2 + \delta - 2\left(\frac{m_b^4}{\delta} + m_b^2\right)\left(\frac{\delta}{m_b^2} - \frac{\delta^2}{2m_b^4} + \mathcal{O}(\delta^3)\right)\right) \\
&= \frac{3G_F}{8\sqrt{2}\pi^2}\left(2m_b^2 + \delta - 2m_b^2 + 2\frac{\delta}{2} - 2\delta + \mathcal{O}(\delta^2)\right) \\
&= \frac{3G_F}{8\sqrt{2}\pi^2}\,\mathcal{O}(\delta^2)
\end{aligned} \tag{261}$$

and indeed vanish for $m_t = m_b$.

The obvious next question is: how do physical modes, which we introduce in the parameterization of the Σ field $\Sigma(x) = \exp(-i\vec{w}/v)$ and which we will describe in more detail in the next section transform under these two different $SU(2)$ symmetries?
Clearly, under the usual $SU(2)_L$ we still find $\Sigma \to U \cdot \Sigma$, the way we actually introduced U earlier. We can write U in terms of the $SU(2)$ generators as $U = \exp(-i\alpha \cdot \tau/2)$. In general, we denote $\vec{w} = w^a\tau^a = w \cdot \tau$ and $\vec{\alpha} = \alpha \cdot \tau$ in terms of the Pauli matrices. We can read off the transformation properties of $\vec{w}$ from

$$\begin{aligned}
U\Sigma &= e^{-i(\alpha\cdot\tau)/2}\, e^{-i(w\cdot\tau)/v} \\
&= e^{-i(\alpha\cdot\tau)/2 - i(w\cdot\tau)/v}\, e^{-\frac{i}{2}[\alpha\cdot\tau)/2,(w\cdot\tau)/v]} \\
&= e^{-i(\alpha\cdot\tau)v/2 + (w\cdot\tau))/v} \\
&= e^{-i(w'\cdot\tau)/v}
\end{aligned} \tag{262}$$

In the second line we have used the Baker-Campbell-Hausdorff formula $e^A e^B = e^{A+B} e^{[A,B]/2}$ which for the Pauli matrices becomes

$$\begin{aligned}
[\tau_i, \tau_j] = 2i\varepsilon_{ijk}\tau_k \quad &\Rightarrow \quad (\vec{\alpha}\cdot\vec{\tau})(\vec{w}\cdot\vec{\tau}) = \vec{\alpha}\cdot\vec{w} + i\vec{\tau}\,(\vec{\alpha}\times\vec{w}) \\
&\Rightarrow \quad [(\vec{\alpha}\cdot\vec{\tau}),(\vec{w}\cdot\vec{\tau})] = 2i\vec{\tau}\,(\vec{\alpha}\times\vec{w})
\end{aligned} \tag{263}$$

From the symmetry requirement $U\Sigma \equiv \Sigma$ we find the transformation property for the physical modes in Σ

$$\boxed{w_a \to w_a' = w_a + \frac{v}{2}\alpha_a} \tag{264}$$

This is a non-linear transformation, in the sense that w_a' is not proportional to w_a. Note that we have derived this shift–symmetry operation only for infinitesimal transformations, so for general transformations we might end up with higher terms in α. The crucial conclusion is the same, though: these modes in Σ shift under the $SU(2)$ transformation, their transformation is not linear. When we construct a symmetric Lagrangian this non–linear transformation forbids mass terms, gauge interactions, Yukawa couplings, and quadratic potential terms for these modes in Σ. Only derivative terms like the kinetic term and derivative couplings are allowed unter the $SU(2)$ symmetry.

Similarly, we can evaluate the transformation of these physical modes under the custodial symmetry group $SU(2)_{L+R}$ and find the linear transformation

$$\boxed{w_a \to w_a' = w_a - \varepsilon_{abc}\alpha_b w_c} \tag{265}$$

In other words, when we transform the physical modes corresponding to the symmetry generators in Σ by the good symmetry $SU(2)_L$ we find a non–linear transformation, while the approximate symmetry $SU(2)_{L+R}$ leads to a linear transformation. A linear transformation for example of a scalar means that we can write a potential for this particle which is symmetric under $SU(2)_{L+R}$ transformations.

This leads us to the definition of Goldstone modes: if we have a global symmetry group which is spontaneously broken into a smaller symmetry group, the broken generators of the original group correspond to physical Goldstone modes. These modes transform non–linearly under the larger group and linearly under the smaller group. If our symmetry groups are gauge groups, Goldstone modes are absorbed into the broken gauge bosons to make them massive. If this spontaneous symmetry breaking involves a vacuum expectation value f, the mass of the heavy gauge bosons which eat the Goldstone modes is of the order f.

A little more tailored towards our later use, we see that because of their non–linear transformation property, Goldstone bosons cannot form a potential symmetric under the original group, so they have to for example be massless. This does not change if we break the original symmetry group spontaneously — potential terms are still forbidden. However, if we also break the larger symmetry group explicitely, for example through a coupling g, potential terms can now occur. They will be proportional to g and proportional to f and can be induced for example through loop effects. In the presence of explicit symmetry breaking the Goldstone modes are called pseudo–Goldstone modes.

B. Little–Higgs Mechanism

Until now we have not talked about any physics beyond the Standard Model. As a matter of fact, we have mostly talked about a watered–down version of the electroweak Standard Model, namely the Σ model. However, first of all it is good to know that we can actually write down a perfectly fine Lagrangian for the electroweak gauge theory including finite W and Z boson masses without introducing a Higgs field, if we are happy with an effective–theory approach. And secondly, the starting point of little–Higgs theories is the attempt to make the Higgs boson a pseudo–Goldstone mode under some broken global symmetry to explain its small mass, and it is a good idea to review this mechanism before diving into the exciting new physics.

1. *Some Goldstone bosons*

In the following, we will track the behavior of different degrees of freedom under $SU(N)$ transformations. We can start with a simple $U(1)$ transformation of a complex scalar field, *i.e.* with two degrees of freedom. For this scalar field $\phi(x)$ we assume a potential $V = V(\phi^*\phi)$ and a global $U(1)$ symmetry transformation $\phi \to e^{i\alpha}\phi$. After expanding the scalar field around its (real) vacuum we find a massive radial mode $r(x)$, with its mass given by the form of the potential around the vacuum. The transformation of the scalar field in terms of these two modes reads

$$\phi \to e^{i\alpha}\phi = e^{i\alpha}\ \frac{v + r(x)}{2}\ e^{i\theta(x)/v} = \frac{v + r(x)}{2}\ e^{i(\theta(x)+v\cdot\alpha)/v} \tag{266}$$

Just as before, we find a non–linear shift of the massless mode in the scalar field: $\theta \to \theta + v\cdot\alpha$. This means $\theta(x)$ has to stay massless, protected by the $U(1)$ symmetry. Only derivative couplings of θ are allowed in a $U(1)$-symmetric Lagrangian.

Unfortunately, we now have to move to the non–abelian case, where we will have to write tons of matrices and any lecturer is bound to get things wrong on the blackboard. First, we can break the global (ungauged) gauge group $SU(N) \to SU(N-1)$ and look at the Goldstone modes associated with the reduced number of degrees of freedom in the symmetry group. We expect

$$(N^2-1)^2 - ((N-1)^2 - 1) = 2N - 1 \tag{267}$$

generators which are not anymore associated with the reduced symmetry group. Think for example of a basis for $SU(3)$ and $SU(2)$, the Gell-Mann and the Pauli matrices. They are traceless hermitian (and unitary) matrices, and generators of the Lie groups $SU(N)$ with $N=2,3$. For $SU(2)$ the three Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{268}$$

with strictly speaking: $SU(2) = \text{span}\{i\sigma^k\}$. The corresponding 8 Gell-Mann matrices can be written in terms of the three Pauli matrices and the remaining degrees of freedom

$$\begin{aligned}
\lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma^1 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad
\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma^2 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad
\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma^3 & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & & 1 \\ & & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad
\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & & -i \\ & & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \text{combined to complex} \quad \begin{pmatrix} 0 & & w_1 \\ & & 0 \\ w_1^* & 0 & 0 \end{pmatrix} \\
\lambda^6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & & 0 \\ & & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad
\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \begin{pmatrix} 0 & & 0 \\ & & -i \\ 0 & i & 0 \end{pmatrix} \qquad \text{combined to complex} \quad \begin{pmatrix} 0 & & 0 \\ & & w_2 \\ 0 & w_2^* & 0 \end{pmatrix} \\
\lambda^8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1\!\!1 & & 0 \\ & & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned} \tag{269}$$

We can arrange all generators of $SU(3)$ which are not generators of $SU(2)$ in the outside column and row of the 3×3 matrix

$$\boxed{U_N \sim \begin{pmatrix} SU(2) & & w_1 \\ & & w_2 \\ w_1^* & w_2^* & w_0 \end{pmatrix}} \tag{270}$$

The entry w_0 is fixed by the requirement that U_N has to be traceless when we add $1\!\!1$ to the $SU(2)$ matrices in the top–left corner. If, as they were introduced in the Σ model, the Goldstone modes describe modes of a system around its broken ground state with a symmetry–breaking scale v, we can collect them in a vector–shaped field ϕ for general $SU(N) \to SU(N-1)$ breaking as

$$\phi = \exp\left\{ -\frac{i}{v} \begin{pmatrix} SU(N-1) & & & w_1 \\ & & & \cdots \\ & & & w_{N-1} \\ w_1^* & & \ldots\; w_{N-1}^* & w_0 \end{pmatrix} \right\} \begin{pmatrix} 0 \\ \cdots \\ 0 \\ v \end{pmatrix} \equiv e^{-i\vec{w}\cdot\vec{\tau}/v} \phi_0 \tag{271}$$

This notation has the advantage that we can write ϕ and ϕ_0 as columns, *i.e.* as fields symmetric under $SU(N)$ or $SU(N-1)$ in the fundamental representation. The vector ϕ then is defined such that its upper $N-1$ component are symmetric under the smaller symmetry group $SU(N-1)$. In the first–order term in the Taylor series in $1/v$ the mass scale v drops out between the exponent and ϕ_0.

Another example for a global symmetry group more similar to our old custodial $SU(2)_{L+R}$ would be $\underline{SU(N) \times SU(N) \to SU(N)}$. The number of Goldstone bosons associated with the broken generators is

$$2(N^2-1) - (N^2-1) = N^2 - 1 \tag{272}$$

Unfortunately, they are not as easily written in matrix form as those of the two gauge groups $SU(N) \to SU(N-1)$. The gauge transformations we know from before: $\phi \to L\phi R^\dagger$. The symmetry–breaking ground state of the combined scalar field is $\langle\phi\rangle \equiv \phi_0 \equiv v 1\!\!1_N$: it is invariant under the diagonal subgroup where we identify the two $SU(2)$ transformations to a simpler $\phi_0 \to U\phi_0 U^\dagger$. The remaining (axial) generators are broken and turn into Goldstone bosons collected in $\phi = \exp(-i(\vec{w}\cdot\vec{\tau})/v)\phi_0 = v\exp(-i(\vec{w}\cdot\vec{\tau})/v)1\!\!1$. The matrices $(\vec{w}\cdot\vec{\tau})$ are traceless hermitian matrices with (N^2-1) degrees of freedom, *i.e.* independent entries.

From our simple examples $SU(2)_L$, $SU(2)_{L+R}$ and $U(1)$ we already have a good idea how to compute the transformation of the Goldstone bosons under broken and unbroken symmetry transformations. We repeat the argument for $SU(N) \to SU(N-1)$, starting with the transformation properties of the scalar field ϕ. This scalar field can be parameterized as $\phi \equiv \exp(-i(\vec{w}\cdot\vec{\tau})/v)\phi_0$ with the generators $\vec{\tau}$ including the broken subgroup $SU(N)/SU(N-1)$. Under the unbroken symmetry group $SU(N-1)$ represented as an $(N \times N)$ matrix the scalar field transform as

$$\begin{aligned}\phi \to U_{N-1}\,\phi &= U_{N-1}\,e^{-i\vec{w}\cdot\vec{\tau}/v}\,\phi_0\\ &= U_{N-1}\,e^{-i\vec{w}\cdot\vec{\tau}/v}\,U^\dagger_{N-1}U_{N-1}\phi_0\\ &= U_{N-1}\,e^{-i\vec{w}\cdot\vec{\tau}/v}\,U^\dagger_{N-1}\phi_0 \qquad (\phi_0 \text{ invariant under } U_{N-1}\text{, but not } U_N\,)\end{aligned} \tag{273}$$

This relation will give us the transformation properties for the Goldstones. We can rewrite the $(N \times N)$ matrix acting on the leading term in ϕ

$$\begin{aligned}&U_{N-1} = \begin{pmatrix}\hat{U}_{N-1} & 0\\ 0 & 1\end{pmatrix}\\ \Rightarrow\quad &U_{N-1}\,e^{-i\vec{w}\cdot\vec{\tau}/v}\,U^\dagger_{N-1} \sim -\frac{i}{v}\begin{pmatrix}\hat{U}_{N-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & \vec{w}\\ \vec{w}^\dagger & 0\end{pmatrix}\begin{pmatrix}\hat{U}_{N-1} & 0\\ 0 & 1\end{pmatrix} = -\frac{i}{v}\begin{pmatrix}0 & \hat{U}_{N-1}\vec{w}\\ (\hat{U}_{N-1}\vec{w})^\dagger & 0\end{pmatrix}\end{aligned} \tag{274}$$

This means the Goldstones transform as $\vec{w} \to \hat{U}_{N-1}\vec{w}$. However, this transformations with $\hat{U}_{N-1}$ from the left is just the usual symmetry transformation for vectors in the fundamental representation of $SU(N-1)$. In the $SU(N-1)$ symmetric Lagrangian we can write any terms for the Goldstones we can write for other states in the fundamental representation.
To compute the more interesting transformation properties under $SU(N)$ we need the fact, that a $SU(N)$ transformation can be written as a product of an $SU(N)/SU(N-1)$ transformation times a $SU(N-1)$ transformation. This means

$$\begin{aligned}\phi \to U_N\phi &= U_N U_*(\vec{w})\phi_0 \qquad \text{with the } SU(N)/SU(N-1) \text{ transformation } U_*(\vec{w})\text{, so } U_N = U_* U_{N-1}\\ &= U_*(\vec{\alpha})U_{N-1}U_*(\vec{w})\phi_0\\ &= U_*(\vec{\alpha})U_{N-1}U_*(\vec{w})U^\dagger_{N-1}U_{N-1}\phi_0\\ &= U_*(\vec{\alpha})U_{N-1}U_*(\vec{w})U^\dagger_{N-1}\phi_0\end{aligned} \tag{275}$$

The combination $U_{N-1}U_*(\vec{w})U^\dagger_{N-1}$ is just what we found above, while the additional $U_*(\alpha) = \exp(-i(\vec{\alpha}\cdot\tau)/2)$ will produce the same behavior we saw in the $SU(2)$ and $U(1)$ cases: if we write out the infinitesimal transformations we find $\vec{w} \to \vec{w}' = \vec{w} + \vec{\alpha}/2$, which forbids Goldstone masses and other potential terms in the Lagrangian and only allows derivative interactions. The Goldstone Lagrangian of mass dimension four with a global $SU(N)$ symmetry will therefore be of the general form

$$\boxed{\mathcal{L} = |\partial_\mu\phi|^2 + \mathcal{O}(\partial^4) + \text{const}} \tag{276}$$

Any mass scale in this spontaneously broken Goldstone Lagrangian is given by the vacuum expectation value f. Constants can for example arise from the gauge–invariant combination $\phi^\dagger\phi = \phi_0^\dagger\phi_0 = f^2$. Note that here we switch from v to f for the same thing, namely the scale responsible for breaking the larger symmetry group, to be consistent with Martin's review. Similarly, we will switch from $-\vec{w}$ to $\vec{\pi}$ for the Goldstones in ϕ.

2. Protecting the Higgs mass

For example from the lecture on supersymmetry or the lecture on extra dimensions you might remember that one of the puzzles of high–energy physics is the question why the Higgs is so light. From general field–theoretical considerations any fundamental scalar should acquire a loop–induced mass of the order of the cutoff of the theory. Clearly, the LEP precision measurements point too a Higgs mass much below the Planck or the unification scales. One way to explain a small Higgs mass would be to make the Higgs a pseudo–Goldstone of a symmetry which is broken at a mass scale around the weak scale. Compared to this scale the Higgs mass has to be small because of the larger symmetry group, which means the Higgs mass cannot diverge quadratically at large energy scales.

This idea has been around for a long time, but for decades people did not know how to construct such a symmetry. Before we solve this problem via the little–Higgs mechanism, let us unsuccessfully start constructing a symmetry which protects the Higgs mass from quadratic divergences at one loop using a global $SU(3)$ as the broken symmetry including $SU(2)_L$. Everything we need to know for this construction we can read off from the general $SU(N) \to SU(N-1)$ case. The $SU(3) \to SU(2)$ Goldstone modes written in the usual matrix form are

$$\vec{\pi} = \begin{pmatrix} SU(2) & h \\ h^\dagger & \eta \end{pmatrix} \tag{277}$$

We of course assume that the $SU(2)_L$ doublet among the $SU(3)$ Goldstones which can acquire a mass once we break $SU(3)$ is the Higgs doublet of the Standard Model. Again, note that to be in agreement with Martin Schmaltz's review we now denote the Goldstone fields as π instead of $\vec{w}$. The additional field η is an $SU(2)$ singlet and can be ignored for now — we will discuss it briefly at the very end of the lecture. To translate the degrees of freedom from the matrix $\vec{\pi}$ to the fields h we are interested in we write the usual matrix representation of the Goldstones with a symmetry–breaking scale f

$$\begin{aligned}
\phi &= \exp\left[\frac{i}{f}\begin{pmatrix} 0_{2\times 2} & h \\ h^\dagger & 0 \end{pmatrix}\right]\begin{pmatrix} 0_2 \\ f \end{pmatrix} \\
&= \left(1\!\!1 + \frac{i}{f}\begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix} - \frac{1}{2f^2}\begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix}\begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix}\right)\begin{pmatrix} 0 \\ f \end{pmatrix} \qquad h = (h_1, h_2) \\
&= \begin{pmatrix} 0 \\ f \end{pmatrix} + \begin{pmatrix} ih \\ 0 \end{pmatrix} - \frac{1}{2f^2}\begin{pmatrix} 0 \\ h^\dagger h f \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ f \end{pmatrix} + \begin{pmatrix} ih \\ -h^\dagger h/(2f) \end{pmatrix}
\end{aligned} \tag{278}$$

Note that only in the first line we indicate which of the zeros in the 3×3 matrix is a 2×2 sub-matrix. This is easy to keep track of if we remember that the Higgs field h is a doublet, while $h^\dagger h$ is a scalar number. This transformation allows us to rewrite the kinetic term as a function of h

$$\begin{aligned}
|\partial_\mu \phi|^2 = (\partial_\mu \phi^*)_i (\partial^\mu \phi)_i &= (-i\partial_\mu h)_i (i\partial^\mu h^*)_i + \frac{1}{4f^2}(\partial_\mu h^\dagger h)_i (\partial^\mu h^\dagger h)_i \\
&= |\partial_\mu h|^2 + \frac{1}{4f^2}(\partial_\mu \sum_j h_j^* h_j)_i (\partial^\mu \sum_j h_j^* h_j)_i \\
&= |\partial_\mu h|^2 + \frac{1}{4f^2}\left(\sum_j (\partial_\mu h_j^*) h_j + \sum_j h_j^* (\partial_\mu h_j)\right)_i \left(\sum_j (\partial^\mu h_j^*) h_j + \sum_j h_j^* (\partial^\mu h_j)\right)_i \\
&= |\partial_\mu h|^2 + \frac{1}{4f^2}\, 4\, |\partial_\mu h|^2\, h^\dagger h \\
&= |\partial_\mu h|^2 \left(1 + \frac{h^\dagger h}{f^2}\right)
\end{aligned} \tag{279}$$

The second term in the parentheses looks like a kinetic term, so it is fine in the Goldstone Lagrangian. However, it includes an additional factor $h^\dagger h$, which corresponds to an outgoing and an incoming Higgs field and which we should have a close look at. These two fields can be linked, giving a one-loop graph which diverges as

$$\int^\Lambda \frac{d^4 q}{(2\pi)^4}\, \frac{1}{q^2} \sim \frac{\Lambda^2}{(4\pi)^2} \tag{280}$$

Comparing the two terms in the parentheses above there is an upper limit to the size of the $h^\dagger h$ term, where this term dominates our theory. This means that our effective theory should will only be valid as long as

$$\boxed{\frac{\Lambda^2}{(4\pi)^2 f^2} \lesssim 1} \tag{281}$$

In other words, the massless Higgs boson has additional high-dimensional Lagrangian terms which become strong for energy scales around $\Lambda \sim 4\pi f$. Above this scale, our effective theory will not be useful.

After we now know how the kinetic term for the massless pseudo–Goldstone–Higgs doublet looks we next have to generate a coupling to the $SU(2)$ gauge bosons and see what happens with the Higgs mass. Of course, from the discussion of Goldstones and pseudo–Goldstones we know that we will not be able to generate the mass or a potential term we want, but it is constructive to see the problems which will arise.

First attempt: We can simply add $g\ (\vec{W}^\mu \vec{\tau})$ in the covariant derivative of the Goldstone. Or in other words, we gauge the $SU(2)$ subgroup of the global $SU(3)$. This automatically creates a 4-point coupling of the kind $|g\vec{W}_\mu h|^2$. Like before, the two W bosons coupling to the Higgs propagator can be linked to a loop and generate a one–loop mass term of the kind

$$\mathcal{L} \supset g^2 \frac{\Lambda^2}{(4\pi)^2}\ h^\dagger h \tag{282}$$

This term is a quadratically divergent Higgs mass. Which means that our operator breaks the shift symmetry $SU(3)$ into $SU(2)$ and at the same time introduces the same kind of mass for which spoils the Standard Model.

Second attempt: We can write the same interaction as in the first attempt in terms of the triplet ϕ, where we simply leave the third entry in the gauge–boson matrix empty

$$\left| g \begin{pmatrix} \vec{W}_\mu \tau & 0 \\ 0 & 0 \end{pmatrix} \phi \right|^2 \tag{283}$$

We can again square this relevant interaction term contributing to the Higgs mass and find (in a suitable $SU(2)$ basis)

$$\phi^\dagger \begin{pmatrix} g\,\mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g\,\mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \phi = \phi^\dagger \begin{pmatrix} g^2\,\mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix} \phi = g^2\ h^\dagger h\ \mathbb{1} \tag{284}$$

which means that the mass terms now read

$$\mathcal{L} \supset g^2 \frac{\Lambda^2}{(4\pi)^2}\ \phi^\dagger \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \phi = g^2\ \frac{\Lambda^2}{(4\pi)^2}\ h^\dagger h \tag{285}$$

This is precisely what we had before. And it is not surprising, because we really only wrote the same thing in a different notation, using $\phi^\dagger\phi$ instead of $h^\dagger h$ and adding zeros into the gauge–boson matrix which in turn acts as a projector onto the $h^\dagger h$ part.

Third attempt: Learning from the previous cases we can instead add a proper covariant derivative not only including the W fields in $SU(2)$, but also the degrees of freedom of the complete $SU(3)$. Closing all of them into loops we obtain again in a proper basis

$$\mathcal{L} \supset g^2 \frac{\Lambda^2}{(4\pi)^2}\ \phi^\dagger \mathbb{1}_3 \phi = g^2 \frac{\Lambda^2}{(4\pi)^2} |\phi_0|^2 = g^2 \frac{\Lambda^2}{(4\pi)^2}\ f^2 \tag{286}$$

There is indeed no Higgs–mass contribution, because our $SU(3)$ gauge bosons ate the Goldstones altogether. This is simple an effect of including a complete set of $SU(3)$ gauge bosons of freedom, where there are no Goldstone degrees of freedom left for the Higgs.

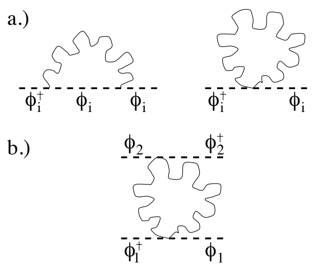

FIG. 1: Feynman diagrams contributing to the Higgs mass in little–Higgs models. This beautiful picture is stolen from Martin Schmaltz's review article.

On the other hand, so this attempt brings us closer to solving Higgs–Goldstone problem. The problem we are stuck in is that either we include only the $SU(2)$ covariant derivative and find quadratic divergences in the Higgs mass or we include the $SU(3)$ covariant derivative and turn the Higgs into a Goldstone mode which gives a mass of scale f to these gauge bosons.

Correct attempt: We obviously have to come up with something better than the usual set of Goldstones from $SU(3)$ breaking. Digesting the unsuccessful attempts we can see a way out: we should use two independent sets of $SU(3)$ generators. These we break to our $SU(2)_L$ gauge group through a combination of spontaneous and explicit breaking. Because of this mixing we will get pseudo–Goldstones which make the $SU(3)$ gauge bosons heavy while the uneaten Goldstones which can form our Higgs. Note that this requires us to only include one set of $SU(3)$ gauge bosons for two $SU(3)$ symmetries, so in a way only one of them will be gauged. Naively, we have $8 + 8 - 3 = 13$ Goldstones degrees of freedom to distribute. However, we have have to be careful not to double count three of them in the case where we identify both $SU(2)$ fractions of the two original sets of $SU(3)$ generators, in which case we are down to ten Goldstone modes. The art will be to arrange the spontaneous and hard symmetry breakings into a workable model.
First, we write each of the set of $SU(3)$ generators into the usual matrices and identify the relevant degrees of freedom in the Goldstone matrix which we hope will become the Higgs

$$\phi_j = \exp\left(\frac{i}{f}\begin{pmatrix} 0_{2\times 2} & h_j \\ h_j^\dagger & 0 \end{pmatrix}\right)\begin{pmatrix} 0 \\ f \end{pmatrix} \qquad j = 1, 2 \tag{287}$$

For simplicity we here set the vevs equal $f_1 \equiv f_2 \equiv f$. At one loop, each of them couples to the set of $SU(3)$ gauge bosons with the usual $SU(3)$ covariant derivative

$$\mathcal{L} \subset |D_\mu \phi_1|^2 + |D_\mu \phi_2|^2 \subset g_1^2 \, |W_\mu \phi_1|^2 + g_2^2 \, |W_\mu \phi_2|^2 \tag{288}$$

These terms can be linked to loop diagrams of the kind shown in Fig. 1(a). From our attempt number three we we know that for universal couplings g_j they read

$$\frac{1}{(4\pi)^2} \, \Lambda^2 \, \left(g_1^2 \, \phi_1^\dagger \phi_1 + g_2^2 \, \phi_2^\dagger \phi_2\right) = \frac{g^2}{(4\pi)^2} \, \Lambda^2 \, 2f^2 \tag{289}$$

However, these are not the only terms we can write down with two sets of Goldstones. For example, we can write diagrams like the one in Fig. 1(b), coupling ϕ_1 to ϕ_2 directly through a gauge-boson loop. Counting the powers in momentum we can guess its contribution to the Lagrangian to be of the kind

$$\frac{g_1^2 g_2^2}{(4\pi)^2} \, \log \frac{\Lambda^2}{\mu^2} \, |\phi_1^\dagger \phi_2|^2 \tag{290}$$

The combination $\phi_1^\dagger \phi_2$ is a scalar and not a matrix and is gauge invariant only under the diagonal $SU(3)$ subgroup of $[SU(3)]^2$. And last but not least, it is not a simple mass term for the ϕ_j, nor is it quadratically divergent, so we simply accept its existence.

In the next step, we have to translate its form into a Lagrangian term in the Higgs fields h_j and see if it gives us a mass term. Its form suggests a reorganization of the h_j, to treat them more symmetrically; if we shift them such that $h_j \to k \pm h$ we find to leading order (neglecting commutators)

$$\begin{aligned}
\phi_1^\dagger \phi_2 &= \left[e^{\frac{i}{f}\vec{k}\vec{\tau}} e^{+\frac{i}{f}\vec{h}\vec{\tau}} \begin{pmatrix} 0 \\ f \end{pmatrix} \right]^\dagger \left[e^{\frac{i}{f}\vec{k}\vec{\tau}} e^{-\frac{i}{f}\vec{h}\vec{\tau}} \begin{pmatrix} 0 \\ f \end{pmatrix} \right] \\
&= \begin{pmatrix} 0 & f \end{pmatrix} e^{-\frac{i}{f}\vec{h}\vec{\tau}} e^{-\frac{i}{f}\vec{k}\vec{\tau}} e^{+\frac{i}{f}\vec{k}\vec{\tau}} e^{-\frac{i}{f}\vec{h}\vec{\tau}} \begin{pmatrix} 0 \\ f \end{pmatrix} \\
&= \begin{pmatrix} 0 & f \end{pmatrix} e^{-\frac{2i}{f}\vec{h}\vec{\tau}} \begin{pmatrix} 0 \\ f \end{pmatrix} \\
&= \begin{pmatrix} 0 & f \end{pmatrix} \left[\mathbb{1} - \frac{2i}{f} \begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix} + \frac{1}{2} \left(\frac{2i}{f} \right)^2 \begin{pmatrix} hh^\dagger & 0 \\ 0 & h^\dagger h \end{pmatrix} + \dots \right] \begin{pmatrix} 0 \\ f \end{pmatrix} \\
&= f^2 \mathbb{1} - \frac{2}{f^2} \begin{pmatrix} 0 & f \end{pmatrix} \begin{pmatrix} 0 \\ h^\dagger h f \end{pmatrix} + \dots \\
&= f^2 \mathbb{1} - 2 h^\dagger h + \dots \qquad \Rightarrow \qquad \boxed{ |\phi_1^\dagger \phi_2|^2 \sim f^4 \mathbb{1} - 4 f^2 h^\dagger h + \dots }
\end{aligned} \tag{291}$$

The Goldstone modes k are $SU(3)$ rotations common to ϕ_1 and ϕ_2 and lead to massive longitudinal $SU(3)$ gauge bosons when we break the $SU(3)$ symmetry spontaneously.
Because of the combination of the spontaneous symmetry breaking of the two $SU(3)$ symmetries and the explicit breaking to the diagonal $SU(3)$ the pseudo–Goldstone field h develops a mass and general potential terms of the kind $|\phi_1^\dagger \phi_2|$. For example its mass term just combining the two above formulae reads

$$\mathcal{L} \subset -\frac{g_1^2 g_2^2 f^2}{(2\pi)^2} \, \log \frac{\Lambda^2}{\mu^2} \, h^\dagger h \tag{292}$$

To summarize, of the two Goldstones $h_1 = k + h$ and $h_2 = k - h$ we use $k = (h_1 + h_2)/2$ to make the gauge bosons of the broken $SU(3)$ heavy. The remaining Goldstones $h = (h_1 - h_2)/2$ are pseudo–Goldstone bosons which can develop a mass and a potential with a mass scale f at which we break $SU(3) \to SU(2)$. Comparing this mass term to the Standard–Model mass scales, we expect or hope for f values which give us

$$\boxed{ M_{\text{weak}} \sim \frac{g^2 f}{2\pi} } \tag{293}$$

The mechanism described above is called collective symmetry breaking. It is a convoluted way of spontaneously and explicitely breaking a global symmetry into a gauged subgroup (here $SU(3)_{\text{diag}}$) and then down to our $SU(2)_L$. Part of the Goldstones from the original global symmetry group will make the additional gauge bosons heavy, with a mass scale f. The remaining Goldstones turn into pseudo–Goldstones because of the explicit breaking of the global symmetry. The reason why this symmetry breaking is called 'collective' is that we need to break two symmetries explicitely to produce mass and potential terms for the pseudo–Goldstone. Only breaking one of them leaves the respective other one as a global symmetry under which the Higgs fields transforms non–linearly. This way we ensure that the Higgs mass and potential terms have a squared g^2 suppression compared to f. As a side remark we notice that while this gives us a suppression of g^2 instead of g, we do not collect additional factors $1/(4\pi)$, because we are still looking at one–loop diagrams.
Looking back, we now have a scale interval where our little–Higgs effective theory does exactly what it is supposed to do: below $g^2 f/(2\pi)$ we have the Standard Model with it usual Higgs mass. Above $4\pi f$ we have a strongly interacting UV completion which we are ignoring at this point, because it might be a mess. In between, there is an energy range $g^2 f/(2\pi), \dots, 4\pi f$ where we can compute effects of the new physics using the little–Higgs theory.

C. Little–Higgs Models

From the last chapters we now know how to generally build models which protect the SM Higgs mass from quadratic divergences at one loop: we pick a global symmetry of which we gauge only a part. Then we break it spontaneously to our $SU(2)_L$ at a scale f and at the same time break it explicitely via gauge or Yukawa couplings. Part of the complete set of Goldstones will make the additional gauge bosons heavy and the remaining pseudo–Goldstones include the SM Higgs sector and protect its low mass.
Because the original global symmetry group is explicitely broken via collective symmetry breaking, the Higgs will develop mass and potential terms governed by the scale f, but doubly loop suppressed (via gauge–boson or fermion loops). It will come as a surprise that this scheme can be realized in many different ways. In the following, we will discuss two realizations, one starting from a global $[SU(3)]^2$ and the other starting from a global $SU(5)$ symmetry.

1. *The simplest little Higgs*

The smallest useful extension of $SU(2)_L$ is $SU(3)$ as discussed before and as Weinberg pointed out decades ago. To protect the Higgs mass a single broken $SU(3)$ symmetry is not sufficient. We instead need a more complex setup, so we postulate a global $[SU(3)]^2$ symmetry and break it in steps down to $SU(2)_L$. We can then express all mass scales in terms of the symmetry–breaking scale f. Starting from the UV the basic structure of our model is

– for $E > 4\pi f$ we know our effective theory in E/f breaks down, so our theory is strongly interacting and/or needs a UV completion.

– below that, the effective Lagrangian obeys a $[SU(3)]^2$ symmetry transformation $U_j \quad (j = 1, 2)$ with two gauge couplings g_j and two Yukawa couplings λ_j. They couple to one set of $SU(3)$ gauge bosons, which contains three $SU(2)$ gauge bosons, plus complex $W'_\pm, W'_0$ with hypercharge 1/2 and a singlet Z'.

– through loop effects gauge and Yukawa couplings explicitely break $[SU(3)]^2 \to SU(3)_{\text{diag}}$. The related pseudo Goldstones give masses of the order $g\,f$ to the heavy $SU(3)$ gauge bosons.

– the other five broken generators of $[SU(3)]^2$ become Goldstones h, η including the Higgs. Terms like $\phi_1^\dagger \phi_2$ give rise to Higgs masses around $g^4 f^2/(2\pi)^2 \equiv M^2_{\text{weak}}$. Fermion loops also lead to a Higgs potential through $\phi_1^\dagger \phi_2 = f^2 - 2h^\dagger h + 2(h^\dagger h/(3f^2)$ which breaks $SU(3)_{\text{diag}} \to SU(2)_L$.

– to introduce hypercharge $U(1)_Y$ we have to postulate another $U(1)_X$, which includes a heavy gauge boson mixing with the $SU(3)/SU(2)$ and the $SU(2)$ gauge bosons, to produce γ, Z, Z'. This will be a problem, because this way we lose the custodial $SU(2)$ which is experimentally so well confirmed.

Because we will definitely need it later, we first compute the one helpful $SU(3)$-invariant term in the Lagrangian after rotating away the eaten Goldstones and to an order higher in $1/f$ than before

$$\phi_1^\dagger \phi_2 = \begin{pmatrix} 0 & f \end{pmatrix} \exp \begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix} = f^2 - 2h^\dagger h + \frac{2}{3f^2}(h^\dagger h)^2 + \mathcal{O}\left(\frac{1}{f^4}\right) \tag{294}$$

Note that we omit the 8th generators of $SU(3)$, diag$(-1, -1, 2)$, and its corresponding Goldstone η and will dicuss it's physics at the end of the lecture. Moreover, we assume $f_1 = f_2 = f$. We will see that such terms can be loop–induced by gauge–boson or top loops, but we can always write them in terms of this combination $\phi_1^\dagger \phi_2$.

The $SU(3)$ gauge interactions sketched in the last sections now include terms like

$$\boxed{\mathcal{L} \supset |g_1 A_\mu \phi_1|^2 + |g_2 A_\mu \phi_2|^2} \tag{295}$$

FIG. 2: Top–quark contributions to the Higgs mass from top loops. Note that the two–point diagram (left) involves a Standard–Model top quark, while the one–point diagram (right) exists only for the heavy top quark.

To study their behavior we can for example set $g_2 = 0$, so that both terms are symmetric under both the two $SU(3)$ symmetries

$$\phi_1 \to U_1 \phi_1 \qquad A_\mu \to U_1^\dagger A_\mu U_1 \qquad\qquad \phi_2 \to U_2 \phi_2 \qquad A_\mu \to U_2^\dagger A_\mu U_2 \tag{296}$$

Switching on g_1 and g_2 in parallel then breaks this $[SU(3)]^2$ symmetry to a single diagonal symmetry $SU(3)_{\rm diag}$

$$\phi_2 \to U \phi_2 \qquad \phi_1 \to U \phi_1 \qquad A_\mu \to U^\dagger A_\mu U \tag{297}$$

As we showed in the last section, the $SU(3)$–gauge–boson loops contribute to the Higgs mass as

$$\mathcal{L} \supset \frac{g_1^2 g_2^2}{(4\pi)^2} \, |\phi_1^\dagger \phi_2|^2 \; \log \frac{\Lambda^2}{\mu^2} \sim - \frac{g_1^2 g_2^2}{(2\pi)^2} \, f^2 \; \log \frac{\Lambda^2}{\mu^2} \, h^\dagger h + \mathcal{O}\left(h^4\right) \tag{298}$$

For a weak-scale Higgs $m_H \sim M_{\rm weak}$ and $SU(2)$-type gauge couplings g_j, this means $f \sim$ TeV, which in turn means that our theory will break down around ~ 10 TeV.

Next, we remember that until now we only dealt with the gauge sector leading to quadratic divergences in the Higgs mass. We obviously need to extend the fermion sector, which otherwise creates quadratic divergences for the Higgs mass proportional to the top Yukawa. So we enlarge the $SU(2)$ heavy–quark doublet Q to an $SU(3)$ triplet $\Psi = (t, b, T) \equiv (Q, T)$. The Yukawa couplings look like $\lambda_j \phi_j^\dagger \Psi t_j^c$, in analogy to the Standard Model, but with two right–handed top singlets t_j^c which will combine to the Standard–Model and to the heavy right–handed tops. We can compute

$$\begin{aligned}
\phi_j^\dagger \Psi &= (0 f) \exp \left[\mp \frac{i}{f} \begin{pmatrix} & h \\ h^\dagger & \end{pmatrix} \right] \begin{pmatrix} Q \\ T \end{pmatrix} \\
&= (0 f) \left[\mathbb{1} \mp \frac{i}{f} \begin{pmatrix} & h \\ h^\dagger & \end{pmatrix} - \frac{1}{2f^2} \begin{pmatrix} h h^\dagger & \\ & h^\dagger h \end{pmatrix} + \dots \right] \begin{pmatrix} Q \\ T \end{pmatrix} \\
&= (0 f) \left[\begin{pmatrix} Q \\ T \end{pmatrix} \mp \frac{i}{f} \begin{pmatrix} h T \\ h^\dagger Q \end{pmatrix} - \frac{1}{2f^2} \begin{pmatrix} h h^\dagger Q \\ h^\dagger h T \end{pmatrix} + \dots \right] \\
&= f T \mp i h^\dagger Q - \frac{1}{2f} \, h^\dagger h T + \dots
\end{aligned} \tag{299}$$

Combining them gives assuming the simplification $\lambda_1 = \lambda_2 = \lambda$

$$\mathcal{L} \supset \lambda f \, \left(1 - \frac{1}{2f^2} \, h^\dagger h \right) T T^c + \lambda \, h^\dagger Q t^c + \dots \tag{300}$$

where we define the SM top quark as $t_2^c - t_1^2 = -i\sqrt{2} t^c$ and where its orthogonal partner $t_1^c + t_2^c = \sqrt{2} T^c$ appears in the T–mass term λf.

Both top quarks contribute to the Higgs mass as shown in Fig. 2. Note the factor 2 in the $T\bar{T}hh$ coupling from the two permutations of the Higgs fields. The scalar integrals involved we know, generally omitting a factor $1/(4\pi)^2$: $B(0; m, m)\big|_{\rm UV} \sim (\Lambda/m)^2$. Adding two fermion propagators with mass m_t and two couplings alters the behavior of the Standard–Model diagram to $-i^4 \, \lambda^2 \, \Lambda^2 = -\lambda^2 \, \Lambda^2$. The second diagram starts from a scalar

UV-divergent $A(m_T)\big|_{\rm UV} \sim \Lambda^2$. Adding one fermion line and the 4-point coupling $-\lambda/f$ yields $-i^2\lambda/f\, m_T\, \Lambda^2 = +\lambda^2\Lambda^2$. From this hand–waving estimate we get an idea how these two top quarks cancel each other's quadratic divergence for the Higgs mass.
If we do this calculation more carefully, we find that indeed, for an $SU(3)$-invariant regulator, the two diagrams cancel. Actually, just like in supersymmetry, only the quadratic divergences cancel, and terms proportional to $\log m_t/m_T$ remain.

Note that again switching off $\lambda_2 = 0$ the Yukawa couplings are symmetric under both $\phi_j \to U_j\phi_j$, $\Psi \to U_j\Psi$. Just as for the gauge couplings, having $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ breaks $[SU(3)]^2 \to SU(3)_{\rm diag}$ as the symmetry of the Yukawa part of the Lagrangian.
Strictly speaking, we could keep the two λ_j separated and would find

$$\begin{aligned} m_T &= \sqrt{\lambda_1^2 f_1^2 + \lambda_2^2 f_2^2} \sim \max_j(\lambda_j f_j) \\ \lambda_t &= \lambda_1\lambda_2 \, \frac{1}{m_T} \, \sqrt{f_1^2 + f_2^2} \end{aligned} \tag{301}$$

Writing down the SM with a protected light Higgs mass requires us to break both groups, $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$,. This makes the Higgs a pseudo–Goldstone and allows only contributions proportional to $\lambda_1\lambda_2$ in the Higgs potential (and the Higgs mass). Strictly speaking, we could even show that only terms proportional to $\lambda_1^2\lambda_2^2$ appear, and terms with four Yukawa couplings never lead to quadratic divergences.

The remaining big mystery in this model is the Higgs potential, and in particular the relation between mass and quartic coupling. We can compare the relative sizes of the mass and self coupling which we get from the fermion loops

$$\begin{aligned} |\phi_1^\dagger\phi_2|^2 &= f^2 - 4f^2(h^\dagger h) + \frac{14}{3}\,(h^\dagger h)^2 + \ldots \equiv -m^2 h^\dagger h + \lambda(h^\dagger h)^2 \\ \Longrightarrow \qquad & \boxed{\left|\frac{m^2}{\lambda}\right| \sim \frac{12}{14}\, f^2 \sim \mathcal{O}\left(\text{TeV}^2\right)} \qquad \text{while} \qquad \left|\frac{m^2}{\lambda}\right|_{\rm SM} = 2v^2 \end{aligned} \tag{302}$$

In other words, compared to the Standard Model, the mass is too large in comparison to the quartic coupling. There is no easy cure to this, so we resort to ad–hoc introducing a μ parameter with the proper sign

$$\mathcal{L} \supset \mu^2\phi_1^\dagger\phi_2 = \mu^2\left(f^2 - 2\,h^\dagger h + \mathcal{O}\left(\frac{1}{f^2}\right)\right) \tag{303}$$

Roughly $\mu \sim M_{\rm weak}$ brings the Higgs mass to the correct value. Note that such a term also breaks the $U(1)$ symmetry linked to the 8th $SU(3)$ generators and gives η a mass of the order $M_{\rm weak}$.

To summarize, we have analyzed the particle spectrum in the μ-model or Schmaltz model or simple–group model, which is necessary to avoid quadratic divergences in the Higgs mass at one loop. In this model we start from a global symmetry group $[SU(3)]^2$. These two symmetries we break spontaneously into a $SU(2)$ each, freeing up 10 Goldstone modes corresponding to the $[SU(3)/SU(2)]^2$ degrees of freedom.
At the same time, gauge and Yukawa interactions break $[SU(3)]^2$ to the diagonal, now gauged, subgroup $SU(3)_{\rm diag}$ which is the one which is really spontaneously broken by a vev f. This means that half of these 10 Goldstone modes are going to be absorbed into massive $SU(3)/SU(2)$ gauge bosons. The other five now pseudo–Goldstones can develop a mass and a potential, but each term has to be proportional to both of the gauge (or Yukawa) couplings. As a check we can switch off one of the two gauge couplings: now we have two exact $SU(3)$ symmetries, one of which is gauged spontaneously broken, and acquires heavy gauge–boson masses of the scale f, while the other one is exact, *i.e.* protecting its Goldstones from acquiring a mass at all.

Apart from the Standard–Model particles and a light protected Higgs we find the particle spectrum

$$\begin{aligned} SU(3) \text{ gauge bosons } W'^{\pm}, W'^{0} \quad &\text{with} \quad m_{W'} = \frac{g^2 f^2}{2} \\ \text{singlet } Z' \quad &\text{with} \quad m_{Z'} = g^2 f^2 \frac{2}{3-t^2} \qquad (t = \tan\theta_w) \\ \text{heavy top } T \quad &\text{with} \quad m_T = \sqrt{2}\lambda_t f \\ \text{Standard Model } Z \quad &\text{with} \quad m_Z = \frac{g^2 v^2}{4}\,(1+t^2) \qquad \text{etc...} \end{aligned} \tag{304}$$

To avoid extending this particle content and correcting for the mass–quartic ratio in the Higgs potential we in addition need a tree–level parameter $\mu^2 \phi_1^\dagger \phi_2$.

2. The littlest Higgs

Combining what we know about sigma models and collective symmetry breaking we can construct another particularly economic little–Higgs model. In the μ model we write two sets of Goldstones in the fundamental representations of $SU(3)$, which are partly gauged and then broken to our $SU(2)_L$ via the high–scale vev f. It is crucial to have two distinct $SU(3)$ gauge groups (and gauge couplings) to forbid one–loop quadratically divergent Higgs self energies. The same trick we can play with two Yukawas, so that a Higgs potential is proportional to $g_1^2 g_2^2$ or to $\lambda_1^2 \lambda_2^2$.

This time, we want to embed two gauge symmetries which overlap by the Standard Model Higgs doublet into one matrix field Σ: in other words, we write a matrix–valued Σ field which includes two copies of $SU(2)$ which are broken to the $SU(2)_L$ and which at the same time includes a pseudo–Goldstone–Higgs doublet. Two $SU(2)$ generators inside a 5×5 matrix could look like

$$Q_1^a = \frac{1}{2}\begin{pmatrix} -\sigma^{a*} & 0_{2x3} \\ 0_{3x2} & 0_{3x3} \end{pmatrix} \qquad Q_2^a = \frac{1}{2}\begin{pmatrix} 0_{3x3} & 0_{2x3} \\ 0_{3x2} & \sigma^{a} \end{pmatrix} \tag{305}$$

The Goldstone modes in the Σ field should include the Higgs doublet in a form which means that neither of the sets of $SU(2)$ generators include it. This means that when we break $SU(5)$ into one of the $SU(2)$ subgroups the Higgs will always stay a (pseudo–) Goldstone, which is the idea of collective symmetry breaking

$$\Sigma = e^{2i(\pi\cdot\hat{T})/f}\langle\Sigma\rangle \qquad\qquad \pi\cdot\hat{T} \sim \frac{1}{\sqrt{2}}\begin{pmatrix} & h^* & \\ h^\dagger & & h^\dagger \\ & h & \end{pmatrix} \tag{306}$$

If the global $SU(5)$ symmetry is broken by $\langle\Sigma\rangle$, this will allow the h doublet to develop a potential, *i.e.* a mass and a self coupling. Note that $\hat{T}$ is indeed hermitian and traceless, so $\exp(2i(\pi\cdot\hat{T})/f)$ is a unitary transformation. The Standard–Model $SU(2)_L$ generators Q^a or the Higgs should of course not be affected by $\langle\Sigma\rangle$, because otherwise they would acquire masses of the order f. Therefore, we write

$$\langle\Sigma\rangle = \begin{pmatrix} & & \mathbb{1}_{2\times 2} \\ & 1 & \\ \mathbb{1}_{2\times 2} & & \end{pmatrix} \tag{307}$$

This vev obviously breaks our global $SU(5)$ symmetry, written in the adjoint representation. $SU(5)$ has $(N^2-1) = 24$ generators $U = \exp(i\theta\cdot T)$ under which the Σ field transforms as $\Sigma \to U\Sigma U^T$. What remains after $\langle\Sigma\rangle$ is an $SO(5)$ symmetry, generated by the antisymmetric tensor with $(4+3+2+1) = 10$ entries. We can use the transformation of Σ to derive the commutation properties of the 10 unbroken generators $\overline{T}$ and the 14 broken generators $\hat{T}$. For the broken generators we find

$$\Sigma = e^{i(\pi\cdot\hat{T})/f}\;\langle\Sigma\rangle\; e^{i(\pi\cdot\hat{T}^T)/f} = e^{2i(\pi\cdot\hat{T})/f}\langle\Sigma\rangle\,, \tag{308}$$

or in other words $\langle\Sigma\rangle\,\hat{T}^T = \hat{T}\,\langle\Sigma\rangle$. For the remaining unbroken, good generators we require

$$\Sigma = e^{i(\pi\cdot\overline{T})/f}\,\langle\Sigma\rangle\,e^{i(\pi\cdot\overline{T}^T)/f} = \langle\Sigma\rangle \tag{309}$$

which translates into $\langle\Sigma\rangle\,\overline{T}^T = -\overline{T}\,\langle\Sigma\rangle$. We can explicitely compute the commutators for the sum of hermitian $SU(2)$ generators $Q^a = Q_1^a + Q_2^a$, to check that they are indeed not broken

$$Q\,\langle\Sigma\rangle = \frac{1}{2}\begin{pmatrix} & -\sigma^* \\ \sigma & \end{pmatrix} \qquad \langle\Sigma\rangle\,Q = \frac{1}{2}\begin{pmatrix} & \sigma^* \\ -\sigma & \end{pmatrix} \qquad \Rightarrow \qquad Q\,\langle\Sigma\rangle = -\langle\Sigma\rangle\,Q^T \tag{310}$$

So the generators Q^a, which we plan to make the generators of our Standard–Model $SU(2)$ gauge group are indeed part of the unbroken set of $SU(5)$ generators $\overline{T}$. The corresponding $U(1)$ generators are the diagonals $\text{diag}(-3,-3,2,2,2)/10$ and $\text{diag}(-2,-2,-2,3,3)/10$.

To compute the spectrum of the littlest Higgs model which breaks $SU(5)\to SO(5)$, we start by writing down the complete set of Goldstones associated with the broken generators in Σ, filling in the remaining 2×2 matrices and the diagonal generator

$$\boxed{\pi\cdot\hat{T} = \begin{pmatrix} \chi_{2\times2} & h^*/\sqrt{2} & \phi^\dagger_{2\times2} \\ h^T/\sqrt{2} & 0 & h^\dagger/\sqrt{2} \\ \phi_{2\times2} & h/\sqrt{2} & \chi^T_{2\times2} \end{pmatrix} + \frac{\eta}{2\sqrt{5}}\begin{pmatrix} 1\!\!1_{2\times2} & & \\ & -4 & \\ & & 1\!\!1_{2\times2} \end{pmatrix}} \tag{311}$$

The form is given by requirement $\langle\Sigma\rangle\,\hat{T}^T = \hat{T}\,\langle\Sigma\rangle$, which links opposite corners of $\pi\cdot\hat{T}$. The $SU(2)$ generators in χ form a hermitian traceless 2×2 matrix, but the combination of χ and χ^T in the opposite corners (instead of $-\sigma^*$ and σ or equivalently $-\sigma^T$ and σ) makes χ part of the broken subset $\hat{T}$. The remaining 2×2 matrix of generators ϕ is not traceless, but complex symmetric. The complex doublet h is hopefully the Standard–Model Higgs doublet, and η is the usual real singlet. Together, these field indeed correspond to the $3+6+4+1=14$ Goldstone degrees of freedom.

Unless something else happens (like collective symmetry breaking) the fields linked to the broken generators $(\pi\cdot\hat{T})$ can either turn into gauge–boson mass terms of the order f or stay massless. In particular, χ will make a set of $SU(2)$ gauge bosons $W'^{\pm}, W'^{0}$ heavy, where η corresponds to the B' field. We can mix the two $SU(2)$ groups described by χ (broken with mass scale f) and σ (unbroken with mass scale v) to the Standard–Model $SU(2)_L$.
For the littlest Higgs collective symmetry breaking occurs just the same way as in the μ model, namely through gauge couplings. The two sets of $SU(2)$ generators Q_j are linked once we remember that the particular combination Q_1+Q_2 is part of the unbroken set of $SU(5)$ generators.

$$D_\mu\Sigma = \partial_\mu\Sigma - i\sum_{j=1,2} g_j(W_{j\mu}Q_j)\Sigma - i\sum_j g_j\Sigma(W_{j\mu}Q_j^T) - i\sum_j g'_j(B_{j\mu}Y_j)\Sigma - i\sum_j g'_j\Sigma(B_{j\mu}Y_j) \tag{312}$$

In other words, the vev $\langle\Sigma\rangle$ again breaks this symmetry $[SU(2)\times U(1)]^2$ to the diagonal $SU(2)_L\times U(1)_Y$ at a scale f. Defining a set of $SU(2)$ and $U(1)$ mixing angles $\tan\Psi^{(\prime)} = g_2^{(\prime)}/g_1^{(\prime)}$ we can write the set of gauge bosons in terms of the Standard–Model and a heavy set of $SU(2)\times U(1)$ bosons

$$\begin{pmatrix} W_H^a \\ W_{SM}^a \end{pmatrix} = \begin{pmatrix} -\cos\Psi & \sin\Psi \\ \sin\Psi & \cos\Psi \end{pmatrix}\begin{pmatrix} W_1^a \\ W_2^a \end{pmatrix} \qquad \begin{pmatrix} B_H \\ B_{SM} \end{pmatrix} = \begin{pmatrix} -\cos\Psi' & \sin\Psi' \\ \sin\Psi' & \cos\Psi' \end{pmatrix}\begin{pmatrix} B_1^a \\ B_2^a \end{pmatrix} \tag{313}$$

As mentioned above the heavy gauge bosons acquire masses though the Goldstones χ

$$M_{W_H} = \frac{gf}{\sin2\Psi} \qquad M_{B_H} = \frac{g'f}{\sqrt{5}\sin2\Psi'} \tag{314}$$

The littlest–Higgs model works for quadratic divergences just like the $SU(3)$ model. Each of the two sets of generators $\{Q_j^a, Y_j\}$ corresponds to a 3×3 matrix of Goldstones in the respective other corner in Σ after breaking $SU(5)$ to $SU(2)$. So if we break the global symmetry down to one of the two $SU(2)$ groups the Higgs doublet will be a broken generator of the global $SU(5)$ and therefore remain massless. If we remove one set of gauge couplings $g_j^{(\prime)} = 0$, we indeed find a global $SU(3)$ symmetry which protects the Higgs from quadratic divergences proportional to the respective other $g_k^{(\prime)}$.

Protecting the Higgs mass from top loop works also similarly to the $SU(3)$ model. We extend the $SU(2)_L$ quark doublet to the triplet $\Psi = (b, -t, T)$ and add a right-handed singlet t'^c. Because we expect mixing between the two top singlets which will give us the Standard–Model and a heavy top quark we write two general Yukawa couplings involving the Σ field (just like we write Yukawa couplings in the usual Σ model)

$$\mathcal{L} \supset \lambda_1 \, f \, \epsilon_{ijk} \Psi_i \, \Sigma_{j4} \Sigma_{k5} \, t_1^c + \lambda_2 \, f \, T t_2^c \tag{315}$$

The Σ-field triplets we take from the 2×3 upper-right corner of the Goldstone matrix

$$\sigma_{jm} = \begin{pmatrix} \phi^\dagger \\ h^\dagger/\sqrt{2} \end{pmatrix} \qquad j = 1, 2, 3 \qquad m = 4, 5 \tag{316}$$

If we set $\lambda_2 = 0$ this Yukawa coupling is symmetric under this $SU(3)$ symmetry, because it is the anti-symmetric combination of three triplets. Again, contributions to the Higgs mass therefore have to be proportional to $\lambda_1^2 \lambda_2^2$ and quadratic divergences are forbidden at one loop.
The two heavy quarks mix to the SM top quark and an additional heavy top

$$t_R = \frac{\lambda_2 t_1 - \lambda_1 t_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \qquad T_R = \frac{\lambda_1 t_1 + \lambda_2 t_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \qquad m_T = \sqrt{\lambda_1^2 + \lambda_2^2} f \tag{317}$$

where we as before have assumed $f = f_1 = f_2$. The actual top–Higgs coupling are given in terms of λ_j

$$\lambda_{ttH} \equiv \lambda_t = \frac{\sqrt{2} \lambda_1 \lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \qquad \lambda_{TTHH} \equiv -\frac{\lambda_T}{\sqrt{2} f} = \frac{-\lambda_1^2}{f \sqrt{\lambda_1^2 + \lambda_2^2}} \tag{318}$$

and ensure that the leading divergences in the Standard–Model two–point diagram and the heavy–top one–point diagram cancel.

An interesting question would be: can we distinguish little–Higgs models for example by relating the parameters in the top sector. After all, the construction of the μ model and the littlest–Higgs model are quite different. Using these expressions above we can write the heavy top mass in the littlest–Higgs model in term of the Yukawas $\lambda_{t,T}$ (modulo factor $\sqrt{2}$?)

$$m_T = f \, \frac{\lambda_t^2 + \lambda_T^2}{\sqrt{2} \lambda_T} \tag{319}$$

In contrast, in the $SU(3)$ model we saw ($f = f_1 = f_2$)

$$\lambda_T = \frac{\lambda_t}{2f} = \frac{\lambda_1 \lambda_2 \, f}{m_T} \qquad m_T = f \, \sqrt{\lambda_1^2 + \lambda_2^2} \tag{320}$$

So the relation between m_T and the $HHTT$ coupling are indeed different.

The heavy spectrum of the littlest Higgs model is

$$\begin{aligned} SU(2) \times U(1) \text{ gauge bosons } B', W'^{\pm}, Z' \quad & \text{with} \quad m_{B',W',Z'} = \mathcal{O}(f) \\ \text{Higgs triplet } \phi = \begin{pmatrix} \phi^{++} & \phi^+ \sqrt{2} \\ \phi^+ \sqrt{2} & \phi^0 \end{pmatrix} \quad & \text{with} \quad m_\phi = \mathcal{O}(f) \\ \text{heavy top } T \quad & \text{with} \quad m_T = \mathcal{O}(f) \end{aligned} \tag{321}$$

As described earlier in the lecture, from B' and ϕ we expect serious violation of custodial $SU(2)$. Electroweak precision data forces us to choose f unusually large in the little–Higgs model. On the other hand, a Higgs triplet with a doubly charged Higgs boson has a smoking–gun signature at the LHC, namely its production in weak–boson fusion: $uu \to ddW^+W^+ \to ddH^{++}$.

In contrast to the μ model, we now do not need a μ term, though. One–loop effects lead to a Coleman–Weinberg potential (which is nothing but a general quartic potential of a massive charged scalar in a gauge theory) with the relative mass scales

$$\frac{m_h^2}{\lambda} \sim \left(\frac{m}{g}\right)^2 \sim (2m)^2 < f^2 \tag{322}$$

after integrating out the heavy ϕ fields.

3. T parity

Looking at the tree–level violation of the custodial $SU(2)$ symmetry (*i.e.* $\rho \neq 1$) and at the benefits of a weakly interacting and stable dark–matter candidate it would be great to introduce a Z_2 symmetry which allows only two heavy little-Higgs particles per vertex. In other words, we would like to define a quantum number with one value for all weak–scale Standard–Model particles and another value for all particles with masses around f. Such a parity will be called T parity.

For the littlest Higgs, we would like to separate the additional heavy $SU(2)$ doublet from our Standard–Model gauge bosons. Assuming $g_1^{(\prime)} = g_2^{(\prime)}$ the Lagrangian involving $D_\mu \Sigma$ is symmetric under the exchange of the two $[SU(2) \times U(1)]$ groups. The eigenstates we can choose as

$$W_\pm = \frac{W_1 \pm W_2}{\sqrt{2}} \qquad\qquad B_\pm = \frac{B_1 \pm B_2}{\sqrt{2}} \tag{323}$$

where W_+, B_+ are Standard–Model gauge bosons, while W_-, B_- are heavy. Exchanging the indices ($1 \leftrightarrow 2$) is an even transformation for W_+, while it is odd for W_-, just as we want. Taking into account all broken generators, we apply a factor $(-)$ to each heavy field, while leaving h unchanged. In proper matrix notation we postulate a symmetry Ω which acts on the broken generators for example in the littlest Higgs model

$$\begin{aligned}
\pi \cdot \hat{T} &= \begin{pmatrix} \chi & h^* & \phi^\dagger \\ h^T & 0 & h^\dagger \\ \phi & h & \chi^T \end{pmatrix} + \eta \begin{pmatrix} 1\!\!1 & & \\ & -4 & \\ & & 1\!\!1 \end{pmatrix} \\
&\to^\Omega - \begin{pmatrix} 1\!\!1 & & \\ & -1 & \\ & & 1\!\!1 \end{pmatrix} \left[\begin{pmatrix} \chi & h^* & \phi^\dagger \\ h^T & 0 & h^\dagger \\ \phi & h & \chi^T \end{pmatrix} + \eta \begin{pmatrix} 1\!\!1 & & \\ & -4 & \\ & & 1\!\!1 \end{pmatrix} \right] \begin{pmatrix} 1\!\!1 & & \\ & -1 & \\ & & 1\!\!1 \end{pmatrix} \\
&= - \begin{pmatrix} 1\!\!1 & & \\ & -1 & \\ & & 1\!\!1 \end{pmatrix} \left[\begin{pmatrix} \chi & -h^* & \phi^\dagger \\ h^T & 0 & h^\dagger \\ \phi & -h & \chi^T \end{pmatrix} + \eta \begin{pmatrix} 1\!\!1 & & \\ & 4 & \\ & & 1\!\!1 \end{pmatrix} \right] \\
&= - \begin{pmatrix} \chi & -h^* & \phi^\dagger \\ -h^T & 0 & -h^\dagger \\ \phi & -h & \chi^T \end{pmatrix} - \eta \begin{pmatrix} 1\!\!1 & & \\ & -4 & \\ & & 1\!\!1 \end{pmatrix} \\
&= \begin{pmatrix} -\chi & h^* & -\phi^\dagger \\ h^T & 0 & h^\dagger \\ -\phi & h & -\chi^T \end{pmatrix} + (-\eta) \begin{pmatrix} 1\!\!1 & & \\ & -4 & \\ & & 1\!\!1 \end{pmatrix}
\end{aligned} \tag{324}$$

This symmetry work perfectly for the additional gauge bosons, including the heavy scalars ϕ. A problem arises when we assign such a quantum number to the heavy tops. Usually, we expand the $SU(2)_L$ doublet to a triplet under Ω and split the fermions into one set transforming under each $[SU(2) \times U(1)]_j$. At this point, we now have to introduce additional fermions and all hell breaks loose, even though the model by definition agrees better with current electroweak precision constraints.

One final remark concerning such a T parity. Recently (hep-ph/0701044) Chris and Richard Hill have shown that such a discrete parity if naively implemented is broken by anomalies, *i.e.* it is not stable after quantum corrections. Obviously, such considerations affect arguments over large time scales, like the formation of dark matter. On the other hand, I am not sure if our model–building friends will get around this problem using a fancier realization of the T parity. Let's wait and see...

D. Pseudo–Axions

Remember that until now we have always neglected the additional diagonal generator of our global symmetry group. In the μ model we saw that is acquires a mass through the μ term $\mu^2 \phi_1^\dagger \phi_2$

$$m_\eta = \left(\frac{f_1}{f_2} + \frac{f_2}{f_1} \right)^{1/2} \mu \gtrsim \sqrt{2}\, \mu \sim M_{\text{weak}} \tag{325}$$

In the littlest Higgs model, in contrast, the same Goldstone mode is eaten by the additional $U(1)_Y$ gauge field, the heavy photon with a mass $m_{B'} \sim f$. Both of these cases are in a sense clever constructions, to avoid the general problem that after breaking a global symmetry group to a lower–rank group, we will typically find diagonal generators which correspond to massless singlet scalars in the low–energy effective theory. Such scalars turn out to be similar to so-called axions.

Fermion coupling: Goldstones we know are protected from becoming massive by to their non–linear shift symmetry $\eta \to \eta + f \cdot \alpha$. This symmetry of course has to be respected by their scalar and pseudo–scalar couplings to fermions, which are of the general form

$$\begin{aligned}
\mathcal{L} \supset\; & g_S \overline{\Psi} \mathbb{1} \Psi\, \eta + g_P \overline{\Psi} \gamma^5 \Psi\, \eta \qquad\qquad \gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \\
&= g_S \Psi^\dagger \gamma^0 \Psi\, \eta + g_P \Psi^\dagger \gamma^0 \gamma^5 \Psi\, \eta \\
&= g_S \left(\overline{\Psi}_L \Psi_R + \overline{\Psi}_R \Psi_L \right) \eta + g_P \left(\overline{\Psi}_L \Psi_R - \overline{\Psi}_R \Psi_L \right) \eta \\
&\to g_S \left(\overline{\Psi}_L \Psi_R + \overline{\Psi}_R \Psi_L \right) (\eta + f \cdot \alpha) + g_P \left(\overline{\Psi}_L \Psi_R - \overline{\Psi}_R \Psi_L \right) (\eta + f \cdot \alpha)
\end{aligned} \tag{326}$$

The first term is obviously not symmetric for $\alpha \neq 0$, so the global symmetry requires $g_S = 0$. The second term we could compute and find that it is actually allowed... So our diagonal generators η or pseudo–axions couple to fermions like pseudo-scalars. Note, however, that the η coupling to fermions does not include an f in the numerator when we write it in terms of the Σ field, so the $t\bar{t}\eta$ coupling will be suppressed by v/f.

Gauge–boson coupling: We can write down operators like $\eta W^+ W^-$, but they are be forbidden at tree level if η is a pseudo scalar. This is, by the way, the same for the heavy MSSM pseudo scalar A^0. Just as in the MSSM, η could couple to gauge bosons via $\eta W_{\mu\nu} \widetilde{W}^{\mu\nu}$, but this CP–odd combination is of mass dimension 5 and therefore loop suppressed as $\frac{v}{f}$. If will for example be induced by heavy top loops.

Mixing with Higgs: Potential terms like $\eta^2 h^2$ are allowed in the Lagrangian. However, they introduce a quadratic divergence in m_H when we link the two η fields to a loop. At one loop we find $\Delta m_h^2 \sim (\Lambda/4\pi)^2 \sim f^2$, which is precisely what we build little–Higgs models to avoid. As usually, $\Delta m_h^2 \sim v^2 \sim m_\eta^2$ is acceptable, which simply corresponds to a mandatory factor $\mathcal{O}(v/f)$ in front of the $\eta^2 h^2$ term.

Signatures for η are similar to the heavy pseudoscalar A^0 in the MSSM; if it is really light, we can see $h \to \eta\eta$ decays, otherwise we rely on production cross sections suppressed by (v^2/f^2) with subsequent decays to Standard–Model gauge bosons or fermions, similar to Higgs signatures. The CP properties of such scalars we can determine either from jet correlations in weak–boson–fusion production or from lepton–correlations in decays to $ZZ \to 4\ell$.

E. Literature

Little–Higgs models have the great advantage that at least I find the original papers very readable. Nevertheless, there are also a few very good review articles on the market...

- for the basics there is Wolfgang Kilian's great book on electroweak symmetry breaking, a very brief and yet complete introduction into the sigma model and strongly interacting theories. It is ridiculously expensive, though.
- there is the usual incredibly useful writeup which for example these lectures are based on. It is Martin Schmaltz' and David Tucker–Smith's review article (hep-ph/0502182). Obviously, it focusses on the μ model or so–called Schmaltz model.
- another equally useful review is Maxim Perelstein's hep-ph/0512128, which starts from the littlest Higgs instead.
- my chapter on T parity is unfortunately very brief. But don't worry, there are very readable papers by Ian Low and collaborators or more phenomenologically by the Cornell group.
- similarly, my chapter on pseudo–axions is too short. You can have a look for example at hep-ph/0411213, in particular for a phenomenological analysis of this general feature of little–Higgs models.
- the collider phenomenology of little–Higgs models you can find in the standard reference hep-ph/0301040. It also includes lots of Feynman rules for those of you who want to calculate for example LHC cross sections.

Acknowledgments: I would like to thank all the people who have tried to explain little–Higgs models to me, in particular Martin Schmaltz. You might have noticed that basically this entire set of note is based on his review. And as usual I would like to thank Maria Ubiali who produced this writeup out of my of unreadable collection of hand–written notes.

IV. MODELS WITHOUT A HIGGS

A. Electroweak symmetry breaking

The usual argument for the existence of a Higgs boson starts from a completely massless Lagrangian of a gauge theory with matter fermions — and the fact that neither gauge–boson nor fermion masses can be simply included without breaking gauge invariance. This is of course correct, but it does not automatically imply the existence of a fundamental scalar Higgs boson. As an introduction to this topic, let us try to give masses to a photon and to fermions and this way break electroweak gauge invariance, but avoiding to postulate a fundamental Higgs boson.

1. Massive photon

As a starting point we choose electrodynamics, *i.e.* a (massless) photon in a locally $U(1)$–symmetric Lagrangian. To its kinetic $F \cdot F$ term we add a photon mass and a real uncharged scalar field without a mass and without a coupling to the photon, but with a scalar–photon mixing term:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}e^2f^2A_\mu^2 - efA_\mu\partial^\mu\phi \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}e^2f^2\left(A_\mu - \frac{1}{ef}\partial_\mu\phi\right)^2 - \frac{1}{2}\left(\partial_\mu\phi\right)^2 \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}e^2f^2\left(A_\mu - \frac{1}{ef}\partial_\mu\phi\right)^2 \end{aligned} \tag{327}$$

e is the usual electric charge, *i.e.* just a c-number without any specific relevance in this interaction–less Lagrangian, while f is a mass scale describing the photon mass as well as the mixing term. The Lagrangian includes only terms with mass dimension four, if we remember that bosonic fields like A_μ and ϕ have mass dimension one. We can define a simultaneous gauge transformation of both fields in the Lagrangian

$$A_\mu \longrightarrow A_\mu + \frac{1}{ef}\partial\chi \qquad\qquad \phi \longrightarrow \phi + \chi \tag{328}$$

under which the Lagrangian is indeed invariant. Here, χ is a real number. If we now re-define the photon field as $B_\mu = A_\mu - \partial_\mu\phi/(ef)$ we can first compare the two kinetic terms

$$\begin{aligned} F_{\mu\nu}\Big|_B &= \partial_\mu B_\nu - \partial_n u B_\mu = \partial_\mu\left(A_\nu - \frac{1}{ef}\partial_\nu\phi\right) - \partial_\nu\left(A_\mu - \frac{1}{ef}\partial_\mu\phi\right) \\ &= \partial_\mu A_\nu - \partial_n u A_\mu = F_{\mu\nu}\Big|_A \end{aligned} \tag{329}$$

and then rewrite the Lagrangian as

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}e^2f^2B_\mu^2 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_B^2B_\mu^2} \tag{330}$$

This Lagrangian effectively describes a massive photon field B_μ, which has absorbed the real scalar ϕ as its additional longitudinal component. Remember that a massless gauge boson A_μ has only two on-shell degrees of freedom, namely left and right–handed polarization, while the massive $B\mu$ has an additional longitudinal polarization degree of freedom. Without any fundamental Higgs boson appearing, the photon has 'eaten' the real scalar field ϕ.

The difference to the usual $SU(2)$ Higgs mechanism is that we have chosen not to introduce a charged $SU(2)$ doublet, so there are no degrees of freedom left after the photon gets is mass. On the other hand, this little trick means that our toy model is not going to well-suited to make $SU(2)$ gauge bosons massive. What is

illustrates is only how by introducing a neutral scalar particle without an interaction but with a mixing term we make gauge bosons heavy. This mechanism we will use later again.
What kind of properties does this field ϕ need to have, so that we can use it to provide a photon mass? From the combined gauge transformation we immediately see that any additional purely scalar terms in the Lagrangian (like a scalar potential $V(\phi)$) need to be symmetric under the linear shift $\phi \to \phi + \chi$, not to spoil gauge invariance. This means that we cannot write down polynomial terms ϕ^n, like a mass or a self coupling of ϕ. Similarly, a regular ϕAA interaction would not be possible, either. Only derivative interactions proportional to $\partial\phi$ to any conserved currents are fine. In that case we can absorb the shift by χ into a total derivative in the Lagrangian.

2. *Fermion masses and chiral symmetry*

Giving a mass to a fermion without a Higgs boson is a little more involved. We start by splitting a Dirac fermion, *i.e.* a 4-spinor, into its left-handed and right-handed projections

$$\psi_L = \frac{\mathbb{1} - \gamma_5}{2}\,\psi \equiv P_L\psi \qquad\qquad \psi_R = \frac{\mathbb{1} + \gamma_5}{2}\,\psi \equiv P_R\psi \tag{331}$$

where $P_{L,R}$ are projectors in the 4×4 Dirac space. The kinetic term of the Dirac fermion can be rewritten as

$$\begin{aligned}
\mathcal{L} \supset \overline{\psi}\, i\, \partial\!\!\!/ \psi &= \overline{\psi}\, i\, \partial\!\!\!/ \,(P_L + P_R)\,\psi \\
&= \overline{\psi}\, i\, \partial\!\!\!/ \left(P_L^2 + P_R^2\right)\psi \\
&= i\overline{\psi}\;(P_R\, \partial\!\!\!/ P_L + P_L\, \partial\!\!\!/ P_R)\,\psi \qquad\qquad \text{with} \qquad \{\gamma_5, \gamma_\mu\} = 0 \\
&= i(\overline{P_L\psi})\; \partial\!\!\!/ (P_L\psi) + i(\overline{P_R\psi})\; \partial\!\!\!/ (P_R\psi) \qquad \text{with} \qquad \overline{\psi} = \psi^\dagger\gamma^0 \\
&= \overline{\psi}_L\, i\, \partial\!\!\!/ \psi_L + \overline{\psi}_R\, i\, \partial\!\!\!/ \psi_R
\end{aligned} \tag{332}$$

Under a global so-called chiral symmetry transformation $U(1)_L \times U(1)_R$ which independently transforms the two chiralities $\phi_{L,R}$

$$\boxed{\psi_L \longrightarrow e^{-i\theta}\psi_L} \qquad\qquad \boxed{\psi_R \longrightarrow e^{-i\omega}\psi_R} \tag{333}$$

this Lagrangian is symmetric. Obviously, we can combine these two parts of the chiral transformation into different basis elements, constructing a vector-type and an axial-vector-type combination:

$$\begin{aligned}
\psi_L &\longrightarrow e^{-i\theta}\psi_L \qquad\qquad & \psi_L &\longrightarrow e^{-i\theta}\psi_L \\
\psi_R &\longrightarrow e^{-i\theta}\psi_R \qquad\qquad & \psi_R &\longrightarrow e^{+i\theta}\psi_R
\end{aligned} \tag{334}$$

A gauge-invariant Lagrangian under one definition of the chiral symmetry will always be invariant under the other.
The same way we can now rewrite a Dirac mass in terms of the two chiralities

$$\begin{aligned}
\mathcal{L} \supset m\,\overline{\psi}\psi &= m\,\overline{\psi}\left(P_L^2 + P_R^2\right)\psi \\
&= m\;(\overline{P_R\psi})(P_L\psi) + m\;(\overline{P_L\psi})(P_R\psi) \\
&= m\;\left(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R\right)
\end{aligned} \tag{335}$$

and immediately notice that the $U(1)_L \times U(1)_R$ symmetry is broken and only its vector combination $\theta = \omega$ remains. The question arises — can we write down a fermion mass while keeping the chiral symmetry intact, and without introducing an additional fundamental Higgs boson.

Just like in the Standard Model we first introduce a complex scalar field Φ with a Yukawa coupling to the fermions:

$$\mathcal{L} \supset \overline{\psi}\; i\; \partial\!\!\!/ \psi - g\;\left(\overline{\psi}_L\psi_R\Phi + \overline{\psi}_R\psi_L\Phi^*\right) + |\partial_\mu\Phi|^2 - V(|\Phi|) \tag{336}$$

If the scalar field transforms under the $U(1)_L \times U(1)_R$ chiral symmetry as

$$\Phi \longrightarrow e^{-i(\theta-\omega)}\ \Phi \tag{337}$$

the Yukawa couplings as well as the kinetic and the potential terms for Φ are gauge invariant. As usual, we now spontaneously break the chiral symmetry by introducing a potential for Φ with a nontrivial (*i.e.* $\Phi \neq 0$) minimum:

$$V = -M^2|\Phi|^2 + \frac{\lambda}{2}|\Phi|^4 = -\frac{\lambda}{2}v^2|\Phi|^2 + \frac{\lambda}{2}|\Phi|^4 = \frac{\lambda}{2}\left(|\Phi|^2 - \frac{v^2}{2}\right)^2 + \text{const} \qquad \text{with} \qquad \langle\Phi\rangle \equiv \frac{v}{\sqrt{2}} = \frac{M}{\sqrt{\lambda}} \tag{338}$$

Note that there are definitions with a factor λ and those with $\lambda/2$ around. I am here sticking to the conventions in the technicolor review. We can define the two on-shell degrees of freedom of a complex scalar (c-number) as $\sqrt{2}\Phi = (v + h(x))\ \exp(i\phi(x)/f)$, again with a dimensionful constant f compensating the mass dimension of the scalar field in the exponent. The Φ-dependent part of the Lagrangian becomes

$$\begin{aligned}
\mathcal{L} \supset\ & \frac{1}{2}(\partial h)^2 + \frac{M^2}{2}(v+h)^2 - \frac{\lambda}{8}(v+h)^4 + \frac{1}{2}(v+h)^2\left|\frac{\partial\phi}{f}\right|^2 \\
=\ & \frac{1}{2}(\partial h)^2 + \frac{M^2}{2}\left(h + \frac{\sqrt{2}M}{\sqrt{\lambda}}\right)^2 - \frac{\lambda}{8}\left(h + \frac{\sqrt{2}M}{\sqrt{\lambda}}\right)^4 + \frac{1}{2}(v+h)^2\left|\frac{\partial\phi}{f}\right|^2 \\
=\ & \frac{1}{2}(\partial h)^2 + \frac{M^2}{2}h^2 + M^2 h\frac{\sqrt{2}M}{\sqrt{\lambda}} - \frac{\lambda}{8}h^4 - \frac{\lambda}{2}h^3\frac{\sqrt{2}M}{\sqrt{\lambda}} - \frac{3\lambda}{4}h^2\frac{2M^2}{\lambda} - \frac{\lambda}{2}h\frac{2M^2}{\lambda}\frac{\sqrt{2}M}{\sqrt{\lambda}} \\
& \qquad + \frac{1}{2}(v+h)^2\left|\frac{\partial\phi}{f}\right|^2 + \text{const.} \\
=\ & \frac{1}{2}(\partial h)^2 - \frac{M^2}{2}h^2 - \sqrt{\frac{\lambda}{2}}Mh^3 - \frac{\lambda}{8}h^4 + \frac{1}{2}(v+h)^2\left|\frac{\partial\phi}{f}\right|^2 + \text{const.}
\end{aligned} \tag{339}$$

Again, the field ϕ has no mass or coupling and only appears as $(\partial\phi)$.
The Higgs field h has a mass M and a self coupling λ. However, in our calculation we have only made use of the finite combination $v = \sqrt{2}M/\sqrt{\lambda}$. As long as v stays finite we can take the combined limit $M \to \infty$ and $\lambda \to \infty$. This way, all terms proportional to $h^n (n = 2,3,4)$ become very large. In contrast, after Fourier-transforming we know that the kinetic term $(\partial h)^2$ will give contributions o f the order of the typical momentum or energy scale E we are probing in a given process. If we make M and with it $\sqrt{\lambda}$ much larger than that, $M \gg E$, we can neglect the kinetic term for the Higgs field in the Lagrangian $(\partial h)^2 L_L M^2 h^2$. Note that this inequality is not really mathematically correct, because for the kinetic term it refers to its size when evaluated for a given process. In that case, our Lagrangian becomes

$$\mathcal{L} \supset -\frac{M^2}{2}h^2 - \sqrt{\frac{\lambda}{2}}Mh^3 - \frac{\lambda}{8}h^4 + \frac{1}{2}(v+h)^2\left|\frac{\partial\phi}{f}\right|^2 \tag{340}$$

Because the Higgs field h does not propagate, we can use its Euler–Lagrange equation $\partial\mathcal{L}/\partial h = 0$ to compute its (constant) field value. If we neglect its appearance in the kinetic term of ϕ (with a prefactor of order vE^2/fL_LM) we see that there is no linear term in h in the Lagrangian, which means that $\partial\mathcal{L}/\partial h$ is proportional to h, so one solution is $h(x) = 0$. Our weak–scale Lagrangian becomes simply the kinetic term for a massless scalar field ϕ. To obtain the correct normalization of this kinetic term for $h = 0$ we need to fix $f^2 = v^2$:

$$\mathcal{L} \supset \frac{1}{2}\frac{v^2}{f^2}(\partial\phi)^2 = \frac{1}{2}(\partial\phi)^2 \tag{341}$$

Going into the limit $M \to \infty$ has one profound consequence for our theory. Usually we attempt to construct renormalizable Lagrangians, *i.e.* Lagrangians which describe physics to arbitrarily high scales. Such a construction ensures for example that any transition amplitude is bounded from above at all energy scales, so that our theory is unitary at all energy scales. Now, in the large-M limit we have explicitely required E/ML_L1, which means that we can still apply our theory to larger and larger energies, but not for a fixed value of M.

We have to make sure that $E/ML_L 1$ always applies. This is the typical condition for an effective field theory — it only produces sensible predictions at energy scales below a given cut–off scale M. Or in other words, our theory is not anymore renormalizable or unitary.
Such a model breaking a gauge symmetry like the chiral symmetry is called a non-linear σ model, because of the non-linear dependence of Φ on the one remaining physical field ϕ. The σ field is our Higgs field, which can be decoupled, while the remaining massless field ϕ is usually referred to as the π field.

Let us now study the Yukawa terms in this limit and see if they still give rise to fermion masses. The original field Φ simply becomes $\sqrt{2}\Phi = f\exp(i\phi/f)$ with one fixed energy scale $f = v$. The complete Lagrangian modulo the potential term becomes

$$\begin{aligned}
\mathcal{L} \supset & \, \overline{\psi}\, i\, \partial\!\!\!/ \psi + \frac{1}{2}(\partial\phi)^2 - \frac{gf}{\sqrt{2}} \left[\overline{\psi}_L \psi_R e^{+i\phi/f} + \overline{\psi}_R \psi_L e^{-i\phi/f} \right] \\
= & \, \overline{\psi}\, i\, \partial\!\!\!/ \psi + \frac{1}{2}(\partial\phi)^2 - \frac{gf}{\sqrt{2}} \left[\overline{\psi}_L \psi_R \left(1 + i\frac{\phi}{f} \right) + \overline{\psi}_R \psi_L \left(1 - i\frac{\phi}{f} \right) \right] + \mathcal{O}\left(\frac{1}{f^2} \right) \\
= & \, \overline{\psi}\, i\, \partial\!\!\!/ \psi + \frac{1}{2}(\partial\phi)^2 - \frac{gf}{\sqrt{2}}\, \overline{\psi}\psi - \frac{ig}{\sqrt{2}}\, \overline{\psi} \left(P_R^2 - P_L^2 \right) \psi\, \phi + \mathcal{O}\left(\frac{1}{f^2} \right) \\
= & \, \boxed{\overline{\psi}\, i\, \partial\!\!\!/ \psi + \frac{1}{2}(\partial\phi)^2 - \frac{gf}{\sqrt{2}}\, \overline{\psi}\psi - \frac{ig}{\sqrt{2}}\, \overline{\psi}\gamma_5 \psi\, \phi + \mathcal{O}\left(\frac{1}{f^2} \right)} \qquad \text{with} \qquad P_R - P_L = \gamma_5
\end{aligned} \tag{342}$$

In this form we can read off that ϕ is a massless pseudoscalar with a coupling strength $ig/\sqrt{2}$ which in terms of the fermion mass $m = fg/\sqrt{2}$ can be written as im/f. This relation between mass and pseudoscalar coupling is called Goldberger–Treiman relation. It can for example be verified in the case of the QCD pion's interaction in comparison to the nucleon masses.
This example of a non-linear sigma model illustrates how using a $SU(2)$ doublet scalar field we can give masses to fermions via Yukawa couplings. The chiral $SU(2)_L \times SU(2)_R$ symmetry is broken by the vacuum expectation value of the scalar field. Its radial excitations around the minimum we can decouple, while the massless scalar becomes a physical mode in our theory. On the other hand, we could of course use such a mode to give masses to gauge bosons, as seen before.

3. *Goldstone's theorem*

Those who know more about spontaneous symmetry breaking have noticed that using these two examples we have illustrated a few vital properties of Nambu–Goldstone bosons (NGB). Such massless physical states appear in many areas of physics and are described by Goldstone's theorem:
If a global symmetry group is spontaneously broken into a group of lower rank, its broken generators correspond to physical Goldstone modes. These fields transform non-linearly under the larger and linearly under the smaller group. They have to be massless, as the non-linear transformation only allows derivative terms in the Lagrangian.
If the spontaneous symmetry breaking induces gauge–boson masses, these massive degrees of freedom are 'eaten' Goldstone modes, and the mass is given by the vev breaking the larger symmetry. If the smaller symmetry is also broken, the NGBs become pseudo-NGB and acquire a mass of the size of this hard-breaking term.

For an alternative introduction into non–linear σ models and into Goldstone modes, you can have a look into the introduction of my little–Higgs lecture notes.

B. Technicolor

Technicolor is a way to break our electroweak symmetry and create masses for gauge bosons essentially using a non-linear sigma model, as we have seen it in the last section. In this example we have given the scalar field Φ a vacuum expectation value v through a potential, which is basically the Higgs mechanism. However, we know another way to break (chiral) symmetries through condensates — QCD. So let us review very few aspects of QCD which we will need later.

First, we should illustrate why an asymptotically free theory like QCD is a good model to explain electroweak symmetry breaking. For this we recall the main theoretical problem with the Higgs mechanism, *i.e.* spontaneous symmetry breaking with a fundamental scalar Higgs boson: If we think of our gauge theories as a stack of fundamental renormalizable field theories with some kind of cutoff scale (like for example the Planck scale) we can compute the quantum corrections to the Higgs mass with this cutoff. We find that the Higgs mass, and *only* the Higgs mass, corrections are quadratically divergent with the cutoff. This behavior is called the hierarchy problem between the electroweak scale v and for example the Planck mass. In other words, we introduce the Higgs boson to construct a renormalizable truly fundamental field theory perturbatively valid to all energies, and the Higgs mass itself spoils the high–energy behavior. The only easy way out is to tune the Higgs–mass counter term to cancel this cutoff dependence order by order, but this way we betray our original idea that small parameters in the Lagrangian cannot just occur, but need to be protected by some kind of symmetry. The alternative would be to postulate a UV completion of the Standard Model which cures this behavior and makes the complete theory consistent again. The most famous such completion is TeV–scale supersymmetry.

How can an interaction which becomes strong at small energies solve this problem — or why have we never heard of the hierarchy problem $\Lambda_{\text{QCD}} L_L M_{\text{Planck}}$? The inherent mass scale of QCD is $\Lambda_{\text{QCD}} \sim 200$ MeV. It describes the scale at which the running QCD coupling constant $\alpha_s = g_s^2/(4\pi)$ becomes strong, *i.e.* perturbation theory in α_s breaks down, and quarks and gluons stop being QCD's physical degrees of freedom. At the leading one-loop level we can easily see where Λ_{QCD} comes from. Summing all gluon self–energy bubbles for example in the s–channel of the process $q\bar{q} \to q'\bar{q}'$ corresponds to the definition of an effective coupling

$$\alpha_s \to \alpha_s \left(1 - \frac{\alpha_s}{4\pi}\beta \log\frac{p^2}{\mu_R^2}\right) \to \alpha_s \left(1 + \frac{\alpha_s}{4\pi}\beta \log\frac{p^2}{\mu_R^2}\right)^{-1} \equiv \alpha_s^{\text{eff}}(p^2) \tag{343}$$

where p^2 is the momentum flowing through the gluon propagator and μ_R is the (artifical) renormalization scale we are forced to introduce because we cannot write down a logarithm of a mass dimension. The form of the β function depends on the particle content of QCD, but not on the particle masses:

$$\beta = \frac{11}{3}N_c - \frac{2}{3}n_f > 0 \qquad \text{with} \qquad N_c = 3, \quad n_f = 5 \quad \text{(below the top threshold)} \tag{344}$$

This way, at large values of p^2 the denominator in parentheses becomes large and the effective running α_s becomes small, *i.e.* QCD is asymptotically free at large energies. We can relate the α_s values at two scales via

$$\frac{1}{\alpha_s(p^2)} = \frac{1}{\alpha_s(p_0^2)} \left(1 + \frac{\alpha_s(p_0^2)\beta}{4\pi} \log\frac{p^2}{p_0^2}\right) = \frac{1}{\alpha_s(p_0^2)} + \frac{\beta}{4\pi} \log\frac{p^2}{p_0^2} \stackrel{!}{=} \frac{\beta}{4\pi} \log\frac{p^2}{\Lambda_{\text{QCD}}^2} \tag{345}$$

and parameterize its energy behavior using one dimensionful parameter Λ_{QCD}. The functional form including Λ_{QCD} only reflects the general polynomial form of the one-loop running $\alpha_s^{-1}(p^2) = C_0 + C_1 \log p^2$. Practically, the value of Λ_{QCD} is extracted for example in a combined with with the parton densities. At leading order we can solve the above definition for Λ_{QCD}:

$$\frac{1}{\alpha_s(p_0^2)} = \frac{\beta}{4\pi}\log\frac{p_0^2}{\Lambda_{\text{QCD}}^2} \qquad \Leftrightarrow \qquad \log\frac{\Lambda_{\text{QCD}}^2}{p_0^2} = -\frac{4\pi}{\beta}\,\frac{1}{\alpha_s(p_0^2)} \qquad \Leftrightarrow \qquad \frac{\Lambda_{\text{QCD}}^2}{p_0^2} = \exp\left[-\frac{4\pi}{\beta}\,\frac{1}{\alpha_s(p_0^2)}\right] \tag{346}$$

This means that because QCD is not scale invariant, *i.e.* we have to introduce a renormalization scale in our perturbative expansion, the running of a dimensionless coupling constant can be translated into an inherent mass scale. This mass scale characterizes the theory, *e.g.* QCD, in the sense that $\alpha_s(p^2 = \Lambda_{\text{QCD}}^2) \sim 1$ and for scales below Λ_{QCD} the theory will become strongly interacting. Note that first of all this scale could not appear if for some reason $\beta \simeq 0$ and that it secondly does not depend on any mass scale in the theory. This phenomenon of a logarithmically running coupling introducing a mass scale in the theory is called dimensional transmutation.

It is the reason why there is no hierarchy problem between Λ_{QCD} and M_{Planck}: if at a high scale we start from a strong coupling in the $10^{-2} \cdots 10^{-1}$ range the QCD scale will arrive at its known value without any need for fine tuning.

Just including the quark doublets and the covariant derivative describing the qqg interaction the QCD Lagrangian reads

$$\mathcal{L}_{\text{QCD}} \supset \overline{\Psi}_L \, i \, \not{D} \Psi_L + \overline{\Psi}_R \, i \, \not{D} \Psi_R \tag{347}$$

We immediately see that it is symmetric under a chiral-type $SU(2)_L \times SU(2)_R$ symmetry. This symmetry forbids quark masses, *i.e.* it acts as a custodial symmetry for the tiny quark masses we measure for example for the valence quarks u, d. Because QCD is asymptotically free, at energies below roughly Λ_{QCD} the essentially massless quarks form condensates, *i.e.* two–quark operators will develop a vacuum expectation value $\langle \overline{\Psi} \Psi \rangle$. This operator spontaneously breaks the $SU(2)_L \times SU(2)_R$ symmetry into the (diagonal) $SU(2)$ of isospin. The valence quarks at low energies develop masses of the order of $m_{\text{nucleon}}/3 \sim \Lambda_{\text{QCD}}$, and the different composite color–singlet mesons and baryons become the relevant physical degrees of freedom. Their masses are of the order of the nucleon masses $m_{\text{nucleon}} \sim 1$ GeV.
The only remaining massless particles are the NGBs from the breaking of $SU(2)_L \times SU(2)_R$, the pions. Their masses are not strictly zero, because the valence quarks do have a small mass of a few MeV. Their coupling strength (or decay rate) is governed by f_π. It is defined by $\langle 0 | j^5_\mu | \pi \rangle = i f_\pi p_\mu$, *i.e.* it parameterizes the breaking of the chiral symmetry via breaking the axial-vector-like $U(1)_A$. The axial current can be computed as $j^5_\mu = \delta \mathcal{L} / \delta(\partial_\mu \pi)$ and in the $SU(2)$ basis reads $j^5_\mu = \overline{\psi} \gamma_\mu \tau \psi / 2$. From the measured decays of the light color–singlet QCD pion into two leptons we know that $f_\pi \sim 100$ MeV.

There are two QCD parameters which we need to adjust when building the simplest technicolor model: the size of the new gauge group and the scale at which the asymptotically free theory becomes strongly interacting. In terms of the two parameters N_c and Λ_{QCD} there are scaling rules in QCD which are based on for example $\beta \propto N_c$ (and which strictly speaking do not hold arbitrarily well):

$$f_\pi \sim \sqrt{N_c} \, \Lambda_{\text{QCD}} \qquad \langle \overline{Q} Q \rangle \sim N_c \, \Lambda^3_{\text{QCD}} \qquad m_{\text{fermion}} \sim \Lambda_{\text{QCD}} \tag{348}$$

The Λ_{QCD} dependence simply follows from the mass dimension. The dimension of the vev is given by the mass dimension 3/2 of each fermion field.
The N_c dependence of f_π can be easily guessed: the pion decay rate is by definition proportional to f_π^2. The Feynman diagrams for this decay are (apart from the strongly interacting complications, parameterized by the appearance of f_π) the same as for the Drell–Yan process $q\bar{q} \to \gamma, Z$. The color structure of this process leads to an explicit factor of $\delta_{ab}\delta_{ab} = N_c$ and an averaging factor of $1/N_c$ for each of the quarks. Together, this gives a factor $1/N_c$ for a color singlet decaying to a non-colored photon, the pion decay rate is proportional to f_π^2/N_c. This means the pion decay constant scales like $f_\pi \sim \sqrt{N_c}$. The vev–operator represents two quarks exchanging a gluon at energy scales small enough for α_s to become large. The color factor (without any averaging over initial states) simply sums over all colors states for the color-singlet condensate, *i.e.* it is proportional to N_c. The fermion masses have nothing to do with color states and hence should not depend on the number of colors. For details you should ask a lattice gauge theorist, but we already get the idea how would should construct our high–scale version of QCD, dubbed technicolor.

1. *Scaling up QCD*

Let us work out the idea that a mechanism just like QCD condensates could be the underlying theory of the non-linear σ model described in the introduction. in contrast to QCD we now have a gauged custodial symmetry of the gauge–boson masses. The longitudinal modes of the massive W and Z bosons are then the NGBs (techni-pions) of the symmetry breaking induced by a condensate. The corresponding mass scale would have to be $\Lambda_{\text{T}} \sim f \sim v = 246$ GeV. Fermion masses we postpone to the next section — in the 70s, when technicolor was developed, all known fermions had masses of the order of GeV or much less, so they were to a good approximation massless compared to the gauge bosons.

To induce W and Z masses we write down the non–linear sigma model in its $SU(2)$ version, at this point without talking about the source of the vacuum expectation value $f_{\rm T}$ appearing in

$$\Phi = \frac{1}{\sqrt{2}}\, e^{i(\pi\cdot\tau)/f_{\rm T}} \begin{pmatrix} f_{\rm T} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} f_{\rm T} + i(\pi\cdot\tau) + \mathcal{O}(f_{\rm T}^{-1}) \\ 0 \end{pmatrix} \tag{349}$$

As basis vectors we use the three Pauli matrices $\{\tau_j,\tau_k\} = 2\delta_{jk}$. We will in a second need their property

$$\left(\sum_j \tau_j\right)\left(\sum_k \tau_k\right) = \sum_{j<k} (\tau_j\tau_k + \tau_k\tau_j) + \sum_j \tau_j^2 = 3\,\mathbb{1} \qquad \Rightarrow \qquad (\tau\cdot\pi_1)\,(\tau\cdot\pi_2) = \sum_j \pi_{1,j}\pi_{2,j} = (\pi_1\cdot\pi_2) \tag{350}$$

The $SU(2)$-covariant derivative in the charge basis of the Pauli matrices

$$(\tau\cdot\pi) \equiv \sum_{(+,-,3)} \tau_j\pi_j = \frac{\tau^1+i\tau^2}{\sqrt{2}}\,\frac{\pi^1-i\pi^2}{\sqrt{2}} + \frac{\tau^1-i\tau^2}{\sqrt{2}}\,\frac{\pi^1+i\pi^2}{\sqrt{2}} + \tau^3\pi^3 = \sum_{(1,2,3)} \tau_j\pi_j \tag{351}$$

gives, when to simplify the formulas we for a moment forget about the $U(1)_Y$ contribution and only keep the non-zero upper entry:

$$\begin{aligned}
iD^\mu\Phi &= \left[i\partial^\mu - \frac{g_2}{2}\,(\tau\cdot W^\mu)\right]\,\frac{1}{\sqrt{2}}\,\left[f_{\rm T} + i(\tau\cdot\pi) + \mathcal{O}(f_{\rm T}^{-1})\right] \\
&= \frac{1}{\sqrt{2}}\,\left[-\partial^\mu(\tau\pi) - \frac{f_{\rm T}g_2}{2}(\tau\cdot W^\mu)\right] \\
(D_\mu\Phi)^\dagger D^\mu\Phi &= \frac{1}{2}\,\left[-\partial_\mu(\tau\pi) - \frac{f_{\rm T}g_2}{2}(\tau\cdot W_\mu)\right]\,\left[-\partial^\mu(\tau\pi) - \frac{f_{\rm T}g_2}{2}(\tau\cdot W^\mu)\right] \\
&\supset \frac{1}{2}(\partial\pi)^2 + \frac{f_{\rm T}g_2}{2}\,(W_\mu\cdot(\partial^\mu\pi))
\end{aligned} \tag{352}$$

If we also include the generator of the hypercharge $U(1)$ we find a mixing term between the techni-pions and the $SU(2)$ gauge bosons

$$\boxed{\mathcal{L} \supset \frac{g_2 f_{\rm T}}{2}\,W_\mu^+\partial^\mu\pi^- + \frac{g_2 f_{\rm T}}{2}\,W_\mu^-\partial^\mu\pi^+ + f_{\rm T}\,\left(\frac{g_2}{2}\,W_\mu^0 + \frac{g_1}{2}\,B_\mu\right)\,\partial^\mu\pi^0} \tag{353}$$

This is precisely the mixing term from the massive–photon example which we need to absorb the NGBs into the massive vector bosons, with $f_{\rm T} = v$ from the known W and Z masses. We have strictly speaking not shown that the $f_{\rm T}$ appearing in the scalar field Φ is really the correctly normalized $f_{\rm T}$, defined as the decay constant of the techni-pions (and there is a lot of confusion about factors $\sqrt{2}$ in the literature which I will ignore in this sketchy argument). But if we assume this correct normalization then $f_{\rm T} \equiv v$ is the scaled-up version of f_π we see that technicolor is something like a scaled-up version of QCD by a factor $v/\Lambda_{\rm QCD} \sim 2000$.

This scaling factor we better compute in the more general case, where technicolor involves a gauge group $SU(N_T)$ instead of $SU(N_c)$ and N_D left-handed fermion doublets in the fundamental representation of $SU(N_T)$. To be able to write down Dirac masses for the fermions at the end of the day we also need $(2N_D)$ right-handed fermion singlets. If instead of one set of techni-pions we have N_D of them, we remember that the W, Z masses arise from the quadratic term associated with the techni-pion mixing above, proportional to g^2v^2. In the sum, the N_D techni-pions need to reproduce the measured mass squares, which means that the vacuum expectation value scales like $v \sim \sqrt{N_D} f_{\rm T}$. The known scaling rules then give:

$$f_{\rm T} \sim \sqrt{\frac{N_T}{N_c}}\,\frac{\Lambda_{\rm T}}{\Lambda_{\rm QCD}}\,f_\pi \qquad\qquad v = \sqrt{N_D} f_{\rm T} \sim \sqrt{\frac{N_D N_T}{N_c}}\,f_\pi \tag{354}$$

We can solve these scaling rules for the unknown technicolor parameters and obtain:

$$f_{\rm T} \sim \frac{v}{\sqrt{N_D}} \qquad\qquad \Lambda_{\rm T} \sim \Lambda_{\rm QCD}\,\frac{f_{\rm T}}{f_\pi}\,\sqrt{\frac{N_c}{N_T}} \sim v\,\frac{\Lambda_{\rm QCD}}{f_\pi}\,\sqrt{\frac{N_c}{N_D N_T}} \qquad \text{with} \quad v = 246~\text{GeV} \tag{355}$$

One simple example for such a technicolor model is the Susskind–Weinberg model. Its gauge group is $SU(N_T) \times SU(3)_c \times SU(2)_L \times U(1)_Y$. The matter fields forming the condensate which in turn breaks the electroweak symmetry we include one doublet ($N_D = 1$) of charged color-singlet techni-fermions $(u^T, d^T)_{L,R}$. In some ways this doublet and the two singlets look like a fourth generation of chiral fermions, but with different charges under all Standard–Model gauge groups: for example, their hypercharges Y need to be chosen such that gauge anomalies do not occur and we do not have to worry about non-perturbatively breaking any symmetries, namely $Y = 0$ for the left–handed doublet and $Y = 1/2, -1/2$ for u_R^T and d_R^T. The formula $Q = I_3 + Y/2$ then gives charges of $\pm 1/2$ to the heavy states u^T and d^T.
The additional $SU(N_T)$ gauge group gives us a running gauge coupling which becomes large at the scale Λ_{T}. As a high-scale boundary condition we can for example choose $\alpha_s(M_{\text{GUT}}) = \alpha_T(M_{\text{GUT}})$. The beta function is modelled after the QCD case

$$\beta_{\text{QCD}} = \frac{11}{3} N_c - \frac{2}{3} n_f \qquad\qquad \beta_{\text{T}} = \frac{11}{3} N_T - \frac{4}{3} N_D \tag{356}$$

keeping in mind that N_D counts the doublets, while $n_f = 6$ counts the number of flavors at the GUT scale. This relation holds for a simple model, where quarks are only charged under $SU(3)_c$ and techniquarks are only charged under $SU(N_T)$. Of course, both of them can carry weak charges. Using the one-loop formula for Λ_{QCD} we can compute

$$\begin{aligned}\frac{\Lambda_{\text{T}}^2}{\Lambda_{\text{QCD}}^2} &= \exp\left[-\frac{4\pi}{\beta_{\text{QCD}}}\,\frac{1}{\alpha_s(m_{\text{GUT}})}\right]\,\exp\left[+\frac{4\pi}{\beta_{\text{T}}}\,\frac{1}{\alpha_T(m_{\text{GUT}})}\right] \\ &= \exp\left[\frac{4\pi}{\alpha_s(m_{\text{GUT}})}\left(\frac{1}{\beta_{\text{T}}} - \frac{1}{\beta_{\text{QCD}}}\right)\right] = \exp\left[\frac{4\pi}{\alpha_s(m_{\text{GUT}})}\,\frac{\beta_{\text{QCD}} - \beta_{\text{T}}}{\beta_{\text{T}}\beta_{\text{QCD}}}\right]\end{aligned} \tag{357}$$

For $N_T = N_D = 4$ and $\alpha_s(M_{\text{GUT}}) \sim 1/30$ we find $\Lambda_{\text{T}} \sim 800\ \Lambda_{\text{QCD}} \sim 165$ GeV. This gives a reasonable $v = 270$ GeV and generates the required hierarchy between v and M_{GUT} via dimensional transmutation.

At this stage, our fermion construction has two global chiral symmetries $SU(2) \times SU(2)$ and $U(1) \times U(1)$ protecting the techni-fermions from getting massive, which we will of course break together with the local weak $SU(2)_L \times U(1)_Y$ symmetry. Details about fermion masses we postpone to the next sections. Let us instead briefly look at the spectrum of our minimal model:
techniquarks — From the scaling rules we know that the techniquark masses will be of the order Λ_{T} as give above. Numerically, the factor $\Lambda_{\text{T}}/\Lambda_{\text{QCD}} \sim 800$ pushes the usual quark constituent mass to around 700 GeV for the minimal model with $N_T = 4$ and $N_D = 1$. Because of the $SU(N_T)$ gauge symmetry there should exist four–techniquark bound states (technibaryons) which are stable due to the asymptotic freedom of the $SU(N_T)$ symmetry. Those are not preferred by standard cosmology, so we should find ways to let them decay.
NGBs — Of course, from the breaking of the global chiral $SU(2) \times SU(2)$ and the $U(1) \times U(1)$ we will have four Goldstone modes. The three $SU(2)$ Goldstones are massless technipions, following our QCD analogy. Because we gauge the remaining Standard–Model subgroup $SU(2)_L$, they become the longitudinal polarizations of the W and Z boson, after all this is the entire idea behind this construction. The remaining $U(1)$ NGB also has an equivalent (η') in QCD, and its technicolor counter part acquires a mass though non–perturbative instanton breaking. Its mass can be estimates to ~ 2 TeV, so we are out of trouble.
more stuff — Just like in QCD we will have a whole zoo of additional technicolor vector mesons and heavy resonances, but all we need to know about them is that they are heavy (and therefore not a problem for example for cosmology) and that at this stage we should really move on and think about fermion masses...

2. Fermion masses: ETC

Before we move on, let us put ourselves into the shoes of the technicolor proponents in the 70s. They knew how QCD gives masses to protons, and the Higgs mechanism had nothing to do with it. Just copying this appealing idea of dimensional transmutation (without any hierarchy problem) once more they explained the measured W and Z masses. And just like in QCD, the masses of the four light quarks and the leptons are well below a

GeV and could be anything, but not linked to weak–scale physics. And then people found the massive bottom quark and the even more massive top quark and it became clear that at least the top mass was very relevant to the weak scale. In this section we will very briefly discuss how this challenge to technicolor basically removed it from the list of models people take seriously — until extra dimensions came and brought it back...

Extended technicolor is a version of the original idea of technicolor which attempts to solve two problems: create fermion masses for three generations of quarks and leptons and let the heavy techniquarks decay, to avoid stable technibaryons. From the introduction we in principle know how to obtain a fermion mass from Yukawa couplings, but to write down the Yukawa coupling to the sigma field or to the TC condensate we need to write down some Standard–Model and technifermion operators. This is what ETC offers a framework for.

First, we need to introduce some kind of multiplets of matter fermions. Just as before, the techniquarks, like all matter particles have $SU(2)_L$ and $U(1)_Y$ or even $SU(2)_R$ quantum numbers. However, there is no reason for them all to have a $SU(3)_c$ charge, because we would prefer not to change β_{QCD} too much. Similarly, the Standard–Model particles do not have a $SU(N_T)$ charge. This means we should write matter multiplets with explicitely assigned color and technicolor charges. This means:

$$\left(Q^T_{a=1..N_T},\ Q^{(1)}_{j=1,...,N_c},\ Q^{(2)}_{j=1,...,N_c},\ Q^{(3)}_{j=1,...,N_c},\ L^{(1)},\ L^{(2)},\ L^{(3)} \right) \tag{358}$$

These multiplets replace the usual $SU(2)_L$ and $SU(2)_R$ singlets and doublets in the Standard Model. The upper indices denote the generation, the lower indices count the N_T and N_c fundamental representations. In the minimal model $N_T = 4$ this multiplet has $4+3+3+3+1+1+1 = 16$ entries. In other words, we have embedded $SU(N_T)$ and $SU(N_c)$ in a local gauge group $SU(16)$. If without further discussion we also extend the Standard–Model group by a $SU(2)_R$ gauge group, the complete ETC symmetry group is $SU(16) \times SU(2)_L \times SU(2)_R$, where we omit the additional $U(1)_{B-L}$ throughout the discussion.
A technicolor condensate will now break $SU(2)_L \times SU(2)_R$, while leaving $SU(3)_c$ untouched. If we think of the generators of the ETC gauge group as (16×16) matrices we can put a (4×4) block of $SU(N_T)$ in the upper left corner and then three (3×3) copies of $SU(N_c)$ on the diagonal. The last three rows/columns can be the unit matrix. Once we break $SU(16)_{\text{ETC}}$ to $SU(N_T)$ and the Standard–Model gauge groups, the NGBs corresponding to the broken generators obtain masses of the order of Λ_{ETC}. This breaking should on the way produce the correct fermion masses. The remaining $SU(N_T) \times SU(2)_L \times U(1)_Y$ will then break the electroweak symmetry through a $SU(N_T)$ condensate and create the measured W and Z masses as described in the last section.

In this construction we will have ETC gauge bosons which for example in the quark sector couple $(\overline{Q^T}\gamma_\mu T_{\text{ETC}} Q^T)$, $(\overline{Q^T}\gamma_\mu T_{\text{ETC}} Q)$ and $(\overline{Q}\gamma_\mu T_{\text{ETC}} Q)$ currents. Here, T_{ETC} stands for the $SU(16)_{\text{ETC}}$ generators. The multiplets Q^T and Q replace the $SU(2)_{L,R}$ singlet and doublets, which means the T_{ETC} include for example the chiral projectors. Below the the ETC breaking scale Λ_{ETC} these currents become four–fermion interactions, just like a Fermi interaction in the electroweak theory:

$$\frac{(\overline{Q^T}\gamma_\mu T^a_{\text{ETC}} Q^T)\,(\overline{Q^T}\gamma^\mu T^b_{\text{ETC}} Q^T)}{\Lambda^2_{\text{ETC}}} \qquad \frac{(\overline{Q^T}\gamma_\mu T^a_{\text{ETC}} Q)\,(\overline{Q}\gamma^\mu T^b_{\text{ETC}} Q^T)}{\Lambda^2_{\text{ETC}}} \qquad \frac{(\overline{Q}\gamma_\mu T^a_{\text{ETC}} Q)\,(\overline{Q}\gamma^\mu T^b_{\text{ETC}} Q)}{\Lambda^2_{\text{ETC}}} \tag{359}$$

The mass scale in this effective theory can be linked to the mass of the ETC gauge bosons and their gauge coupling and should be of the order $1/\Lambda_{\text{ETC}} \sim g_{\text{ETC}}/M_{\text{ETC}}$. Let us see what these kind of interactions predict at energy scales below Λ_{ETC}, which means somewhere around the weak scale, where we have data. Because currents are much harder to interpret, we first Fierz–rearrange these operators and then pick out three relevant classes of scalar operators.

Maybe at this stage I should very briefly repeat without proof what a Fierz transformation is. We start from scalar operators based on spinors in a Lagrangian. The complete set is defined schematically written as:

$$\mathcal{L} \supset \left(\overline{\psi} A_j \psi\right)\ \left(\overline{\psi} A^j \psi\right) \qquad \text{with} \qquad A_j = \mathbb{1},\ \gamma_5,\ \gamma_\mu,\ \gamma_5\gamma_\mu,\ \sigma_{\mu\nu} \tag{360}$$

The multi-index j implies summing over all open indices in the diagonal combination $A_j A^j$. These five types of (4×4) matrices form a basis of all real (4×4) matrices which can occur in the Lagrangian. Note that in the

equation above we have not specified anything about the spinors. If they carry charges, the $\overline{\psi}$ and the ψ have to cancel in the entire term, but of course not inside each current, *i.e.* there is more than one scalar operator of this type with a given set of spinors.
If we now specify the spinors and exchange them in one of the terms in the Lagrangian, we should be able to write the new (1,4,3,2) scalar term (or any new scalar term, for that matter) as a linear combination of the scalar basis elements (1,2,3,4):

$$\left(\overline{\psi}_1 A_i \psi_4\right) \, \left(\overline{\psi}_3 A_i \psi_2\right) = \sum_j C_{ij} \left(\overline{\psi}_1 A_j \psi_2\right) \, \left(\overline{\psi}_3 A_j \psi_4\right) \tag{361}$$

Note that in this notation we have ignored the normal-ordering of the spinors in the Lagrangian. It is easy to show $C \cdot C = \mathbb{1}$. All we need to know is the value of the coefficients C_{ij}, which I will list for completeness reasons, but without using them at all in the technicolor context:

	$\mathbb{1}$	γ_5	γ_μ	$\gamma_5\gamma_\mu$	$\sigma_{\mu\nu}$
$\mathbb{1}$	$-1/4$	$-1/4$	$-1/4$	$1/4$	$-1/8$
γ_5	$-1/4$	$-1/4$	$1/4$	$-1/4$	$-1/8$
γ_μ	-1	1	$1/2$	$1/2$	0
$\gamma_5\gamma_\mu$	1	-1	$1/2$	$1/2$	0
$\sigma_{\mu\nu}$	-3	-3	$1/2$	0	$1/2$

(362)

Applying this transformation to the three quark–techniquark four–fermion operators listed above we certainly obtain scalar ($A = \mathbb{1}$) operators by Fierz–transforming the three current ($A = \gamma_\mu$) operators listed above. Because we are model builders, these are the only operators we will discuss in this context, and which will give us all the information we need:

$$\boxed{\frac{(\overline{Q^T}\, T^a_{\rm ETC}\, Q^T)\; (\overline{Q^T}\, T^b_{\rm ETC}\, Q^T)}{\Lambda^2_{\rm ETC}}} \qquad \boxed{\frac{(\overline{Q^T_L}\, T^a_{\rm ETC}\, Q^T_R)\; (\overline{Q_R}\, T^b_{\rm ETC}\, Q_L)}{\Lambda^2_{\rm ETC}}} \qquad \boxed{\frac{(\overline{Q_L}\, T^a_{\rm ETC}\, Q_R)\; (\overline{Q_R}\, T^b_{\rm ETC}\, Q_L)}{\Lambda^2_{\rm ETC}}} \tag{363}$$

Note that we have now picked certain chiralities of the Standard Model fields and the technifermions. Let us go through these operators once after the other in the following section.

3. *Killing technicolor*

From the title of this part it is fairly obvious that not all of the operators listed above will be our friends. On the other hand, we need them to give masses to the Standard–Model fermions, which means we have to live with their additional constraints:

(1) Once technicolor becomes strongly interacting and forms condensates of the kind $\left\langle \overline{Q^T} Q^T \right\rangle \propto \Lambda^3_{\rm T}$ the first operator will lead to masses for all TC generators which do not commute with the (broken) ETC generators. Without going into the details we know from the scalar operators that these masses have to be proportional to $1/\Lambda_{\rm ETC}$. The TC condensate will be proportional to N_T, which means that by dimensional analysis these masses will be $m \sim N_T \Lambda^2_{\rm T}/\Lambda_{\rm ETC}$. This mechanism will be very useful once we go beyond the minimal $N_T = 4, N_D = 1$ structure of technicolor, which predicts massless pseudoscalar NGBs which do not get eaten by the weak gauge bosons, so-called techni-axions. ETC has a mechanism to give these particles a mass of the order $\Lambda_{\rm T}$. So the first scalar operator is our friend.

(2) Condensating the techniquarks in the second operator will according to the QCD scaling rules give us fermion mass terms of the kind

$$\mathcal{L} \supset \frac{N_T \Lambda^3_{\rm T}}{\Lambda^2_{\rm ETC}}\, \overline{Q_L} q_R \equiv m_Q\, \overline{Q_L} q_R \qquad \Leftrightarrow \qquad \Lambda_{\rm ETC} \sim \sqrt{\frac{N_T \Lambda^3_{\rm T}}{m_Q}} \sim \begin{cases} 2 \text{ TeV} & m_Q = 1 \text{ GeV} \\ 200 \text{ GeV} & m_Q = 100 \text{ GeV} \end{cases} \tag{364}$$

for $N_T = 4$ and $\Lambda_{\rm T} = 100$ GeV. Remember that Dirac mass terms involve a left–right mixing, which means that they form an $SU(2)$ doublet, which in turn means that gauge invariance forces us to couple them to a techniquark doublet as well. From the numbers above we see that this operator appears to be our friend for light quarks, but it becomes problematic for the top quark, where $\Lambda_{\rm ETC}$ needs to be probably too low for current constraints.
Moreover, the operator responsible for the top mass can be fierzed into a fermion–technifermion current which can occur for either chirality

$$\left(\overline{Q_L^T} Q_R^T\right)\ \left(\overline{Q_R} Q_L\right) \quad \rightarrow \quad \left(\overline{Q_L^T} \gamma_\mu Q_L\right)\ \left(\overline{Q_R} \gamma^\mu Q_R^T\right) \quad \text{and} \quad \left(\overline{Q_L^T} \gamma_\mu Q_L\right)\ \left(\overline{Q_L} \gamma^\mu Q_L^T\right) \tag{365}$$

where we omitted the prefactor $g_{\rm ETC}^2/M_{\rm ETC}^2$. Of course, until now we have identified the right–handed Standard Model field with the right-handed top singlet. But because of the $SU(2)_R$ symmetry which as we will see later it necessary to avoid electroweak precision data as a custodial symmetry, we can rotate this $t_{L,R}$ into a $b_{L,R}$. So the operator we are looking at it of the kind

$$\frac{g_{\rm ETC}^2}{M_{\rm ETC}^2} \left(\overline{Q_L^T} \gamma_\mu b_L\right)\ \left(\overline{Q_L} \gamma^\mu Q_L^T\right) \tag{366}$$

where the techniquarks carry the index T. This operator induces a coupling of a charged ETC gauge boson to $T_L b_L$. Such a diagram contributes to the decay $Z \to b\bar{b}$, where the two outgoing b quarks exchange a heavy charged ETC gauge boson and this propagator is pinched after integrating out the ETC gauge bosons. It contributes to the effective bbZ coupling

$$g_L = \frac{e}{s_w c_w}\left(-\frac{1}{2} + \frac{s_w^2}{3}\right) \rightarrow g_L - \frac{\xi^2}{4}\frac{\Lambda_{\rm T}^2}{\Lambda_{\rm ETC}^2}\frac{e}{s_w c_w} = g_L - \frac{\xi^2}{4}\frac{m_t}{N_T \Lambda_{\rm T}}\frac{e}{s_w c_w} \tag{367}$$

The angle ξ describes a possible mixing between the W and the ETC gauge boson. Unless we find a good argument why the different gauge boson cannot mix at all, this contribution will be considerably too big for the LEP measurement of $R_b = \Gamma_Z(b\bar{b})/\Gamma_Z(\text{hadrons})$. Note that this constraint from B decay will affect any theory which induces a top mass through a partner of the top quark and allows for a general set of (fierzed) operators corresponding to this mass term, not just extended technicolor.
The way out of these problem with $1/M_{\rm ETC}$ operators we can read off the formula: we need to increase $\Lambda_{\rm ETC}$ while at the same time still getting the correct m_t. This can be achieved by so-called walking technicolor, which we will not discuss here, though.

(3) The third operator on the list does not include any techniquarks, but all combinations of four–fermion couplings of light quarks. In the Standard Model such operators are very strongly limited, in particular when they involve different flavors of quarks. Typical operators of this form which are strongly constrained are

$$\frac{1}{\Lambda_{\rm ETC}^2}\left(\bar{s}\gamma^\mu d\right)\ \left(\bar{s}\gamma_\mu d\right) \qquad\qquad \frac{1}{\Lambda_{\rm ETC}^2}\left(\bar{\mu}\gamma^\mu e\right)\ \left(\bar{e}\gamma_\mu \mu\right) \tag{368}$$

They are examples for flavor–changing neutral currents, *i.e.* couplings of a neutral gauge boson to two different fermion generations. Note that if we only allow two different generations in any of the operators, Fierz transformations will distribute them into all other operators. The currently strongest constraints come from kaon physics, for example the mass splitting between the K^0 and the $\overline{K}^0$. Its limit $\Delta M_K \lesssim 3.5 \cdot 10^{-12}$ MeV implies $M_{\rm ETC}/(g_{\rm ETC}\theta_{sd}) \gtrsim 600$ TeV in terms of the Cabibbo angle θ_{sd}. We can translate such a lower bounds on $\Lambda_{\rm ETC}$ into an upper bound on fermion masses we can construct in our minimal model. $\Lambda_{\rm ETC} > 10^3$ TeV simply translates in a maximum fermion mass which we can explain in this model: $m \lesssim 4$ MeV for $\Lambda_{\rm T} \lesssim 1$ TeV. This is obviously not good news.

The last problem ETC runs into has to do with electroweak precision data, namely the two parameters S and T. While I will probably not be able to cover this in the lecture, let met briefly sketch a really nice introduction into electroweak precision observables from Csaba Csaki's lecture which I believe he found in an article by Cliff Burgess.

If we allow for deviations from the Standard–Model gauge sector, but limit ourselves to only dimension–four operators in the Lagrangian we can write down the additional terms

$$\mathcal{L} \supset -\frac{\Pi'_{\gamma\gamma}}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - \frac{\Pi'_{WW}}{2}\hat{W}_{\mu\nu}\hat{W}^{\mu\nu} - \frac{\Pi'_{ZZ}}{4}\hat{Z}_{\mu\nu}\hat{Z}^{\mu\nu} - \frac{\Pi'_{\gamma Z}}{4}\hat{F}_{\mu\nu}\hat{Z}^{\mu\nu} - \Pi_{WW}\,\hat{m}_W^2\hat{W}^+_\mu\hat{W}^{-\mu} - \frac{\Pi_{ZZ}}{2}\,\hat{m}_Z^2\hat{Z}^+_\mu\hat{Z}^{-\mu} \tag{369}$$

The field strengths $\hat{F}_{\mu\nu}, \hat{W}_{\mu\nu}, \hat{Z}_{\mu\nu}$ correspond to the photon and the W and Z gauge bosons, *i.e.* the fields $\hat{A}_\mu, \hat{W}_\mu, \hat{Z}_\mu$. The hats on the field are necessary, because these kinetic terms and therefore the fields do not (yet) have the canonical normalization. If we assume that the parameters $\Pi'_{\gamma\gamma}, \Pi'_{WW}, \Pi'_{ZZ}$ and $\Pi'_{\gamma Z}$ are small, we can express the hatted gauge–boson fields in terms of the properly normalized fields as

$$\hat{A}_\mu = \left(1 - \frac{\Pi'_{\gamma\gamma}}{2}\right) A_\mu + \Pi'_{\gamma Z} Z_\mu \qquad \hat{W}_\mu = \left(1 - \frac{\Pi'_{WW}}{2}\right) W_\mu \qquad \hat{Z}_\mu = \left(1 - \frac{\Pi'_{ZZ}}{2}\right) Z_\mu \tag{370}$$

which means for example for the terms proportional to $\Pi'_{\gamma Z}$:

$$\begin{aligned}
-\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\Big|_{\gamma Z} &= -\frac{1}{4}\left(\partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu\right)\left(\partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu\right)\Big|_{\gamma Z} \\
&= -\frac{1}{4}\left(\partial_\mu(A + \Pi'_{\gamma Z}Z)_\nu - \partial_\nu(A + \Pi'_{\gamma Z}Z)_\mu\right)\left(\partial_\mu(A + \Pi'_{\gamma Z}Z)_\nu - \partial_\nu(A + \Pi'_{\gamma Z}Z)_\mu\right)\Big|_{\gamma Z} \\
&= -\frac{\Pi'_{\gamma Z}}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial_\mu Z_\nu - \partial_\nu Z_\mu\right) - \frac{\Pi'_{\gamma Z}}{4}\left(\partial_\mu Z_\nu - \partial_\nu Z_\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + \mathcal{O}(\Pi'^2) \\
&= -\frac{\Pi'_{\gamma Z}}{2}\left(\partial_\mu Z_\nu - \partial_\nu Z_\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + \mathcal{O}(\Pi'^2) \\
&= -\frac{\Pi'_{\gamma Z}}{2} Z_{\mu\nu}F^{\mu\nu} + \mathcal{O}(\Pi' x^2) = -\frac{\Pi'_{\gamma Z}}{2}\hat{Z}_{\mu\nu}\hat{F}^{\mu\nu} + \mathcal{O}(\Pi'^2)
\end{aligned} \tag{371}$$

So the two contributions to $Z - \gamma$ mixing indeed cancel each other. This brings the kinetic terms in the Lagrangian given above into the canonical form

$$\mathcal{L} \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - (1 + \Pi_{WW} - \Pi'_{WW})\,\hat{m}_W^2 W^+_\mu W^{-\mu} - \frac{1}{2}(1 + \Pi_{ZZ} + \Pi'_{ZZ})\,\hat{m}_Z^2 Z^+_\mu Z^{-\mu} \tag{372}$$

The Z mass is given in terms of the additional small parameters $m_Z^2 = (1 + \Pi_{ZZ} + \Pi'_{ZZ})\,\hat{m}_Z^2$. Just as in the usual Lagrangian we can link the two gauge–boson masses through the (hatted) weak mixing angle $\hat{m}_W = \hat{c}_w\hat{m}_Z$, and in terms of this mixing angle we can compute the muon decay constant. The relation we obtain is:

$$\hat{s}_w^2 = s_w^2\left[1 + \frac{c_w^2}{c_w^2 - s_w^2}\left(\Pi'_{\gamma\gamma} - \Pi'_{ZZ} - \Pi_{WW} + \Pi_{ZZ}\right)\right] \tag{373}$$

With all these corrections the W–mass term in the Lagrangian reads

$$\begin{aligned}
\mathcal{L} \supset &-(1 + \Pi_{WW} - \Pi'_{WW})\,\hat{m}_W^2\, W^+_\mu W^{-\mu} = -(1 + \Pi_{WW} - \Pi'_{WW})\,\hat{c}_w^2\,\hat{m}_Z^2\, W^+_\mu W^{-\mu} \\
&= -(1 + \Pi_{WW} - \Pi'_{WW})\left[1 - \frac{s_w^2}{c_w^2 - s_w^2}\left(\Pi'_{\gamma\gamma} - \Pi'_{ZZ} - \Pi_{WW} + \Pi_{ZZ}\right)\right] c_w^2\,(1 - \Pi_{ZZ} + \Pi'_{ZZ})\, m_Z^2\, W^+_\mu W^{-\mu} \\
&= \left[1 - \Pi'_{WW} + \Pi'_{ZZ} + \Pi_{WW} - \Pi_{ZZ} - \frac{s_w^2}{c_w^2 - s_w^2}\left(\Pi'_{\gamma\gamma} - \Pi'_{ZZ} - \Pi_{WW} + \Pi_{ZZ}\right)\right] m_Z^2\, W^+_\mu W^{-\mu} \\
&= \left[1 - \frac{\alpha S}{2(c_w^2 - s_w^2)} + \frac{c_w^2\alpha T}{c_w^2 - s_w^2} + \frac{\alpha U}{4 s_w^2}\right] m_Z^2\, W^+_\mu W^{-\mu}
\end{aligned} \tag{374}$$

In the last step we have defined three typical combinations of the different correction factors as

$$\begin{aligned}
\alpha S &= 4 s_w^2 c_w^2\left(-\Pi'_{\gamma\gamma} + \Pi'_{ZZ} - \Pi'_{\gamma Z}\frac{c_w^2 - s_w^2}{c_w s_w}\right) \\
\alpha T &= \Pi_{WW} - \Pi_{ZZ} \\
\alpha U &= 4 s_w^4\left(\Pi'_{\gamma\gamma} - \frac{\Pi'_{WW}}{s_w^2} + \Pi'_{ZZ}\frac{c_w^2}{s_w^2} - 2\Pi'_{\gamma Z}\frac{2c_w}{s_w}\right)
\end{aligned} \tag{375}$$

These three so-called Peskin–Takeuchi can be understood fairly easily: the S parameter corresponds to a shift of the Z mass. This is not quite as obvious because it seems to also involve anomalous terms involving the photon's kinetic term, but we have to remember that the weak mixing angle is defined such that the photon is massless (*i.e.* corresponds to the unbroken $U(1)_Q$), while all mass terms are absorbed in the Z boson. The T parameter obviously compares contributions to the W and Z masses. Since the custodial $SU(2)$ symmetry precisely protects this mass ration, usually referred to as $\rho = 1$, the T parameter measures the amount of custodial symmetry violation. To get an idea how additional fermions contribute to S and T I just quote the contributions from the heavy fermion doublet:

$$\Delta S = \frac{N_c}{6\pi} \left(1 - 2Y \log \frac{m_t^2}{m_b^2} \right) \qquad \Delta T = \frac{N_c}{4\pi s_w^2 c_w^2 m_Z^2} \left(m_t^2 + m_b^2 - \frac{2 m_t^2 m_b^2}{m_t^2 - m_b^2} \log \frac{m_t^2}{m_b^2} \right)$$

$$\begin{aligned} \Delta\rho &= \frac{N_c G_F}{8\sqrt{2}\pi^2} \left(m_t^2 + m_b^2 - \frac{2 m_t^2 m_b^2}{m_t^2 - m_b^2} \log \frac{m_t^2}{m_b^2} \right) \\ &= \frac{N_c}{8\sqrt{2}\pi^2} \, \frac{\sqrt{2} e^2}{8 s_w^2 c_w^2 m_Z^2} \left(2 m_b^2 + \delta - \frac{2(m_b^2 + \delta) m_b^2}{\delta} \log \left(1 + \frac{\delta}{m_b^2} \right) \right) \qquad m_t^2 = m \\ &= \frac{N_c}{4\pi s_w^2 c_w^2 m_Z^2} \, \frac{e^2}{16\pi} \, \left(1 + \mathcal{O}(\delta^2) \right) \end{aligned} \tag{376}$$

Where $Y = 1/6$ for Standard–Model quarks and $Y = -1/2$ for Standard–Model leptons. The ρ parameter is defined in terms of the W and Z masses and is one at tree level

$$\rho = \frac{m_W^2}{c_w^2 m_Z^2} = 1 \tag{377}$$

One of the main differences between ρ and T is the reference point, where $\rho = 1$ refers to its tree-level value and $T = 0$ is often chosen for some kind of light Higgs mass and including the Standard–Model top-bottom corrections. For a slightly longer discussion of such contributions to the ρ parameter or ΔT, just have a look into my little–Higgs notes.

Let us now get to the constraints on technicolor models from the very strongly constrained S, T plane. The central point in this plane $S = T = 0$ is somewhat conventional, because the Standard Model predicts for example two sources for finite T: the Higgs boson itself as well as the mass splitting between up-type and down-type quarks (like the bottom and top quarks). Moreover, the electroweak precision constraints typically form a diagonal ellipse in the $S - T$ plane. But unless we can rely on a clear correlations, we can assume that models which to not predict $-0.15 < \Delta S < 0.25$ and $-0.1 < \Delta T < 0.3$ on the diagonal are ruled out with 95% C.L. For $S = 0$ or $T = 0$ the range of the respective other parameter is typically out to ± 0.1.
From the formulas we know that all we need to compute for S and T are the photon and W, Z self energies. Self energies from a field theoretical point of view can be considered part of the renormalization of a field, because whatever we do we need to reproduce the canonically normalized kinetic terms. If we introduce new particles with $SU(2)_L \times U(1)_Y$ quantum numbers, all of these particles will contribute to these self–energy loops. From the appearance of N_c in the formulas above we see that all these contributions simply add, unless the up-type and down-type contributions cancel. This is for example the case for a chiral fourth generation, just as a side remark.
In technicolor models, the singlet techniquarks will contribute to the S parameter each with a factor $N_T/(6\pi) \sim N_T/20 \sim 0.2$, assuming the minimal model with $N_D = 1$. This number can barely be tolerated if it is accompanied with $\Delta T \sim 0.2$, due to the diagonal ellipse structure of the current constraints. Constructing an appropriate model with an up-type and down-type is a challenge to technicolor model building in the minimal models. More complex models easily get to $\Delta S \sim \mathcal{O}(1)$, which is firmly ruled out, no matter what kind of ΔT we manage to obtain. These electroweak constraints are typically considered the last blow to technicolor models, even though we should mention that good model builders will find ways to construct models around almost any constraint, even the deadly list of technicolor constraints listed above. Only once we see (or do not see) a fundamental light Higgs at the LHC will we know...

C. Symmetry breaking by boundary conditions

A much more recent idea of electroweak symmetry breaking which will, however, have to deal with the same kind of experimental constraints, is electroweak symmetry breaking from a fifth dimension. In other words, we extend our usual picture of space-time by an additional spacial coordinate, *i.e.* $\mu = 0, 1, 2, 3$ becomes $M = 0, 1, 2, 3, 5$. Giving the additional fifth dimension the index '5' instead of '4' is meant to avoid confusion. Of course, we have to construct our model such that for example gravitational measurements cannot detect the fact that there is this additional dimension. This will be one of the requirements on the extra dimension, which at this stage we will not discuss. For a very simple introduction into extra–dimensional theories and their benefits in solving the hierarchy problem you could have a look into my lecture notes. In the following three lectures we will limit ourselves to a new mechanism of breaking electroweak symmetry without introducing a Higgs field. In a way, this concept is more revolutionary than technicolor, because as we have seen in the very beginning, we can always think of a non-linear sigma model as the special case of a decoupled fundamental Higgs boson. Using extra–dimensional boundary condition really does not resemble the usual Higgs mechanism anymore.

Before we break electroweak symmetry, we need to get a general feel for a field theory which involves a higher–dimensional space (called bulk) and four–dimensional boundaries. Therefore, let us look at the action of a simple scalar field in five dimensions. Naively, we just write down a Lagrangian which we integrate over five dimensions of space-time:

$$S_{\text{bulk}} = \int d^4x \int dy \, \mathcal{L}_5 = \int d^4x \, \mathcal{L}_4 \tag{378}$$

Already from this formula we know that our counting of powers of mass will be different - if the action still has mass dimension zero, then the Lagrangian $\mathcal{L}_5$ now has to have mass dimension five instead of four.

Gravitational constraints suggest that the extra dimension cannot be arbitrarily large, because it would modify Newton's gravity at very large distances (or very low energies), and such modifications are ruled out by everything we know about how our solar system or our galaxy works. Moreover, to get any mileage out of boundary conditions we need to give our extra dimension such boundaries, which means a finite size. A finite–size additional dimension we can obtain from an infinite dimensions two ways: either we think of it as a repeated interval, or we think of it as running around a circle, where the ends are simply identified. The latter leads us to the concept of an orbifold compactification which defines a brane. However, in comparison to the most general boundaries, such an orbifold compactification limits the set of possible boundary conditions, so we will instead stick to a general boundary setup. In both cases we can write the size of the fifth dimension as $y = 0...\pi R$.

The simplest field we can write down is a scalar field with a kinetic term and a potential, so our action reads:

$$S_{\text{bulk}} = \int d^4x \int_0^{\pi R} dy \, \left(\frac{1}{2} (\partial^M \phi)^2 - V(\phi) \right) = \int d^4x \int_0^{\pi R} dy \, \left(\frac{1}{2} (\partial^\mu \phi)^2 - \frac{1}{2} (\partial^5 \phi)^2 - V(\phi) \right) \tag{379}$$

Because the additional dimensions is a space dimension the metric tensor g_{MN} is $(+, -, -, -, -)$. The trouble with this Lagrangian is that the kinetic term means that this scalar field has a mass dimension 3/2, but on the other hand it is not clear what we could do instead.

1. *Fields on the boundary*

Trying to derive the equations of motion from this action will bring in the boundaries. The variation of the action is

$$\begin{aligned} 0 \stackrel{!}{=} \delta S_{\text{bulk}} &= \int d^4x \int_0^{\pi R} dy \, \left((\partial^\mu \phi)(\partial_\mu \delta\phi) - \frac{\partial}{\partial \phi} V(\phi) \, \delta\phi - (\partial^5 \phi)(\partial_5 \delta\phi) \right) \\ &= \int d^4x \left[\int_0^{\pi R} dy \, \left(-\partial^\mu \partial_\mu \phi - \frac{\partial}{\partial \phi} V(\phi) + \partial^5 \partial_5 \phi \right) \delta\phi - (\partial_5 \phi) \delta\phi \Big|_0^{\pi R} \right] \\ &= \int d^4x \left[\int_0^{\pi R} dy \, \left(-\partial^M \partial_M \phi - \frac{\partial}{\partial \phi} V(\phi) \right) \delta\phi - (\partial_5 \phi) \delta\phi \Big|_0^{\pi R} \right] \end{aligned} \tag{380}$$

We have simply integrated by parts in all five dimensions. In contrast to the four usual dimension where our Hilbert space is defined such that all fields vanish at the infinite boundary we cannot require such a thing for the fifth dimension. Instead, we need to keep the surface term in the variation of the action, which will generically give us boundary terms from the originally five–dimensional Lagrangian. The first condition we read off this variation is the five–dimensional bulk equation of motion $\partial_M \partial^M \phi = -\partial V/\partial\phi$.

In addition, the boundary term if the variation of the action has to vanish, which gives us the choice of two boundary conditions:

$$\boxed{\partial_5 \phi \Big|^{0,\pi R} = 0} \qquad \text{(Neumann)} \qquad \text{or} \qquad \boxed{\phi \Big|^{0,\pi R} = 0} \qquad \text{(Dirichlet)} \tag{381}$$

There is in principle be a third possibility, namely that the contributions from both boundaries cancel, but this would force is to treat the two boundaries equal, which as we will see later is not what we want.

From this short argument we see that it would be useful to study the behavior of additional Lagrangian terms only on the boundary, to modify such boundary conditions. For example, what happens, if we add a boundary mass term?

$$S = S_{\text{bulk}} - \int d^4x \, \frac{1}{2} M\phi^2 \Big|^0 - \int d^4x \, \frac{1}{2} M\phi^2 \Big|^{\pi R} \tag{382}$$

The masses on the two boundaries can of course be different. Looking at the formula above we have gotten ourselves into trouble, because the usual four–dimensional mass terms would be M^2. However, this M^2 would need to have mass dimension one to arrive at the usual dimension–four Lagrangian in four dimensions. The variational principle gives us

$$\begin{aligned} \delta S &= \delta S_{\text{bulk}} - \int d^4x \, M\phi \, \delta\phi \Big|^0 - \int d^4x \, M\phi \, \delta\phi \Big|^{\pi R} \\ &= \int d^4x \left[\int_0^{\pi R} dy \, (\cdots) \, \delta\phi - \partial_5\phi \, \delta\phi \Big|_0^{\pi R} - M\phi \, \delta\phi \Big|^0 - M\phi \, \delta\phi \Big|^{\pi R} \right] \\ \Leftrightarrow \qquad & \partial_5\phi - M\phi \Big|^0 = 0 \qquad \text{and} \qquad \partial_5\phi + M\phi \Big|^{\pi R} = 0 \end{aligned} \tag{383}$$

This form is interesting, because it interpolates between the two possible boundary conditions in the absence of the mass term: for $M = 0$ we recover the Neumann BC, while for $M \to \infty$ we are left with the Dirichlet BC. Note again that these conditions really do not look like equations of motion on the boundary because of mass dimension of the scalar field. In fact, they look much more like a Dirac equation, which makes no sense for scalars, but then they are not equations of motion either.

Moving on, let us try a boundary kinetic term on one of the boundaries:

$$S = S_{\text{bulk}} + \int d^4x \, \frac{1}{2M} (\partial_\mu\phi)(\partial^\mu\phi) \Big|^{\pi R} \tag{384}$$

Note that on the four–dimensional boundary we are using the four–dimensional derivative of course. The variational principle now gives us — as usually integrating by parts and keeping the factor two from the symmetric squared kinetic term:

$$\begin{aligned} \delta S &= \delta S_{\text{bulk}} + \int d^4x \, \frac{1}{M} (\partial_\mu\phi)(\partial^\mu\delta\phi) \Big|^{\pi R} \\ &= \int d^4x \left[\int_0^{\pi R} dy \, (\cdots) \, \delta\phi - \partial_5\phi \, \delta\phi \Big|_0^{\pi R} - \frac{1}{M} (\partial_\mu\partial^\mu\phi) \, \delta\phi \Big|^{\pi R} \right] \quad \Leftrightarrow \quad \partial_5\phi = -\frac{1}{M} \partial_\mu\partial^\mu\phi \Big|^{\pi R} = 0 \end{aligned} \tag{385}$$

Remembering the bulk equation of motion $\partial_M \partial^M \phi = 0$ we can re-write this boundary conditions as $\partial_5\phi = -(\partial_5)^2\phi/M$. On the other boundary, the relative sign would simply change. This form has an interesting consequence: if we want the second-derivative operator $\phi'' \equiv (\partial_5)^2\phi$ to be hermitian $(f,g'') = (f'',g)$ we have to redefine the scalar product on the space of five–dimensional wave functions including a boundary term. Csaba nicely derives this in his lecture.

As the final step we will move away from the scalar toy model and introduce a five–dimensional photon field into our theory:

$$S = \int d^5x \, \left(-\frac{1}{4} F_{MN} F^{MN} \right) = \int d^5x \, \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} F_{\mu 5} F^{\mu 5} \right) \tag{386}$$

There would be the additional F_{55} term, but it vanishes due to the antisymmetric nature of $F_{MN} = \partial_M A_N - \partial_N A_M$. The additional term including the fifth component of the gauge field becomes

$$\begin{aligned}
S &= -\frac{1}{2} \int d^4x \int_0^{\pi R} dy \; F_{\mu 5} F^{\mu 5} \\
&= -\frac{1}{2} \int d^4x \int_0^{\pi R} dy \; (\partial_\mu A_5 - \partial_5 A_\mu) \; \left(\partial^\mu A^5 - \partial^5 A^\mu\right) \\
&= -\frac{1}{2} \int d^4x \int_0^{\pi R} dy \; \left[+\partial_\mu A_5 \, \partial^\mu A^5 + \partial_5 A_\mu \, \partial^5 A^\mu - 2\partial_\mu A_5 \, \partial^5 A^\mu \right] \\
&= -\frac{1}{2} \int d^4x \int_0^{\pi R} dy \; \left[-A_5 \, \partial_\mu \partial^\mu A^5 + \partial_5 A_\mu \, \partial^5 A^\mu + 2A_5 \, \partial_\mu \partial^5 A^\mu \right] \\
&= -\frac{1}{2} \int d^4x \int_0^{\pi R} dy \; \left[-A_5 \, \partial_\mu \partial^\mu A^5 - A_\mu \, \partial_5 \partial^5 A^\mu - 2\partial^5 A_5 \, \partial_\mu A^\mu \right] - \frac{1}{2} \int d^4x \; \left[A_\mu \, \partial^5 A^\mu + 2A_5 (\partial_\mu A^\mu) \right]_0^{\pi R}
\end{aligned} \tag{387}$$

again after integrating by parts first in the four–dimensional space (with vanishing boundary terms) and then in the fifth dimension . The first term in the last line is obviously a kinetic terms for the scalar field A_5. The second term will after a Kaluza–Klein decomposition (*i.e.* a discrete Fourier transform in the periodic fifth dimension) become a mass term for our photon in five dimensions. We can schematically write the five–dimensional wave functions by separating variables

$$A_\mu(x,y) = \bar{A}_\mu(x) f(y) \sim \sum_n \hat{A}_\mu^{(n)}(x) \; e^{iny/R} \quad \Rightarrow \quad \partial_5^2 A_\mu(x,y) = \sum_n \partial_5^2 \hat{A}_\mu^{(n)}(x) \; e^{iny/R} = -\sum_n \frac{m^2}{R^2} \hat{A}_\mu^{(n)}(x) \; e^{iny/R} \tag{388}$$

Which means that if we write our five–dimensional photon field as an effective theory in four dimensions we obtain towers of massive photons whose mass is given by the inverse size of our fifth dimension. Note, however, that we have to clearly distinguish between two kinds of photon masses. The KK excitations will be massive, but this does not mean that we break the symmetry of our Lagrangian. In particular, there will be a zero mode $n = 0$ with vanishing mass. Which means we still have to find a mechanism for electroweak symmetry breaking. The role of the KK excitations will become obvious later, when we discuss unitarity in these models.

From the formula above it also becomes clear what role boundary conditions play: Dirichlet boundary conditions $(A = 0)$ mean sine-type behavior, while Neumann boundary conditions $(\partial_5 A = 0)$ mean cosine at the respective boundaries. This implies that if we want to write down a zero mode, *i.e.* a constant wave function in the y dimensions which corresponds to $\exp(ny/R)$ for $n = 0$, we need a Neumann-Neumann setup on the two boundaries.

For the third term in eq.(387) we have to briefly remember something about gauge theories which I also had to read again for example in the book by Peskin and Schroeder. It obviously mixes the scalar field and the photon. The same thing happens if we write down the usual Higgs mechanism: the NGB will mix with the transverse degrees of freedom of the gauge boson which will then eat it as its longitudinal component. Such a term we do not want in the Lagrangian — the definition of the gauge boson should instead absorb this term into the massive gauge boson. This we can achieve in a general $(R - \xi)$ gauge: gauge fixing means including a gauge–fixing term with including the Lagrangian multiplier $1/\xi$. You can find a discussion of this gauge

for the Standard–Model Higgs mechanism in Peskin & Schroeder section 21.1. For example, for an abelian massive photon we introduce a gauge–fixing term $(\partial_\mu A^\mu - \xi e v \phi)^2/(2\sqrt{\xi})$ to cancel the photon–NGB mixing and fix the photon gauge at the same time. The third term from the gauge fixing gives us a mass for the NGB $m_\phi^2 = \xi(ev)^2 = \xi m_A^2$ (in terms of the photon mass). Since this mass is gauge dependent the NGB it not a well-defined physical degree of freedom, and it can be decoupled by choosing $\xi \to \infty$, which is called unitary gauge. In that gauge the NGB survives only as the longitudinal component of the massive photon, but does not appear in the Lagrangian anymore.
Precisely the same way we now introduce a gauge–fixing term in the five dimensional space (bulk):

$$\begin{aligned} S_{\text{GF,bulk}} &= \frac{1}{2\xi} \int d^4x \int_0^{\pi R} dy \; \left(\partial_\mu A^\mu - \xi\, \partial_5 A^5\right)^2 \\ &= \int d^4x \int_0^{\pi R} dy \; \left[\frac{1}{2\xi} (\partial_\mu A^\mu)^2 - \partial_\mu A^\mu\, \partial_5 A^5 + \frac{\xi^2}{2} (\partial_5 A^5)^2 \right] \\ &= \int d^4x \int_0^{\pi R} dy \; \left[\frac{1}{2\xi} (\partial_\mu A^\mu)^2 - \partial_\mu A^\mu\, \partial_5 A^5 - \frac{\xi^2}{2} A_5 \partial_5 \partial^5 A^5 \right] + \frac{\xi^2}{2} \int d^4x \; A_5\, \partial_5 A^5 \Big|_0^{\pi R} \end{aligned} \tag{389}$$

The usual gauge fixing term $(\partial_\mu A^\mu)^2$ appears for the transverse degrees of freedom of the massless photon. The second term cancels the mixing term between A_μ and A_5. What is interesting is the last term in $S_{\text{GF,bulk}}$: there is no need to fix the gauge for the scalar field A^5, and if we compute the equation of motion for A_5 using the variational principle for the contributions to δS proportional to δA_5 it includes a term $\xi^2 \partial_5 A^5\, \partial^5(\delta A_5)$. After integrating by parts this leads to $\xi^2 (\partial_5)^2 A^5$ appearing in the equation of motion for A_5, which is nothing but a massive KK tower. The KK masses will become infinitely large in unitary gauge $\xi \to \infty$, so that the entire A_5 tower as a physical mode decouples from the theory. Instead, its degrees of freedom now give KK masses to the excitation of the four–dimensional gauge field A_μ. Note that a possible zero mode in the A_5 tower would be linked to a finite mass for the lowest (*i.e.* Standard Model) gauge boson. Dependent on the boundary conditions such a zero might or might not appear. We will discuss the role of such a A_5 zero term when we discuss ways to break electroweak symmetry.

We know that we are not living in five but in four dimensions. Which means that we should have a careful look at the action on the boundaries in eq.(387). After fixing the gauge in the bulk, there is also a dangerous boundary mixing term of the type $A_5(\partial_\mu A^\mu)$. Again, we have to introduce a gauge fixing term, now on the boundary

$$\begin{aligned} S_{\text{GF,bound}} &= \frac{1}{2\hat{\xi}} \int d^4x \; \left(\partial_\mu A^\mu \pm \hat{\xi} A_5\right)^2 \Big|^{0,\pi R} \\ &= \int d^4x \; \left[\frac{1}{2\hat{\xi}} (\partial_\mu A^\mu)^2 \Big|^{0,\pi R} + \frac{\hat{\xi}}{2} A_5^2 \Big|^{0,\pi R} - (\partial_\mu A^\mu) A_5 \Big|^{0} + (\partial_\mu A^\mu) A_5 \Big|^{\pi R} \right] \\ &= \int d^4x \; \left[\frac{1}{2\hat{\xi}} (\partial_\mu A^\mu)^2 \Big|^{0,\pi R} + \frac{\hat{\xi}}{2} A_5^2 \Big|^{0,\pi R} + (\partial_\mu A^\mu) A_5 \Big|_0^{\pi R} \right] \end{aligned} \tag{390}$$

Note the difference between the upper and lower notation of the boundary terms. The last term precisely cancels the boundary mixing term. We can now combine S_{bound} from the original Lagrangian and from the two gauge fixing terms:

$$\begin{aligned} S_{\text{bound}} &= -\frac{1}{2} \int d^4x \; \left[A_\mu\, \partial^5 A^\mu + 2 A_5 (\partial_\mu A^\mu) \right]_0^{\pi R} + \frac{\xi^2}{2} \int d^4x \; A_5\, \partial_5 A^5 \Big|_0^{\pi R} \\ &\quad + \int d^4x \; \left[\frac{1}{2\hat{\xi}} (\partial_\mu A^\mu)^2 \Big|^{0,\pi R} + \frac{\hat{\xi}}{2} A_5^2 \Big|^{0,\pi R} + (\partial_\mu A^\mu) A_5 \Big|_0^{\pi R} \right] \\ &= -\frac{1}{2} \int d^4x \; A_\mu\, \partial^5 A^\mu \Big|_0^{\pi R} + \frac{\xi^2}{2} \int d^4x \; A_5\, \partial_5 A^5 \Big|_0^{\pi R} + \int d^4x \; \left[\frac{1}{2\hat{\xi}} (\partial_\mu A^\mu)^2 \Big|^{0,\pi R} + \frac{\hat{\xi}}{2} A_5^2 \Big|^{0,\pi R} \right] \end{aligned} \tag{391}$$

For this action we can compute the variation, which needs to be zero. The two gauge parameters ξ in the bulk and $\hat{\xi}$ on the boundary do not have to be identical. To simplify the results we can use the unitary gauge on the boundary $\hat{\xi} \to \infty$ and find for the terms proportional to the variation of A_5

$$0 \stackrel{!}{=} \delta S_{\text{bound}}\Big|_{A^5} = \int d^4x \left[\frac{\xi}{2} \partial_5 A^5 \, \delta A_5 \Big|_0^{\pi R} + \frac{\xi}{2} A^5 \, \partial_5 (\delta A_5) \Big|_0^{\pi R} + \hat{\xi} A^5 \, \delta A_5 \Big|^{0,\pi R} \right]$$
$$\sim \hat{\xi} \int d^4x \; A^5 \, \delta A_5 \Big|^{0,\pi R} \qquad \Leftrightarrow \qquad \boxed{A^5 \Big|^{0,\pi R} = 0} \tag{392}$$

while the condition on $\partial_5 A^5$ we would have gotten from the gauge fixing in the bulk does not contribute anymore. The second term proportional to $\partial_5 \delta A_5$ looks funny at first, but it is taken care of by the boundary condition $A^5 = 0$. Secondly, the variational contributions proportional to the regular photon field A^μ are:

$$0 \stackrel{!}{=} \delta S_{\text{bound}}\Big|_{A^\mu} = \int d^4x \left[-\frac{1}{2} \delta A_\mu \, \partial_5 A^\mu \Big|_0^{\pi R} - \frac{1}{2} A_\mu \, \partial_5 \delta A^\mu \Big|_0^{\pi R} + \frac{1}{\hat{\xi}} (\partial_\nu A^\nu) \, (\partial_\mu \delta A^\mu) \Big|^{0,\pi R} \right]$$
$$= \int d^4x \left[-\frac{1}{2} \delta A_\mu \, \partial_5 A^\mu \Big|_0^{\pi R} - \frac{1}{2} A_\mu \, \partial_5 \delta A^\mu \Big|_0^{\pi R} - \frac{1}{\hat{\xi}} (\partial_\mu \partial_\nu A^\nu) \, \delta A^\mu \Big|^{0,\pi R} \right]$$
$$\sim \int d^4x \left[-\frac{1}{2} \partial_5 A^\mu \, \delta A_\mu \Big|_0^{\pi R} - \frac{1}{2} A_\mu \, \partial_5 \delta A^\mu \Big|_0^{\pi R} \right] \qquad \Leftrightarrow \qquad \boxed{\partial_5 A^\mu \Big|^{0,\pi R} = 0} \tag{393}$$

Because we fix $\partial_5 A^\mu$ on the boundaries, it does not contribute in the second term of δS_{bound}, like any other constant would not contribute. According to our very brief look at zero modes this set of boundary conditions means that after Fourier–transforming the fifth dimension there will be a zero mode for the photon A_μ, while due to the Dirichlet boundary conditions the scalar mode A_5 will not have a zero mode. It will only occur with finite KK masses, which are eaten by the massive KK gauge bosons. In other words, we expect a massless Standard–Model photon with a massive KK tower, but no additional A_5 fields.
Looking back at S_{bound} we see that the two sets of boundary conditions and in addition the boundary unitary gauge $\hat{\xi} \to \infty$ implies $S_{\text{bound}} = 0$. All we have to consider for our five–dimensional QED is the bulk action in eq.(387).

$$S = \int d^5x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \left(\partial_\mu A_5 \, \partial^\mu A^5 + \partial_5 A_\mu \, \partial^5 A^\mu - 2 \partial_\mu A_5 \, \partial^5 A^\mu \right) \right] \tag{394}$$

2. *Breaking the gauge symmetry on the boundaries*

Since we now know how the basics of a five–dimensional version of QED, let us see what happens if we break the gauge symmetry — in this case the $U(1)$ — on the boundaries. From the introduction we know how to do this; let us add a non-linear sigma model on the two boundaries.

$$S = \int d^4x \, \mathcal{L}_4 \qquad \mathcal{L}_4 \supset \int_0^{\pi R} dy \left[|D_\mu \Phi|^2 - \frac{\lambda}{2} \left(|\Phi|^2 - \frac{v^2}{2} \right)^2 \right] \qquad \Phi \sim \frac{v}{\sqrt{2}} \, e^{i\pi/v} \tag{395}$$

Again, in these notes am using the technicolor version of the Higgs potential with a prefactor $\lambda/2$, instead simply λ as I use it in my Higgs notes or Csaba uses it as well... In the last step we have already decoupled the physical Higgs field and chosen $\lambda \to \infty$, with finite v. The two Higgs fields on the two boundaries should of course be labelled differently, and the parameters λ and v will not be the same for both of them. To keep things short I will only spell out the action for $y = \pi R$. This gives us the bulk contributions we computed before, remembering that in unitary gauge and with given boundary conditions $\mathcal{L}_{\text{bound}} = 0$:

$$\mathcal{L}_4 = \mathcal{L}_{4,\text{bulk}} + \mathcal{L}_{4,\sigma}$$
$$= \int_0^{\pi R} dy \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A_5)^2 - \frac{1}{2} (\partial_5 A_\mu)^2 + \partial_\mu A_5 \partial^5 A^\mu \right] + \frac{1}{2} \left(\partial_\mu \pi - v A_\mu \right)^2 \Big|^{\pi R} \tag{396}$$

The sigma-field contribution we simply copy from eq.(327) with $ef \to v$. Note that writing down the boundary terms we can see that if A_μ now has mass dimension 3/2, we need to assign mass dimension $[v] = M^{1/2}$ and $[\pi] = M^1$. In contrast to our earlier discussion we now use a general $(R-\xi)$ gauge, which means we need to introduce gauge–fixing terms to cancel the $(A_5 - A_\mu)$ and $(\pi - A_\mu)$ mixing terms

$$\begin{aligned}\mathcal{L}_{4,\mathrm{GF}} &= -\frac{1}{2\xi}\int_0^{\pi R} dy\ \left(\partial_\mu A^\mu - \xi\partial_5 A^5\right)^2 - \frac{1}{2\hat{\xi}}\left(\partial_\mu A^\mu + \hat{\xi}(v\pi + A_5)\right)^2 \Big|^{\pi R} \\ &= -\frac{1}{2\xi}\int_0^{\pi R} dy\ \left[(\partial_\mu A^\mu)^2 - \xi\,\partial_\mu A^\mu\ \partial_5 A^5 + \xi^2(\partial_5 A^5)\right] - \frac{1}{2\hat{\xi}}\left(\partial_\mu A^\mu + \hat{\xi}(v\pi + A_5)\right)^2 \Big|^{\pi R} \\ &= -\frac{1}{2\xi}\int_0^{\pi R} dy\ \left[(\partial_\mu A^\mu)^2 - \xi\,\partial_\mu A^\mu\ \partial_5 A^5 - \xi^2 A_5\partial_5^2 A^5)\right] - \frac{\xi}{2}A_5\partial_5 A^5\Big|_0^{\pi R} - \frac{1}{2\hat{\xi}}\left(\partial_\mu A^\mu + \hat{\xi}(v\pi + A_5)\right)^2 \Big|^{\pi R}\end{aligned} \tag{397}$$

Again, we have copied the bulk contribution from eq.(389) and added the appropriate term needed for the NGB contributions. After adding these gauge–fixing terms the bulk action involving only the gauge field A_μ is

$$\begin{aligned}\mathcal{L}_{A^\mu} &= \int_0^{\pi R} dy\ \left[-\frac{1}{4}\left((\partial_\mu A_\nu)^2 + (\partial_\nu A_\mu)^2 - 2(\partial_\mu A_\nu)(\partial^\nu A^\mu)\right) - \frac{1}{2}(\partial_5 A_\mu)^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right] \\ &= \frac{1}{2}\int_0^{\pi R} dy\ \left[-(\partial_\mu A_\nu)^2 + (\partial_\mu A_\nu)(\partial^\nu A^\mu) - (\partial_5 A_\mu)^2 - \frac{1}{\xi}(\partial_\mu A^\mu)^2\right] \\ &= \frac{1}{2}\int_0^{\pi R} dy\ \left[A_\nu\partial^\mu\partial_\mu A^\nu - A_\nu\partial_\mu\partial^\nu A^\mu + A_\mu\partial_5\partial^5 A^\mu + \frac{1}{\xi}A_\nu\partial^\nu\partial_\mu A^\mu\right] \quad - \frac{1}{2}A_\mu\partial_5 A^\mu\Big|_0^{\pi R} \\ &= \frac{1}{2}\int_0^{\pi R} dy\ A_\nu\left[g^{\mu\nu}(\partial^\rho\partial_\rho - \partial^\mu\partial^\nu + g^{\mu\nu}\partial_5\partial^5 + \frac{1}{\xi}\partial^\nu\partial^\mu\right] A_\mu \quad - \frac{1}{2}A_\mu\partial_5 A^\mu\Big|_0^{\pi R} \\ &= \frac{1}{2}\int_0^{\pi R} dy\ A_\nu\left[g^{\mu\nu}(\partial^\rho\partial_\rho + \partial_5\partial^5) - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right] A_\mu \quad - \frac{1}{2}A_\mu\partial_5 A^\mu\Big|_0^{\pi R}\end{aligned} \tag{398}$$

What we see in the last line it simply the gauge–boson propagator in $(R-\xi)$ gauge, now including the KK term. **tp: for some reason this ∂_5^2 has a weird sign...?** The corresponding bulk equation of motion for the scalar component in the absence of any additional mass terms arises from gauge fixing: $\mathcal{L}_5 \supset -\xi/2\ (\partial_5 A^5)^2$.

To compute the boundary conditions for A_μ, we can for example collect all boundary contributions at $y = \pi R$ after removing the $(A_5 - A_\mu)$ mixing:

$$\begin{aligned}\mathcal{L}_{A^\mu} &= -(\partial_\mu\pi)(vA_\mu) + \frac{1}{2}(vA_\mu)^2 \quad - \frac{1}{2\hat{\xi}}(\partial_\mu A^\mu)^2 - \frac{1}{\hat{\xi}}(\partial_\mu A^\mu)\,\hat{\xi}\,(v\pi) \quad - \frac{1}{2}A_\mu\partial_5 A^\mu \\ &= -(\partial_\mu\pi)(vA_\mu) + \frac{1}{2}(vA_\mu)^2 - \frac{1}{2\hat{\xi}}(\partial_\mu A^\mu)^2 + A^\mu\partial_\mu(v\pi) - \frac{1}{2}A_\mu\partial_5 A^\mu \\ &= \frac{1}{2}v^2 A_\mu A^\mu - \frac{1}{2\hat{\xi}}(\partial_\mu A^\mu)^2 - \frac{1}{2}A_\mu\partial_5 A^\mu \\ &\sim \frac{1}{2}A_\mu\left(v^2 - \partial_5\right)A^\mu\end{aligned} \tag{399}$$

In unitary gauge, this determines the boundary condition at πR and correspondingly at $y = 0$ to be

$$\boxed{(\partial_5 \mp v^2)A_\mu\Big|^{0,\pi R} = 0} \tag{400}$$

Remember that now $[v] = M^{1/2}$. From the general scalar boundary–mass case we expect that adding a boundary–mass for the photon indeed means that the new boundary conditions will become an interpolation of Dirichlet and Neumann conditions. What is new in this formula is that the mass scale is given by v, the vacuum expectation value breaking electroweak symmetry on the boundaries. In other words, in the unbroken phase $v = 0$ the photon field has to obey Neumann boundary conditions $\partial_5 A_\mu = 0$, while in the broken phase $v \neq 0$ it will follow Dirichlet boundary conditions $A_\mu = 0$. We know that this means that only in the unbroken phase it will have a zero mode. We can turn this argument around: a physical photon field with a Dirichlet boundary condition $A_\mu = 0$ and hence without a zero mode is indeed a sign for a broken symmetry on the respective boundary.

If a Dirichlet boundary condition for the physical gauge–boson field is indeed a sign for a broken symmetry, some combination of A_5 and the NGB π has to provide the degrees of freedom to make the photon (including its zero mode) massive. The boundary terms for A_5 and π after removing all mixing terms and including a boundary mass m with $[m] = M^1$ for π are

$$\begin{aligned}\mathcal{L}_{A^5,\pi} &= \frac{1}{2}(\partial_\mu \pi)^2 - \frac{\xi}{2} A_5 \partial_5 A^5 - \frac{\hat{\xi}}{2}\left(v\pi + A_5\right)^2 - \frac{m^2}{2}\pi^2 \\ &\sim -\frac{\xi}{2} A_5 \partial_5 A^5 - \frac{\hat{\xi}}{2}\left[\left(v^2 + \frac{m^2}{\hat{\xi}}\right)\pi^2 + 2v\pi A_5 + A_5^2\right]\end{aligned} \tag{401}$$

We then find for the π's boundary conditions in combination with A_5

$$0 \stackrel{!}{=} \frac{\partial}{\partial \pi}\Big[\cdots\Big] = 2\left(v^2 + \frac{m^2}{\hat{\xi}}\right)\pi + 2vA_5 \qquad \Leftrightarrow \qquad \left(v^2 + \frac{m^2}{\hat{\xi}}\right)\pi + vA_5\Big|^{0,\pi R} = 0 \tag{402}$$

The same way we can compute the boundary conditions for A_5 in terms of both scalar fields:

$$\begin{aligned} 0 &\stackrel{!}{=} -\frac{\xi}{2}\partial^5 A_5 - \frac{\hat{\xi}}{2}\left[2v\pi + 2A_5\right] \\ &= -\frac{\xi}{2}\partial^5 A_5 - \hat{\xi}A_5 + \hat{\xi}v\,\frac{vA_5}{v^2 + m^2/\hat{\xi}} \\ &= -\left[\frac{\xi}{2}\,\partial^5 + \hat{\xi}\frac{m^2/\hat{\xi}}{v^2 + m^2/\hat{\xi}}\right] A_5 \end{aligned} \tag{403}$$

From there we can read off the boundary condition for the scalar component A_5

$$\boxed{\left(\partial_5 \mp \frac{\hat{\xi}}{\xi}\,\frac{m^2/\hat{\xi}}{v^2 + m^2/\hat{\xi}}\right) A_5\Big|^{0,\pi R} = 0} \tag{404}$$

For unitary gauge on the boundaries $\hat{\xi} \to \infty$ we know from the last example without boundary scalars that indeed we should find Dirichlet boundary conditions $A_5 = 0$. In that limit the NGB mass terms become suppressed, because these degrees of freedom are not physical and should be eaten by the gauge bosons.

If we want to study the behavior of the NGB in the bulk we can go into unitary gauge in the bulk $\xi \to \infty$. We see that breaking the symmetry on the boundaries shifts the A_5 boundary conditions from originally Dirichlet ($A_5 = 0$) in eq.(393) to Neumann ($\partial_5 A_5 = 0$). This means that A_5 now can develop a zero mode, which provides the necessary degree of freedom for the photon which in the presence of v cannot include a zero mode any longer!
In general we see a pattern for the boundary conditions of the gauge boson and of the scalar A_5 when we break the symmetry on the boundaries. In the unbroken symmetry the gauge boson will have a zero mode, which corresponds to Neumann BC, while we have seen that the scalar mode's Dirichlet BC do not allow for a zero mode. After symmetry breaking, the Dirichlet BC for the gauge boson forbids their zero mode, but the scalar A_5 can include a zero mode, provided the symmetry is broken on both boundaries. The necessary degree of freedom for this zero mode comes from the boundary scalar π. This implies that the boundary conditions for the scalar component have to be the opposite of the vector's conditions, simply exchanging Neumann and Dirichlet BCs. It also means, that in the absence of massless scalars we should concentrate on Neumann–Neumann and Dirichlet–Neumann boundary conditions on our two boundaries $y = 0, \pi R$.

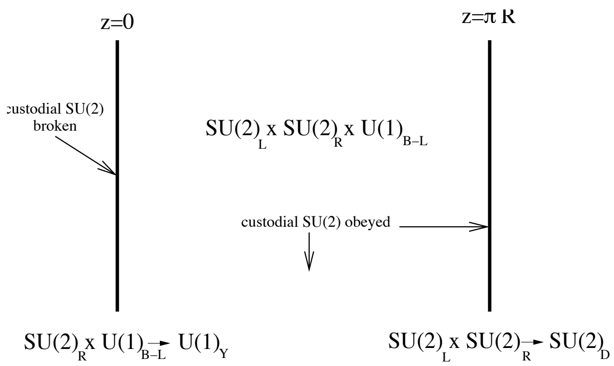

FIG. 3: Symmetry–breaking pattern of the Higgsless toy model, Figure stolen from Csaba's notes.

This mechanism now allows us to write down a very simple toy model for breaking a gauge symmetry by boundary conditions. We start with three gauge bosons, corresponding to a $SU(2)$ gauge group in the bulk. We will try to make two of them (W, Z) heavy while leaving the third (γ) massless. The massless photon is simple, because we know that we need Neumann–Neumann boundary conditions:

$$\partial_5 A^3_\mu \Big|^{0,\pi R} = 0 \qquad \Rightarrow \qquad \hat{A}^3_\mu \sim \cos\frac{ny}{R} \qquad \Rightarrow \qquad \boxed{m^{(n)}_{A^3} = \frac{n}{R} = 0, \frac{1}{R}, \frac{2}{R} \cdots} \tag{405}$$

For the other two gauge bosons it is sufficient to require a Dirichlet boundary condition at at least one of the boundaries. The choice of the second boundary condition will then affect the mass of the first KK excitation and the mass ratios of the higher excitations:

$$\partial_5 A^{1,2}_\mu \Big|^{0} = 0 \qquad A^{1,2}_\mu \Big|^{\pi R} = 0 \qquad \Rightarrow \qquad \hat{A}^{1,2}_\mu \sim \cos\frac{(2n+1)y}{2R} \qquad \Rightarrow \qquad \boxed{m^{(n)}_{A^{1,2}} = \frac{n+1/2}{R} = \frac{1}{2R}, \frac{3}{2R} \cdots} \tag{406}$$

As discussed above, the boundary conditions for the scalar components are exactly the opposite of those for the vectors derived here. We have deliberately not chosen any pure Dirichlet–Dirichlet boundary setup for the gauge fields, because the corresponding scalar would then have purely Neumann boundary conditions, which would imply an unwanted massless scalar zero mode in the model.
This means, we indeed built a model with a massless photon and a W and Z with the same mass terms. Because of the factors of two between the ZZ and the W^+W^- mass terms in the Lagrangian we predict $m_Z/m_W = 2$ and for their first KK modes $m_{Z'}/m_Z = 2$ and $m_{W'}/m_W = 3$.
Nothing of that is anywhere close to reality, but we also have many aspect of the model to play with, so let us see what we can do better. At this point we can for the first time see why knowing technicolor and its problems helps us building models which break electroweak symmetry through boundary conditions: if we want to survive the electroweak precision constraints we need to protect the relevant observables using symmetries in our model.

3. A toy model with custodial symmetry

From the section on electroweak precision data we know that the S and T parameters in the gauge sector are very small. We also remember that the parameter T measures the different contributions to the W and Z masses from quantum corrections to their propagators. In the Standard Model there are two sources of this global $SU(2)$ symmetry breaking: in the Feynman diagrams contributing to m_Z we either find pure bottom or pure top loops, while m_W corrections include mixed bottom–top contributions. Modulo prefactors we can either say $\Delta T \sim 0$ or $\rho \sim 1$ defined as $\rho = m_W^2/(c_w^2 m_Z^2)$. For $m_b \neq m_t$ we find the contributions shown in eq.(376). In addition, electroweak symmetry breaking giving the Higgs doublet a vev in one doublet component also breaks the $SU(2)$ symmetry protecting $T = 0$.
We can think of the complete symmetry of the Lagrangian with a protected value of $T = 0$ as $SU(2)_L \times SU(2)_R$. At this stage, none of them needs to be gauged, even though we know that $SU(2)_L$ at some point will be gauged.

If both global $SU(2)_{LR}$ are unbroken, the left-right mixing Dirac masses of quark doublets will be degenerate $m_b = m_t$. If following the example of the chiral $U(1)_L \times U(1)_R$ symmetry we are now willing to re-align the two $SU(2)$ symmetries such that Dirac masses only break one of the combinations, there will be a remaining (diagonal) $SU(2)_D$ to protect T. To construct a realistic model of electroweak symmetry breaking we need to combine the electroweak symmetry and the custodial $SU(2)_D$ symmetry.

Let us first collect the maximal symmetry structure of the Standard Model. We start from the $SU(2)_L$ symmetry of the unbroken Lagrangian and expand it to $SU(2)_L \times SU(2)_R$ which protects the ρ parameter. In contrast to $SU(2)_L$ we do not need to gauge the global $SU(2)_R$, since we know there are no $SU(2)_R$ gauge bosons. But there is an additional gauged $U(1)_Y$ which we need for the abelian electromagnetic symmetry, under which left-handed and right-handed fermions are charged. So our unbroken electroweak symmetry can be viewed as a subset of the left-right symmetry $SU(2)_L \times SU(2)_R \supset SU(2)_L \times U(1)_Y$, where $SU(2)_R$ now needs to be gauged. In the presence of fermions we finally need to add another global symmetry which gives us the fermions' hypercharges. They need to be protected by a global symmetry to avoid anomalies, *i.e.* quantum effects violating the $(B-L)$ number conservation. Again, this $U(1)_{B-L}$ does not need to be gauged, unless we embed $U(1)_Y \subset SU(2)_R \times U(1)_{B-L}$. In our model we will start from this complete unbroken $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ gauge symmetry in the bulk. The five–dimensional gauge bosons we denote as $A_M^{(L)}, A_M^{(R)}, B_M$. On the two boundaries we will break this maximal symmetry group into the electroweak $SU(2)_L \times U(1)_Y$ and into the custodial $SU(2)_D$ subgroups.

We know how to break symmetries on the boundaries from the last section. For the massless B gauge boson we require Neumann BCs while for the massive $SU(2)_L$ gauge bosons we assume a mixed set:

$$\partial_5 B_\mu \Big|^{0,\pi R} = 0 \qquad \partial_5 A_\mu^{(L)} \Big|^{0} = 0 \qquad A_\mu^{(L)} \Big|^{\pi R} = 0 \tag{407}$$

This is the same model as before, which means it will wrongly give us $m_Z/m_W = 2$, so we have to modify this setup.
What we would hope to achieve is implementing the custodial $SU(2)_D$ on the boundary which describes our TeV-scale physics. For $y = \pi R$ we therefore replace $A^{(L,R)}$ by $(cA^{(R)} + sA^{(L)})$ and $(-sA^{(R)} + cA^{(L)})$ where the '+' combination corresponds to the unbroken $SU(2)_D$. The mixing angle we write in terms of $c \equiv g_{5,R}$ and $s \equiv g_{5,L}$. For the boundary conditions at $y = \pi R$ this implies:

$$\partial_5 B_\mu \Big|^{\pi R} = 0 \qquad \partial_5 \left(g_{5,L} A^{(L)} + g_{5,R} A^{(R)} \right) \Big|^{\pi R} = 0 \qquad \left(g_{5,L} A^{(L)} - g_{5,R} A^{(R)} \right) \Big|^{\pi R} = 0 \tag{408}$$

The still unbroken electroweak symmetry $SU(2)_L \times SU(2)_R \times U(1)_{B-L} \to SU(2)_L \times U(1)_Y$ we realize on the other boundary. Note that we will not discuss in detail how the large symmetry in the bulk will be broken on the two boundaries, but we do know how to write a non-linear sigma model on the boundaries. Also note that the electroweak symmetry $SU(2)_L \times U(1)_Y$ is not broken anywhere directly to $U(1)_Y$, this will happen by the boundary conditions automatically.
Our setup of the two $U(1)$ symmetries implies a mixing between B_μ and $A_\mu^{(R)}$. Again, we define the unbroken $U(1)_Y$ gauge boson as one of the linear combinations $(cA^{(R,3)} + sB)$ and break the other linear combination $(-sA^{(R,3)} + cB)$. The mixing angles are now $c \equiv g'_5$ and $s \equiv g_{5,R}$. This gives us for $y = 0$:

$$\partial_5 A_\mu^{(L)} \Big|^{0} = 0 \qquad \partial_5 \left(g_{5,R} B + g'_5 A^{(R,3)} \right) \Big|^{0} = 0 \qquad \left(g'_5 B - g_{5,R} A^{(R,3)} \right) \Big|^{0} = 0 \qquad A_\mu^{(R,12)} \Big|^{0} = 0 \tag{409}$$

The one remaining question is what to do with the two remaining fields $A^{(R)}$ at $y = 0$. We do not want a zero mode for the corresponding gauge fields, so we give them a Dirichlet BC there. This setup produces precisely the symmetries in Csaba's Fig. 3. Notice that we do not have to specify the boundary conditions for the scalar fifth components, because they are as usually fixed by exchanging Neumann and Dirichlet conditions.

We first see that each five–dimensional field combines different types of boundary conditions and that by construction the zero-mode photon will be built out of components of B_μ and $A_\mu^{(R,3)}$ mixed at $y = 0$ and $A_\mu^{(L,3)}$ and $A_\mu^{(R,3)}$ mixed at $y = \pi R$. This linear combination is the only field with purely Neumann boundary conditions. The physical Z boson will come from the '$-$' combination of $A_\mu^{(L,3)}$ and $A_\mu^{(R,3)}$ at $y = \pi R$, which has mixed boundary conditions. Its mass we can in principle compute:

$$m_Z^{(n)} = m_0 + \frac{n}{R} = m_0, \left(m_0 + \frac{1}{R} \right) \cdots \qquad\qquad m_0 = \frac{1}{\pi R} \arctan \sqrt{1 + \frac{g'^2}{g^2}} \tag{410}$$

The mass scale lifting the first Z mode off the zero mode is given in terms of the gauge couplings, which acted as the mixing angles in the rotation to a massless photon. This is really the same thing we know as the weak mixing angle in the Standard Model. Similarly, we can compute the W boson masses. To make the analysis of the KK states easier we can identify $g_{5,L} \equiv g_{5,R} = g$. This allows us to combine $A^{(L)}$ and $A^{(R)}$ into $A^{(\pm)}$, which should describe the $W^\pm$ gauge bosons.

$$m_W^{(n)} = \frac{2n+1}{4R} = \frac{1}{4R}, \frac{3}{4R} \cdots \qquad \Rightarrow \qquad \frac{m_W^{(0)}}{m_Z^{(0)}} = \frac{\pi^2}{16} \left[\arctan \sqrt{1 + \frac{g'^2}{g^2}} \right]^{-2} \sim 0.85 \qquad \Rightarrow \qquad \boxed{\rho = \frac{m_W^{(0)}}{c_w^2 m_Z^{(0)}} \sim 1.10} \tag{411}$$

This is really not bad a result. At this stage we will have to believe that we can adjust the result by for example bending our flat space and incorporating our setup in a Randall–Sundrum model. Such a model is built as a five–dimensional theory with two branes, usually referred to as a Planck brane at $y = 0$ and a TeV brane at $y = \pi R$. The difference will be that we cannot simply write sine and cosine Fourier series for the wave functions in the warped fifth dimension, but that we have to solve a differential equation which will give us Bessel functions (except for zero modes like the photon). In the usual RS language we can then play around with the location $y = b$ of the TeV brane and the warp factor k, to adjust the gauge boson masses. In Csaba's lecture he replaces the warp factor in the metric $\exp(-A(z)) = 1/(1 + kz)^2$ by $(R/z)^2$ with $R \sim 1/M_{\text{Planck}}$. The TeV-scale in the RS models arises as $M_{\text{Planck}} \exp(-kb)$ which can be written as $R' \sim 1/\text{TeV}$. In that case the KK mass scale is given in terms of $1/R'$, but including logarithms of the type $\log R/R'$ from the Bessel functions, so we have parameters to play with. Instead of discussing in detail how such a Randall–Sundrum embedding works we will move on and see how the KK towers of massive electroweak gauge bosons behave in the usual unitarity argument for a light fundamental Higgs boson.

4. Unitarity and KK excitations

One of the ways to introduce a Higgs boson is the complete unitarization of a theory with massive gauge bosons, *e.g.* from a non-linear sigma model. The classical example is the scattering process of longitudinal $W_L W_L \to W_L W_L$, where we can express the W polarization vector in terms of the energy and momentum as

$$\epsilon_\mu = \left(\frac{|\vec{p}|}{M}, \frac{E}{M} \frac{\vec{p}}{|\vec{p}|} \right) \propto E \qquad\qquad p_\mu^{(\text{in})} = \left(E, 0, 0, \pm\sqrt{E^2 - M^2} \right) \propto E$$

$$p_\mu^{(\text{out})} = \left(E, \pm\sqrt{E^2 - M^2} \sin\theta, 0, \pm\sqrt{E^2 - M^2} \cos\theta, \right) \propto E \tag{412}$$

We have indicated the energy behavior of the longitudinal components. If we now compute the scattering amplitude at high energies we find that for example the contact interaction contributes proportional to the maximum power $\mathcal{A} \propto E^4$. However, with the s, t, u-channel gauge-boson exchange diagrams this E^4 term cancels due to gauge invariance. What we are left with is $\mathcal{A} \propto E^2$, which still means that the transition amplitude diverges at high energies and will at some point violate perturbative unitarity. The old argument for the existence of a Higgs boson with a mass smaller than the scale at which unitarity is violated (the TeV scale) is that such a Higgs boson with all the proper couplings will unitarize the $W_L W_L \to W_L W_L$ scattering process. In my notes on Higgs searches you can see for example how to compute this behavior using the equivalence theorem between gauge bosons and Goldstone bosons. The obvious question is: how will our theory without any fundamental Higgs boson cure this fundamental problem with massive gauge bosons?

Csaba explicitly writes the form for the leading E^4 term in the amplitude of four Standard–Model gauge boson with index n and the exchanged KK tower k:

$$\mathcal{A}^{(4)} = i \left(g_{nnnn}^2 - \sum_k g_{nnk}^2 \right) \left[f^{abe} f^{cde} \left(3 + 6\cos\theta - \cos^2\theta \right) + 2 f^{ace} f^{bde} \left(3 - \cos^2\theta \right) \right] \tag{413}$$

No masses appear in this form, only coupling constants. This dangerous contribution vanishes only if the couplings fulfill the appropriate sum rule. The coupling between different KK modes is given by the overlap of their wave functions in the fifth dimension

$$g_{mnk} = g_5 \int dy \; f_m(y) f_n(y) f_k(y) \qquad\qquad g_{mnkl} = g_5^2 \int dy \; f_m(y) f_n(y) f_k(y) f_l(y) \tag{414}$$

The Fourier transforms of the wave functions have a completeness relation

$$\sum_k f_k(y) f_k(z) = \delta(y-z) \tag{415}$$

which we can use to show the couplings sum rule starting from the left-hand side

$$\begin{aligned} \sum_k g_{nnk}^2 &= g_5^2 \sum_k \left(\int_0^{\pi R} dy \; f_n^2(y) f_k(y) \right) \left(\int_0^{\pi R} dz \; f_n^2(z) f_k(z) \right) \\ &= g_5^2 \int_0^{\pi R} dy \int_0^{\pi R} dz \; f_n^2(y) f_n^2(z) \left(\sum_k f_k(y) f_k(z) \right) \\ &= g_5^2 \int_0^{\pi R} dy \int_0^{\pi R} dz \; f_n^2(y) f_n^2(z) \delta(y-z) \\ &= g_5^2 \int_0^{\pi R} dy \; f_n^4(y) \qquad \Rightarrow \qquad \boxed{\sum_k g_{nnk}^2 = g_{nnnn}} \end{aligned} \tag{416}$$

Assuming that this sum rule — which really does not have anything to do with a Higgs boson, only with gauge invariance between 3-point and 4-point couplings — we can write a compact form of the second diverging term in the amplitude:

$$\mathcal{A}^{(2)} = \frac{i}{m_n^2} \left(4 g_{nnnn} m_n^2 - 3 \sum_k g_{nnk}^2 m_k^2 \right) \left[- f^{abe} f^{cde} \sin^2\frac{\theta}{2} + f^{ace} f^{bde} \right] \stackrel{!}{=} 0 \tag{417}$$

Again, there is a mass–couplings sum rule given by the first parentheses. It involves KK masses as well as the gauge couplings, which is different from the Higgs mechanism. In other words, the KK tower with all couplings fixed properly plays the role of the Higgs boson in the Standard Model. The problem is that while the Higgs mass can be chosen such that its effects come in beyond the scale of unitarity violation, the KK tower involves an infinite sum over states with arbitrarily high masses. This implies a cutoff scale of our effective theory, but then we always knew there would be such a cutoff, namely the fundamental Planck scale, above which we cannot use the KK effective theory to compute scattering effects.

If we had more time we would at this point need to talk about fermion masses in this model. The problem starts long before writing down Yukawa terms in five dimensions, namely with the extension of chiral fermions into more than four dimensions. In four dimensions spinors are another representation of the Lorentz group. We can express the 4×4 matrices γ_μ in terms of the 2×2 Pauli matrices σ_j and $-\mathbb{1}$ and define the transformation

$$x^\mu \to [x] = x^0 - x^j \sigma^j = \begin{bmatrix} x^0 - x^3 & -x^1 + i x^2 \\ -x^2 - i x^2 & x^0 + x^3 \end{bmatrix} \tag{418}$$

which is nothing but a Lorentz transformation. When we write fermions in five dimensions we need to extend the corresponding Dirac gamma–matrix basis $\gamma_\mu \to \gamma_M$. There is even a candidate for the fifth gamma matrix, namely γ_5. The problem is that this γ_5 appears in the chiral projectors $(\mathbb{1} \pm \gamma_5)/2$, which means that it mixes chiralities. This means that Lorentz transformations to not respect chirality. If we write the Dirac equation in five dimensions, the derivative ∂_5 will just like the mass term mix left-handed and right-handed Weyl fermions. Once we Fourier-transform the fifth dimensions into a KK tower this is not surprising — after all ∂_5 is nothing but a mass term. But to learn more about writing down Yukawa couplings and making them into fermion masses you will need to read Csaba's review or some of the original papers for example by Tim Tait and friends...

One last word concerning these fermions. From extended technicolor we remember that giving the top quark a mass using a dimension–six operator leads to problems with the effective $Zb\bar{b}$ coupling. This happens because of the $SU(2)_L$ symmetry in combination with a chiral or custodial $SU(2)_R$ symmetry. In extra–dimensional models we will define a mass for all fermions via their position in the fifth dimension and a wave–function overlap with something playing the role of a sigma field. By construction, we incorporate the $SU(2)_L$ and the $SU(2)_D$ symmetries, which means we will run into precisely the same problem as extended technicolor did. Unless our really bright model–building colleagues manage to solve this problem at some stage.

D. Literature

In particular on the first part of the course there are many lectures available by all the experts on the field. As usual, I find the TASI lecture notes the most useful, but not the only good source

– A very extensive introduction into technicolor and its successors can be found in hep-ph/0203079. Note that this writeup is almost 200 pages long, but at least the first half of them are really instructive. Most of my notes on technicolor are based on this review.

– A shorter and also very modern introduction into technicolor is Sekhar Chivucula's hep-ph/0011264. If you have already understood something and would like to refresh your memory on the ideas behind technicolor, it is great.

– The short introduction on electroweak symmetry breaking from boundary conditions is based on Csaba Csaki's, Jay Hubisz's and Patrick Meade's TASI lecture hep-ph/0510275. I have no idea how Csaba managed to teach all this material in four lectures, but I always had the suspicion that he is an extraordinarily good teacher.

– And finally, for an introduction to electroweak precision data there is the usually nicely written TASI lecture, in that case by James Wells: hep-ph/0512342. James even teaches how to compute loops leading to S and T contributions, so go and have a look.