Perturbative Renormalizability of ϕ_6^3 by renormalization group differential equations*

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Abstract

I discuss the setup and details of proofs of perturbative renormalizability by renormalization group differential equations. As an example, I show that ϕ^3 theory in six dimensions is perturbatively renormalizable.

1 Introduction

In these notes, I give a largely selfcontained introduction to the flow equation method as a tool for proving perturbative renormalizability of field theories. I apply it to ϕ^3 theory in 6 dimensions because this is an example at which Connes and Kreimer exhibited the Hopf algebra structure underlying BPHZ renormalization. Since then it the Hopf algebra method has been applied to many other examples; ϕ^3 was probably chosen because it has the simplest graphical structure.

In many ways, the method presented here is complementary to the Connes–Kreimer method. One of its main advantages is that one can do all proofs without even talking of Feynman graphs, but that one can also use it to generate the Feynman graph expansion in all detail. Indeed, the Brydges–Kennedy tree formula [1], a natural outcome of integrating the RG differential equations, provides a particularly convenient arrangement of perturbation theory.

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From the point of view of nonperturbative quantum field theory, the focus on all–order graphical expansions seems almost anachronistic in bosonic theories, and also, dimensional regularization should not be regarded as fundamental to renormalization because it is up to now a method that is strictly tied to perturbation theory in graphical terms, whereas other methods of renormalization have already been applied in a nonperturbative setting. It would certainly be a major breakthrough if dimensional regularization could be established as a nonperturbative method. One hope in this direction comes from noncommutative field theory.

An aspect not captured in formal perturbation expansions is the need to split field space in small–field and large–field regions. It is possible to disentangle this split, which is fundamental in bosonic theories, from the renormalization problem of putting counterterms as it arises in perturbation theory. This has been done in pioneering work and used for the mathematical construction of the infrared limit of ϕ^4 theory in four dimensions (i.e. the theory with a fixed UV cutoff, but zero mass) and the Gross–Neveu model in 2 and $2+\varepsilon$ dimensions [7, 8, 9]. The Wilsonian flow equation method discussed here shares at least the basic idea with these hard analysis proofs, but it is substantially simpler because it is applied only in a perturbative setting here.

For the example to be discussed below, ϕ^3 theory, the partition function cannot even be written in a completely regularized version of the field theory on a Euclidean lattice with finitely many points because the ϕ^3 action is not bounded below, so this example is usually not considered. For perturbative considerations this problem is inessential: one can add a term $\delta\,\phi^4$, $\delta > 0$ very small, to the action at cutoff scale. Then the action is bounded below, the partition function makes sense, and one can take $\delta \to 0$ at fixed cutoff in the perturbative equations for the Green functions. Below I shall also show that one can add a ϕ^4 term with an inverse power of the ultraviolet cutoff instead of δ as a prefactor without spoiling the property of renormalizability.

In Section 2 I introduce the generating functions of field theory and describe the renormalization problem very briefly. In Section 3 I give a short derivation of the flow equation and in Section 4 I discuss its application to ϕ^3_6. I will not provide a general introduction to renormalization since this has been done at many places. The RGDE method of renormalization has been described in detail in [2, 3]. I will use notations and conventions from [2].

2 The problem of renormalizability

In this section, I describe renormalizability of field theories in the sense of finiteness of the Green functions of the theory in the limit as a regulating momentum space cutoff is removed. This is the minimal condition for a theory to be called renormalizable. In theories with large symmetry groups of the classical Lagrangian, such as gauge theories, finiteness of the limit is only a starting point. One then has to prove that the limiting function satisfy the Ward or Slavnov-Taylor identities to know that they are really gauge theories. Proofs of this are rather nontrivial add–ons to the few statements shown here even for the simplest gauge theories. Such proofs have, however, been given using the method described here [17, 18, 19, 20].

2.1 Generating functions

The combinatorial structure of quantum field theories is best captured by using generating functions, e.g. those for the connected Green functions or the vertex functions. To make these generating functions well–defined, I put the theory on a finite lattice of lattice spacing ε and sidelength L. Then the functional integral for the generating function becomes an ordinary integral, and functional derivatives simply become ordinary partial derivatives, scaled by an inverse power of ε. This integral for the generating function is convergent provided the action is bounded below. All this is explained in detail e.g. in [2]. Given this finite–dimensional integral, one can then work with generating functions that are well–defined and derive equations for the Green functions.

The first natural mathematical question about the ultraviolet problem of a thus defined regularized field theory is whether the function $\mathcal{G}$ exists in the continuum limit $\varepsilon \to 0$ *if the interaction* $\mathcal{V}$ *is adjusted appropriately as a function of* ε (without this adjustment, the limit does not exist). The limit has been proven to exist for a variety of theories in two and three dimensions but not yet for a four–dimensional theory. The reason why this is so in ϕ^4 theory is that the theory is not asymptotically free in the ultraviolet, and most likely to be noninteracting [10, 11, 12]. In Yang–Mills theory, the technical problems remain formidable despite pioneering work in which stability was shown [13].

Here I only study perturbative renormalizability, i.e. the following question. Let g be a formal parameter and replace $\mathcal{V}$ by $g\mathcal{V}$. Can one prove that the continuum limit of $\mathcal{G}$ exists as a formal power series in g, provided that only terms of a fixed functional form (usually a local polynomial of fixed degree in the field and its derivatives) are added to the bare interaction? As everyone knows, this question has been answered in various schemes, most of which do not refer to the lattice regularization. A direct proof that the continuum limit of a lattice theory exists in formal perturbation theory to all orders was given using BPHZ methods [14]. A signficiant complication in that proof as compared to continuum proofs is that on the lattice the theory does not have full Euclidian invariance, so there are many more possible counterterms.

To avoid these problems for the discussion of the flow equation proof, I first introduce a momentum space ultraviolet cutoff Λ_0. In presence of this cutoff, the lattice regulator can be removed rather easily and Euclidian symmetry is restored in the limit. In a second step, the cutoff Λ_0 is removed using the counterterms of the continuum theory and a flow equation technique. Thus, in the present setting, the only purpose of the lattice is to provide well–defined generating functions in a straightforward way. Once equations for the cutoff Green functions have been obtained, the infinite–volume and continuum limit will be taken, and the dependence on Λ_0 is studied afterwards.

Let $\varepsilon > 0$, and $L \gg 1$ be such that $L/(2\varepsilon) \in \mathbb{N}$. The finite lattice is the torus $\Gamma_{L,\varepsilon} = \varepsilon\mathbb{Z}^d/L\mathbb{Z}^d$, i.e. I choose periodic boundary conditions with period L. A field configuration $\phi \in \mathbb{R}^{\Gamma_{L,\varepsilon}}$ of a real scalar field is simply a map from $\Gamma_{L,\varepsilon}$ to $\mathbb{R}$. We assume the action to be of the form

$$S_0(\phi) = \frac{1}{2}(\phi, Q\phi) + \mathcal{V}(\phi) \tag{1}$$

where Q defines a strictly positive quadratic form and $\mathcal{V}$ is bounded below: $\mathcal{V}(\phi) \geq -v$ for all $\phi \in \mathbb{R}^{\Gamma_{L,\varepsilon}}$. In the examples of interest, the action is the discretization of a local continuum action: let ∇ denote the forward lattice derivative $(\nabla f)_k(x) = \varepsilon^{-1}(f(x+\varepsilon e_k) - f(x))$ (with e_k the unit vector in direction $k \in \{1, \ldots, d\}$). Then

$$(\phi, Q\phi) = \int_{\Gamma_{L,\varepsilon}} \left((\nabla\phi)^2 + \mu^2\phi^2\right) \tag{2}$$

with m the mass of the scalar field and

$$\mathcal{V}(\phi) = \int_{\Gamma_{L,\varepsilon}} \mathrm{d}x \; V(\phi(x)) \tag{3}$$

where $V : \mathbb{R} \to \mathbb{R}$ is bounded below. Here I have used the shorthand notation $\int_{\Gamma_{L,\varepsilon}} \mathrm{d}x \; f(x) = \varepsilon^d \sum_{x\in\Gamma_{L,\varepsilon}} f(x)$. I also denote $(f, g) = \int_{\Gamma_{L,\varepsilon}} \bar{f} g$.

The generating function W for the connected Green functions is defined by

$$W(J) = -\log \int \prod_{x \in \Gamma_{L,\varepsilon}} \mathrm{d}\phi(x)\ \mathrm{e}^{-S(\phi)+(J,\phi)} \tag{4}$$

The source terms J can be used to calculate m–point functions by taking derivatives with respect to J in the usual way. A function arising naturally in the Wilsonian flow is the Wilsonian effective potential

$$\mathcal{G}(\phi) = -\log \int \mathrm{d}\mu_C(\phi')\ \mathrm{e}^{-\mathcal{V}(\phi'+\phi)} \tag{5}$$

where $\mathrm{d}\mu_C$ is the normalized Gaussian measure on $\mathbb{R}^{\Gamma_{L,\varepsilon}}$ with mean zero and covariance $C = Q^{-1}$. Since Q was assumed to be strictly positive, C exists. By completion of the square,

$$\mathcal{G}(\phi) = W(C^{-1}\phi) + \frac{1}{2}(\phi, C^{-1}\phi) + \log\det(\pi C)^{1/2} \tag{6}$$

so studying W and $\mathcal{G}$ is equivalent. Up to the explicit ϕ–independent term from the normalization and the quadratic term, the only difference between the two functions is that the source terms in $\mathcal{G}(\phi)$ and $W(J)$ are related by $J = C^{-1}\phi$. In graphical expansions this means that external propagators are amputated by the free propagator. $\mathcal{G}$ is therefore the generating function for the connected *amputated* Green functions. In the following I shall focus on $\mathcal{G}$ and its Wick ordered counterpart (called $\mathcal{H}$ below).

Let $C = (-\Delta + \mu^2)^{-1}$ be the propagator of the scalar field with mass $\mu > 0$ defined above. Let Λ_0 be an ultraviolet cutoff, imposed by changing the propagator C such that it vanishes unless momentum is smaller than a constant times Λ_0 (specified below). Assume that g is a formal variable and the interaction $\mathcal{V}$ is given by

$$\mathcal{V}(\phi) = g \int \mathrm{d}x\ \left(\zeta_{\Lambda_0}(\nabla\phi(x))^2 + \nu_{\Lambda_0}\phi(x)^2 + \mathcal{P}_{\Lambda_0}(\phi(x))\right) \tag{7}$$

where $\mathcal{P}_{\Lambda_0}$ is a polynomial of a fixed degree p whose coefficients are formal power series in g that are allowed to depend on Λ_0. The coefficients ζ_{Λ_0} and ν_{Λ_0} are formal power series in g with Λ_0–dependent coefficients, too. The highest coefficient in $\mathcal{P}$ is required to be of the form $g+O(g^2)$. Expand $\mathcal{G}(\phi)$ as a formal power series in g and in the fields. I show below that this expansion is

$$\mathcal{G}(\phi) = \sum_{r=1}^{\infty} g^r \sum_{m=0}^{\bar{m}_r} \int \mathrm{d}x_1 ... \mathrm{d}x_m\ G^{\Lambda_0}_{m,r}(x_1, ..., x_m)\phi(x_1)...\phi(x_m) \tag{8}$$

(i.e. in every order in g the dependence on ϕ is polynomial) and that for any $\mathcal{P}_{\Lambda_0}$ that is bounded below, the limits $L \to \infty$ and $\varepsilon \to 0$ of the $G^{\Lambda_0}_{m,r}$ exist at fixed Λ_0 and are Euclidian, i.e. translation and $O(d)$, invariant. The general perturbative renormalizability question is then: for given d, what is the maximal degree of a polynomial whose coefficients can be adjusted as functions of Λ_0 such that the limit of all $G^{\Lambda_0}_{m,r}$ as $\Lambda_0 \to \infty$ exists; how many parameters have to be included in the polynomial, and in what sense are they unique? (norms in which the limit is to be taken are given in Subsection 3.8).

The answer is well–known: For $d = 2$, any degree p will do. For $d = 3$, $p \leq 6$, for $d = 4$, $p \leq 4$, and for $d = 6$, $p \leq 3$. Whenever the degree is the maximally allowed one, all coefficients of monomials of degree $\leq p$ that are not forbidden by Euclidian symmetry or the symmetry $\phi \to -\phi$ (if present) need to be adjusted in a Λ_0–dependent way to obtain a finite limit of the $G^{\Lambda_0}_{m,r}$. If the degree is not maximal (*superrenormalizable* theories) some terms are not needed; e.g. ϕ^4 in $d = 3$ requires only a mass counterterm ν_{Λ_0} but no field renormalization ζ_{Λ_0}. Uniqueness is achieved by imposing one condition for each parameter in the action that is necessary for finiteness (i.e. each counterterm). Posing such *renormalization conditions* is possible consistently to all orders in g.

For a RGDE proof of these statements for $d \leq 4$, see [2]. In this paper I give the proof for ϕ^3 in $d = 6$. It will also become clear in the proof that higher powers of ϕ can always be added, *provided they are scaled with an appropriate inverse power of the ultraviolet cutoff* Λ_0.

3 The RG equation

3.1 Basic identities for Gaussian integration

These identities are standard (see, e.g. [2]) so I will just recapitulate them very briefly.

Wick's theorem. Let P be an arbitrary polynomial in the fields. Then

$$(\mu_C * P)(\phi) = \int \mathrm{d}\mu_C(\phi')\; P(\phi' + \phi) = \mathrm{e}^{\frac{1}{2}\Delta_C} P(\phi) \tag{9}$$

where

$$\Delta_C = \left(\frac{\delta}{\delta\phi}, C\frac{\delta}{\delta\phi} \right) = \int_{\Gamma_{L,\varepsilon}} \mathrm{d}x \int_{\Gamma_{L,\varepsilon}} \mathrm{d}y\; \frac{\delta}{\delta\phi(x)} C(x,y) \frac{\delta}{\delta\phi(y)} \tag{10}$$

Here $\frac{\delta}{\delta\phi(x)} = \varepsilon^{-d}\frac{\partial}{\partial\phi(x)}$, as already discussed above, so that this Laplacian is finite–dimensional, and the exponential is defined by its expansion (which is well–defined since it terminates after a finite number of terms when acting on a polynomial).

For a proof, see, e.g. [2]. In a nutshell, this identity holds because Gaussian convolution describes the solution of a heat equation associated to this Laplacian. It is called Wick's theorem because the formulation of Wick's theorem in terms of contractions follows from it simply by expanding the exponential and doing the derivatives. One advantage over the usual formulation of Wick's theorem is that the same identity holds for fermions, with the only change that the derivatives and monomials live on a Grassmann algebra.

Equation (9) also makes it particularly obvious that Gaussian convolution defines a semigroup: if $C = C_1 + C_2$, then

$$\mu_C * P = \mu_{C_1} * (\mu_{C_2} * P). \tag{11}$$

This simple identity, applied many times, allows us to split covariances that have singularities into many regular pieces and define a renormalization group flow as an iteration of Gaussian convolutions, sometimes combined with suitable extraction and rescaling operations. The continuous RG generated by the RGDE belongs to a continuous decomposition of the covariance.

It is tempting to conclude from Wick's theorem that

$$\mathrm{e}^{-\mathcal{G}(\phi)} = \mathrm{e}^{\frac{1}{2}\Delta_C}\mathrm{e}^{-\mathcal{V}(\phi)} \tag{12}$$

but this identity is formal because the exponentials are not polynomials, and indeed the expansion is divergent for bosons(for fermions on a finite lattice, it still makes sense since the maximal degree in a finitely generated Grassmann algebra is finite). If $\mathcal{V}$ is replaced by $g\mathcal{V}$ and g is regarded as a formal expansion variable, (12) uniquely defines the formal power series

$$\mathcal{G}(\phi) = \sum_{r=1}^{\infty} g^r G^{(r)}(\phi) \tag{13}$$

by recursive solution (in r) of

$$\left[\exp\left(-\sum_{s=1}^{r} g^s G^{(s)}(\phi)\right)\right]_{O(g^r)} = \mathrm{e}^{\frac{1}{2}\Delta_C}(-g\mathcal{V}(\phi))^r. \tag{14}$$

with the subscript indicating that the series is truncated to the polynomial of degree r in g. Given that this is a recursion, it is clear that $\mathcal{G}^{(r)}$ is of degree at most rp if $\mathcal{V}$ is a polynomial of degree p in ϕ. The recursive structure of the equation is also obvious: the left hand side equals

$$1 - g^r \mathcal{G}^{(r)}(\phi) + \left[\exp\left(-\sum_{s=1}^{r-1} g^s G^{(s)}(\phi) \right) \right]_{O(g^r)} \tag{15}$$

so it can be solved easily for $\mathcal{G}^{(r)}$ in terms of the $\mathcal{G}^{(s)}$ with $s < r$. The solution of the recursion is the *connected part* of the right hand side which also has a tree representation [4]. The connections by the lines of the tree decrease the degree of $\mathcal{G}^{(r)}$ further (see below). A suitable topology for defining convergence of iterations on formal power series rings is given in Appendix A of [2].

3.2 Polchinski's equation

Let $C = C_\Lambda$ depend differentiably on another parameter Λ, but $\mathcal{V}$ remain independent of Λ. Then the effective potential $\mathcal{G}$ also depends on Λ; I denote it by $\mathcal{G}_\Lambda$. Differentiating the exponential of the Laplacian and using that the Laplacian commutes with its derivative gives Polchinski's equation [5]

$$\dot{\mathcal{G}}_\Lambda = \frac{1}{2} \Delta_{\dot{C}_\Lambda} \mathcal{G}_\Lambda - \frac{1}{2} \left(\frac{\delta \mathcal{G}_\Lambda}{\delta \phi}, \dot{C}_\Lambda \frac{\delta \mathcal{G}_\Lambda}{\delta \phi} \right) \tag{16}$$

(derivatives with respect to Λ are denoted by dots). Formally (and for fermions, rigorously) this follows by applying a derivative w.r.t. Λ to (12). It can also be shown easily using the definition of $\mathcal{G}$ as a Gaussian convolution.

Polchinski's equation was known to others before; however, he deserves credit for giving an explicit argument for perturbative renormalizability using this equation. I will in the following discuss a rigorous version of his argument, which is further simplified by using Wick ordering. For this it is now convenient to go to the specific choice of the action (2) and also to specify the way the parameter Λ enters as a cutoff. Most results do not depend on that specific choice, and using a flow parameter Λ that is not a cutoff parameter can be very useful, see, e.g. [6].

3.3 Ultraviolet cutoff and flow parameter

Fix $\Lambda_0 > 0$ (very large since it plays the role of an ultraviolet cutoff). For $0 \le \Lambda \le \Lambda_0$, let

$$D_\Lambda(p) = \int \mathrm{d}p \; C(p)\chi_\Lambda(p) \tag{17}$$

where $\chi_\Lambda \in C_0^\infty(\mathbb{R}^d, [0,1])$ is equal to one in the ball of radius Λ around zero and zero outside the ball of radius 2Λ around zero, and

$$C_\Lambda(p) = D_{\Lambda_0}(p) - D_\Lambda(p). \tag{18}$$

By the properties of χ_Λ, C_Λ is nonzero only for momenta in the range $\Lambda \le |p| \le 2\Lambda_0$. Thus Gaussian convolution with $\mathrm{d}\mu_{C_\Lambda}$ means, roughly speaking, integrating the fields with momenta in that range.

3.4 The Wick ordered RGDE

Because $C_\Lambda + D_\Lambda = D_{\Lambda_0}$, $\dot{C}_\Lambda = -\dot{D}_\Lambda$, so Polchinski's equation reads in terms of D_Λ

$$\dot{\mathcal{G}}_\Lambda = -\frac{1}{2}\Delta_{\dot{D}_\Lambda}\mathcal{G}_\Lambda + \frac{1}{2}\left(\frac{\delta \mathcal{G}_\Lambda}{\delta\phi}, \dot{D}_\Lambda \frac{\delta \mathcal{G}_\Lambda}{\delta\phi}\right) \tag{19}$$

To expand in Wick ordered monomials, I define

$$\mathcal{H}_\Lambda = \mathrm{e}^{\frac{1}{2}\Delta_{D_\Lambda}}\; \mathcal{G}_\Lambda \tag{20}$$

Then the linear term cancels in the equation for $\mathcal{H}_\Lambda$ and

$$\dot{\mathcal{H}}_\Lambda = \frac{1}{2}\mathrm{e}^{\frac{1}{2}\Delta_{D_\Lambda}}\left(\mathrm{e}^{-\frac{1}{2}\Delta_{D_\Lambda}}\frac{\delta \mathcal{H}_\Lambda}{\delta\phi}, \dot{D}_\Lambda \mathrm{e}^{-\frac{1}{2}\Delta_{D_\Lambda}}\frac{\delta \mathcal{H}_\Lambda}{\delta\phi}\right). \tag{21}$$

I now use a combinatorial trick, writing

$$\left(\frac{\delta A}{\delta\phi}, \dot{D}_\Lambda \frac{\delta B}{\delta\phi}\right) = \mathbb{E}_{12}\left[\Delta^{(1,2)}_{\dot{D}_\Lambda} A(\phi^{(1)})B(\phi^{(2)})\right] \tag{22}$$

where $\mathbb{E}_{12}\left[\cdot\right]$ is evaluation at $\phi^{(1} = \phi^{(2)} = \phi$ and

$$\Delta^{(i,j)}_{\dot{D}_\Lambda} = \left(\frac{\delta}{\delta\phi^{(i)}}, \dot{D}_\Lambda \frac{\delta}{\delta\phi^{(j)}}\right). \tag{23}$$

Because

$$\frac{\delta}{\delta\phi}\mathbb{E}_{12}\left[A(\phi^{(1)})B(\phi^{(2)})\right] = \mathbb{E}_{12}\left[\left(\frac{\delta}{\delta\phi^{(1)}} + \frac{\delta}{\delta\phi^{(2)}}\right) A(\phi^{(1)})B(\phi^{(2)})\right] \tag{24}$$

the right hand side of (21) can now be written as

$$
\begin{aligned}
& \frac{1}{2} \mathrm{e}^{\frac{1}{2}\Delta_{D_\Lambda}} \mathbb{E}_{12} \left[\Delta^{(1,2)}_{\dot{D}_\Lambda} \mathrm{e}^{-\frac{1}{2}\Delta_{D^{(1,1)}_\Lambda}} \mathrm{e}^{-\frac{1}{2}\Delta_{D^{(2,2)}_\Lambda}} \mathcal{H}_\Lambda(\phi^{(1)}) \mathcal{H}_\Lambda(\phi^{(2)}) \right] \\
= & \frac{1}{2} \mathbb{E}_{12} \left[\mathrm{e}^{\frac{1}{2}\sum_{i,j=1}^{2} \Delta^{(i,j)}_{D_\Lambda}} \Delta^{(1,2)}_{\dot{D}_\Lambda} \mathrm{e}^{-\frac{1}{2}\Delta_{D^{(1,1)}_\Lambda}} \mathrm{e}^{-\frac{1}{2}\Delta_{D^{(2,2)}_\Lambda}} \mathcal{H}_\Lambda(\phi^{(1)}) \mathcal{H}_\Lambda(\phi^{(2)}) \right] \\
= & \frac{1}{2} \mathbb{E}_{12} \left[\Delta^{(1,2)}_{\dot{D}_\Lambda} \mathrm{e}^{\Delta_{D^{(1,2)}_\Lambda}} \mathcal{H}_\Lambda(\phi^{(1)}) \mathcal{H}_\Lambda(\phi^{(2)}) \right]. \qquad (25)
\end{aligned}
$$

Thus

$$
\dot{\mathcal{H}}_\Lambda(\phi) = \frac{1}{2} \mathbb{E}_{12} \left[\partial_\Lambda \left(\mathrm{e}^{\frac{1}{2}\Delta_{D^{(1,2)}_\Lambda}} - 1 \right) \mathcal{H}_\Lambda(\phi^{(1)}) \mathcal{H}_\Lambda(\phi^{(2)}) \right]. \qquad (26)
$$

Wick ordering removes self–contractions with “soft” lines D_Λ. In the limit $\Lambda \to 0$, $D_\Lambda \to 0$, so $\mathcal{H}_0 = \mathcal{G}_0$. Thus studying $\mathcal{H}$ is the same as studying $\mathcal{G}$ in the limit $\Lambda \to 0$. At positive Λ, $\mathcal{G}_\Lambda$ can be obtained from $\mathcal{H}_\Lambda$ by moving the exponential of the Laplacian to the other side of (20).

3.5 Expansion in the fields

The coefficients of an expansion of $\mathcal{H}_\Lambda$ in the fields are the connected amputated Wick–ordered Green functions. The expansion is

$$
\mathcal{H}_\Lambda(\phi) = \sum_{r=1}^{\infty} g^r \sum_{m=0}^{\bar{m}_r} \int \mathrm{d}x_1 \dots \mathrm{d}x_m \; H^{\Lambda\Lambda_0}_{m,r}(x_1, \dots, x_m) \phi(x_1) \dots \phi(x_m) \qquad (27)
$$

Here I have included a superscript Λ_0 to exhibit the dependence of these functions on Λ_0. I also write this in a more shorthand notation, using $X = (x_1, \dots, x_m)$, as

$$
\mathcal{H}_\Lambda(\phi) = \sum_{r=1}^{\infty} g^r \sum_{m=0}^{\bar{m}_r} \int \mathrm{d}^m X H^{\Lambda\Lambda_0}_{m,r}(X) \phi^m(X). \qquad (28)
$$

For every r, the summation over m is finite because in an expansion

$$
\mathcal{H}_\Lambda(\phi) = \sum_{r=1}^{\infty} g^r \; \mathcal{H}^{(r)}(\phi) \qquad (29)
$$

every $\mathcal{H}^{(r)}(\phi)$ is a polynomial in ϕ. This follows immediately from an expansion in g as discussed after (12), since I am considering the case where $\mathcal{V}(\phi)$ is a polynomial of a fixed degree.

Graphically, these functions are connected, so a bound for $\bar{m}_r$ can be obtained by considering tree graphs with r vertices. If the degree of the interaction as a polynomial in ϕ is p, then each vertex can have at most p legs, and

$$\bar{m}_r \le rp - 2(r-1) = 2 + (p-2)r \tag{30}$$

(a tree on r vertices has $r-1$ lines, each of which binds two legs).

Thus, in formal perturbation theory, no convergence questions for expansions in the fields arise because all the functions appearing are polynomials in ϕ. Because the fields commute, I choose the coefficient functions to be symmetric under permutations:

$$H_{m,r}^{\Lambda\Lambda_0}(x_{\pi(1)}, \ldots, x_{\pi(m)}) = H_{m,r}^{\Lambda\Lambda_0}(x_1, \ldots, x_m) \tag{31}$$

for all permutations π of $\{1, \ldots, m\}$. For fermions, the $H_{m,r}^{\Lambda\Lambda_0}$ would be chosen antisymmetric. This choice allows to compare coefficients and to obtain an equation for the $H_{m,r}^{\Lambda\Lambda_0}$ in which no fields appear any more. The algebra to do this is straightforward: expand

$$\mathrm{e}^{\Delta_{D_\Lambda}^{(1,2)}} - 1 = \sum_{k \ge 1} \frac{1}{k!} \left(\Delta_{D_\Lambda}^{(1,2)} \right)^k \tag{32}$$

and use that by symmetry

$$\begin{aligned} &\frac{\delta^k}{\delta\phi(x_1)\ldots\delta\phi(x_m)} \int \mathrm{d}^m X\, H_{m,r}^{\Lambda\Lambda_0}(X)\phi^m(X) \\ &= k! \binom{m}{k} \int \mathrm{d}^{m-k} X' H_{m,r}^{\Lambda\Lambda_0}(x_k, \ldots, x_1, X')\phi^{m-k}(X'). \end{aligned} \tag{33}$$

Reordering, symmetrizing and comparing coefficients gives

$$\begin{aligned} \partial_\Lambda H_{m,r}^{\Lambda\Lambda_0}(X) \;=\; & \frac{1}{2}\mathbb{S}_m^{(\pm)} \sum_{\substack{r_1 \ge 1, r_2 \ge 1 \\ r_1 + r_2 = r}} \sum_{\substack{m_1 \ge 1, m_2 \ge 1, k \ge 1 \\ m_1 + m_2 = m + 2k}} k! \binom{m_1}{k} \binom{m_2}{k} \\ & \int \mathrm{d}^k Y \int \mathrm{d}^k Y' \, \partial_\Lambda \left(\prod_{i=1}^k D_\Lambda(y_i, y_i') \right) \\ & H_{m,r}^{\Lambda\Lambda_0}(y_1, \ldots, y_k, X^{(1)}) H_{m,r}^{\Lambda\Lambda_0}(y_k', \ldots, y_1', X^{(2)}) \end{aligned} \tag{34}$$

with $X^{(1)} = (x_1, \ldots, x_{m_1-k})$ and $X^{(2)} = (x_{m_1-k+1}, \ldots, x_m)$. Here $\mathbb{S}_m^{(\pm)}$ denotes (anti)symmetrization in $(x_1, \ldots, x_m)$.

Equation (34) is the RGDE in component form. Together with the initial interaction imposed by the choice of theory and counterterms and the renormalization conditions, which fix the counterterms, it contains the information about the model to all orders r in formal perturbation theory.

Although the equation itself looks complicated, it has a number of very nice features. Most importantly, it is recursive in r: the conditions $r_1 + r_2 = r$ and $r_i \geq 1$ in the sum on the right hand side imply $r_i < r$. Statements about the Green functions in perturbation theory are statements about the $H_{m,r}^{\Lambda\Lambda_0}$ for every r. These statements will be proven by induction; in the inductive step of showing them for $H_{m,r}^{\Lambda\Lambda_0}$, the inductive hypothesis for $H_{m,r_i}^{\Lambda\Lambda_0}$ can then be used on the right hand side.

Both the functional and the component form of the RG have their advantages. The functional form is structurally and conceptually simple. The component form contains only functions of finitely many arguments, so the auxiliary lattice regulator introduced above can be removed. Once this is done, one could take the point of view that the limiting system of equations, together with the initial condition given as the form of the cutoff interation and with the renormalization conditions, *defines* the theory. This is possible because the system of differential equations can be solved recursively to give the perturbation expansion. This way one can avoid any mention of functional integrals etc., but in my opinion, the functional approach provides much insight in the structure of the theory, just as generating functions generally do in combinatorics.

3.6 Translation invariance and momentum conservation

The proof of the following simple lemma is a baby example how the recursive structure of the RGDE is used in inductive proofs.

Lemma 1 *Assume that translation invariance holds for the initial interaction and the propagator: for all $r \in \mathbb{N}$ and all $a \in \varepsilon\mathbb{Z}^d$,*

$$H_{m,r}^{\Lambda_0\Lambda_0}(x_1 + a, \ldots, x_m + a) = H_{m,r}^{\Lambda_0\Lambda_0}(x_1, \ldots, x_m) \tag{35}$$

and for all Λ, $D_\Lambda(x + a, y + a) = D_\Lambda(x, y)$. Then for all $\Lambda \leq \Lambda_0$, $H_{m,r}^{\Lambda\Lambda_0}$ is translation invariant, i.e.

$$H_{m,r}^{\Lambda\Lambda_0}(x_1 + a, \ldots, x_m + a) = H_{m,r}^{\Lambda\Lambda_0}(x_1, \ldots, x_m). \tag{36}$$

Proof: Induction on r, with the conclusion of the lemma as the inductive hypothesis. For $r = 1$, the RHS of (34) is zero. Thus $H_{m,r}^{\Lambda\Lambda_0} = H_{m,r}^{\Lambda_0\Lambda_0}$ for all $\Lambda \leq \Lambda_0$, and the statement is obvious from the hypotheses of the lemma. Assume the statement to hold for all $r' < r$. The inductive hypothesis and a change of variables $y_i \to y_i + a, y_i' \to y_i' + a$ in (34) imply that

$$\partial_\Lambda H^{\Lambda\Lambda_0}_{m,r}(x_1 + a, \ldots, x_m + a) = \partial_\Lambda H^{\Lambda\Lambda_0}_{m,r}(x_1, \ldots, x_m). \tag{37}$$

The statement for r now follows by integration from Λ_0 to Λ and using that the initial interaction is translation invariant. ■

The structure of the renormalizability proof is the same; what will change is that the integration step becomes less trivial.

I consider only translation invariant interactions and propagators, so the hypotheses of Lemma 1 are satisfied. Thus

$$\tilde{H}^{\Lambda\Lambda_0}_{m,r}(p_1, \ldots p_m) = \int \mathrm{d}^m X \; H^{\Lambda\Lambda_0}_{m,r}(X) \, \mathrm{e}^{\mathrm{i}(p_1 \cdot x_1 + \ldots + p_m \cdot x_m)} \tag{38}$$

contains a momentum conservation delta function, i.e.

$$\tilde{H}^{\Lambda\Lambda_0}_{m,r}(p_1, \ldots p_m) = \hat{H}^{\Lambda\Lambda_0}_{m,r}(p_1, \ldots p_{m-1}) \; \delta_L(p_1 + \ldots + p_m). \tag{39}$$

where $\delta_L(p) = L^d$ if $p = 0$ and 0 otherwise. The variables p are elements of the dual lattice of $\Gamma_{L,\varepsilon}$ (for details, see [2]). The function $\hat{H}^{\Lambda\Lambda_0}_{m,r}$ inherits permutation symmetry as follows: for all permutations π of $\{1, \ldots, m\}$,

$$\hat{H}^{\Lambda\Lambda_0}_{m,r}(p_{\pi(1)}, \ldots, p_{\pi(m-1)}) = \hat{H}^{\Lambda\Lambda_0}_{m,r}(p_1, \ldots, p_{m-1}) \tag{40}$$

with $p_m = -p_1 - \ldots - p_{m-1}$.

The RGDE in terms of the $\hat{H}^{\Lambda\Lambda_0}_{m,r}$ reads

$$\begin{aligned}
&\partial_\Lambda H^{\Lambda,\Lambda_0}_{m,r}(p_1, \ldots, p_{m-1}) = \\
&\qquad \frac{1}{2}\mathbb{S}_m \sum_{\substack{r_1, r_2 \geq 1 \\ r_1 + r_2 = r}} \sum_{l \geq 0} \sum_{\substack{m_1, m_2 \geq l+1 \\ m_1 + m_2 = m + 2l + 2}} \binom{m_1}{l+1} \binom{m_2}{l+1} (l+1)\,(l+1)! \\
&\qquad \int \left(\prod_{k=1}^{l} \mathrm{d}q_k \; D_\Lambda(q_k) \right) \; \partial_\Lambda D_\Lambda(\tilde{q}) \\
&\qquad \hat{H}^{\Lambda,\Lambda_0}_{m_1,r_1}(p_1, \ldots, p_{m_1-l-1}, q_1, \ldots, q_l) \\
&\qquad \hat{H}^{\Lambda,\Lambda_0}_{m_2,r_2}(p_{m_1-l}, \ldots, p_{m-1}, q_1, \ldots, q_l)
\end{aligned} \tag{41}$$

with $\tilde{q} = -p_1 - \ldots - p_{m_1-l-1} - \sum_{k=1}^{l} q_k$.

From now on, all considerations will be in momentum space, and I therefore drop the hats on all functions and write $H^{\Lambda,\Lambda_0}_{m_1,r_1}$ instead of $\hat{H}^{\Lambda,\Lambda_0}_{m_1,r_1}$.

I introduce the following shorthand notation which keeps only those pieces explicit that are essential for the following arguments. I abbreviate the RHS of the RGDE as

$$R^{\Lambda,\Lambda_0}_{m,r}(P) = \left\langle \int \mathrm{d}\rho_\Lambda(Q)\ \partial_\Lambda D_\Lambda(\tilde{q}) H^{\Lambda,\Lambda_0}_{m_1,r_1}(P^{(1)}, Q) H^{\Lambda,\Lambda_0}_{m_2,r_2}(P^{(2)}, Q) \right\rangle \tag{42}$$

so that

$$\partial_\Lambda H^{\Lambda,\Lambda_0}_{m,r} = R^{\Lambda,\Lambda_0}_{m,r} \tag{43}$$

The notations are $P = (p_1, \ldots, p_{m-1})$, $Q = (q_1, \ldots, q_l)$, etc., and

$$\mathrm{d}\rho_\Lambda(Q) = \prod_{i=1}^{l} \mathrm{d}q_i\ D_\Lambda(q_i). \tag{44}$$

The angular brackets include the summation over $r_1 \geq 1$, $r_2 \geq 1$, $m_1 \geq 1$, $m_2 \geq 1$, $l \geq 1$ with the restriction

$$m_1 + m_2 = m + 2l + 2, \quad r_1 + r_2 = r, \tag{45}$$

with the combinatorial factors, as well as symmetrization with respect to the external momenta $(p_1, \ldots, p_{m-1})$ as in (41). In other words, $\langle\cdot\rangle$ is a mean with respect to a positive (discrete) measure.

Note again that the angular brackets include summation over the r_i and m_i, so that the result indeed depends only on m and r. I choose this notation in favour of just writing a bilinear form $\mathcal{B}_{mr}(H^{\Lambda,\Lambda_0}, H^{\Lambda,\Lambda_0})$ because the inductive hypothesis will depend on m and r and it is therefore useful to exhibit the dependence on the "internal" variables m_i and r_i as well.

3.7 Choice of scaled propagator

To be specific, I now take the propagator of our scalar field theory and choose the cutoff function $K \in C^\infty(\mathbb{R}^+_0, [0,1])$, $K(x) = 1$ for $x \in [0,1]$, $K(x) = 0$ for $x \geq 4$, $K'(x) < 0$ for $x \in (1,4)$. Let

$$\hat{D}_\Lambda(p) = \frac{1}{p^2+\mu^2} K\left(\frac{p^2+\mu^2}{\Lambda^2}\right) \tag{46}$$

and $D_\Lambda(x, x') = \int \mathrm{d}p \; \mathrm{e}^{ip\cdot(x-x')} \hat{D}_\Lambda(p)$. The initial propagator C is defined as $C = D_{\Lambda_0}$. It has an ultraviolet cutoff $2\Lambda_0$ because the cutoff function K vanishes for $|p| \geq 2\Lambda_0$. The D_Λ in the decomposition used above is given by (46). For $\Lambda \leq \Lambda_0$, this then implies that the propagator C_Λ is given by

$$\hat{C}_\Lambda(p) = \frac{1}{p^2+\mu^2}\left(K\left(\frac{p^2+\mu^2}{{\Lambda_0}^2}\right) - K\left(\frac{p^2+\mu^2}{\Lambda^2}\right)\right) \tag{47}$$

i.e. it has ultraviolet cutoff Λ_0 and infrared cutoff Λ. Thus the effective potential $\mathcal{G}_\Lambda$ describes the result of integrating fields with momenta roughly between Λ and $2\Lambda_0$ ("roughly" because of the smooth cutoff function and the mass parameter μ that also appears in the cutoff function). Also, $C_\Lambda(p) \geq 0$ for all p as it must be for the covariance of a well–defined Gaussian measure.

The scale parameter Λ is the flow parameter of our RG method. Its initial value is Λ_0. At Λ_0, $C_{\Lambda_0} = 0$, so nothing has been integrated yet, and $\mathcal{G}_{\Lambda_0} = \mathcal{V}$. Thus the initial value for the effective potential is the interaction $\mathcal{V}$ at the cutoff scale. The final value of Λ is zero: at $\Lambda = 0$, $\mathcal{G}_0$ contains the effect of integrating over all degrees of freedom with momenta up to $2\Lambda_0$, i.e. indeed the full cutoff functional integral.

To avoid confusion one should keep in mind that Λ is an auxiliary parameter that has no direct physical significance. This means on the one hand that Λ indeed has to be sent to zero to get results about physical observables, on the other hand that one has some freedom about how to treat it, in particular how to choose the cutoff function, as long as one makes sure that in the end, all degrees of freedom get integrated. I have already used this freedom by including the mass μ in the cutoff function (one could also have chosen simply $K(p^2/\Lambda^2)$ as a cutoff function. The choice in (46) is convenient because

$$K\left(\frac{p^2+\mu^2}{\Lambda^2}\right) = 0 \text{ for all } \Lambda < \frac{\mu}{2}. \tag{48}$$

Thus in particular $\partial_\Lambda D_\Lambda = 0$ for $\Lambda < \mu/2$, and therefore the right hand side of the RGDE vanishes for $\Lambda < \mu/2$. Thus all $H_{m,r}^{\Lambda,\Lambda_0}$ become independent of Λ for $\Lambda < \mu/2$, and in particular

$$H_{m,r}^{0,\Lambda_0} = H_{m,r}^{\mu/2,\Lambda_0} \tag{49}$$

so that

$$H_{m,r}^{\Lambda,\Lambda_0} = H_{m,r}^{0,\Lambda_0} + \int_{\mu/2}^{\Lambda} \mathrm{d}\ell R_{m,r}^{\ell,\Lambda_0} \tag{50}$$

Therefore, to give bounds for $H^{0,\Lambda_0}_{m,r}$, it suffices to bound $H^{\mu/2,\Lambda_0}_{m,r}$, which is convenient because the bounds I shall use contain inverse powers of Λ.

I choose this cutoff function for technical convenience. One can equally well do all proofs with the choice $K(p^2/\Lambda^2)$ (and the early proofs were done with that choice), but the choice in (46) provides a slight simplification since one does not have to stop the flow at an intermediate scale Λ_1. With the cutoff function including μ, $\partial_\Lambda D_\Lambda(p) = -\frac{2}{\Lambda^3} K'(\frac{p^2+\mu^2}{\Lambda^2})$ so that

$$\|\partial_\Lambda D_\Lambda\|_\infty \leq 2\|K'\|_\infty \, \Lambda^{-3} \tag{51}$$

3.8 Norms and Estimates

Integration of the RGDE downwards from Λ_0 to Λ gives

$$H^{\Lambda\Lambda_0}_{mr} = H^{\Lambda_0\Lambda_0}_{mr} - \int_\Lambda^{\Lambda_0} \mathrm{d}\ell \; R^{\ell\Lambda_0}_{mr} \tag{52}$$

Let $\xi, \eta > 0$. For $F : (p_1, \ldots, p_{m-1}) \to F(p_1, \ldots, p_{m-1})$ define the seminorms

$$\begin{aligned} A_{\xi,\eta}(F) &= \|F \chi_{\max\{\xi,\eta\}}\|_\infty & (53)\\ &= \sup\{|F(p_1, \ldots, p_{m-1})| : \forall i \; |p_i| \leq \max\{\xi, \eta\}\} & (54) \end{aligned}$$

$A_{\xi,\eta}(F)$ is increasing in ξ and η. It is a seminorm and not a norm because $m = 0$ is left out in the sum. However, $m = 0$ terms never contribute to the RHS of the flow equation for $m \geq 1$ so this seminorm is appropriate for bounding the m–point functions for $m \geq 1$. The $m = 0$ terms can afterwards be bounded by a constant times the volume L^d.

Like any seminorm, $A_{\xi,\eta}$ fulfils the triangle inequality so

$$A_{\xi,\eta}(\int \mathrm{d}\nu(s) \; F(s)) \leq \int \mathrm{d}\nu(s) \; A_{\xi,\eta}(F(s)) \tag{55}$$

for any positive measure ν, and therefore, if $\langle \cdot \rangle$ is as in (42),

$$A_{\xi,\eta}\left(\langle F \rangle\right) \leq \langle A_{\xi,\eta}(F) \rangle. \tag{56}$$

Moreover

$$A_{\xi,\eta}(FG) \leq A_{\xi,\eta}(F) \; A_{\xi,\eta}(G) \tag{57}$$

By (42), the above inequalities and because $D_\Lambda(q_i) = 0$ for $|q_i| > 2\Lambda$,

$$A_{2\Lambda,\eta}(R^{\Lambda\Lambda_0}_{mr}) \leq \left\langle \|\partial_\Lambda D_\Lambda\|_\infty \int \mathrm{d}\rho_\Lambda(Q) \; A_{2\Lambda,\eta}(H^{\Lambda,\Lambda_0}_{m_1,r_1}) A_{2\Lambda,\eta}(H^{\Lambda,\Lambda_0}_{m_2,r_2}) \right\rangle. \tag{58}$$

The integral over the q_i now factorizes, so

$$A_{2\Lambda,\eta}(R^{\Lambda\Lambda_0}_{mr}) \le \frac{2}{\Lambda^3}\|K'\|_\infty \left\langle \|D_\Lambda\|_1^l A_{2\Lambda,\eta}(H^{\Lambda,\Lambda_0}_{m_1,r_1}) A_{2\Lambda,\eta}(H^{\Lambda,\Lambda_0}_{m_2,r_2}) \right\rangle . \tag{59}$$

Estimates for derivatives are obtained by taking derivatives with respect to momentum (α a multiindex) and using the Leibniz rule

$$\partial^\alpha(\dot{C} H_1 H_2) = \sum_{\substack{\alpha_0,\alpha_1,\alpha_2 \\ \alpha_0+\alpha_1+\alpha_2=\alpha}} \frac{\alpha!}{\alpha_0!\alpha_1!\alpha_2!}\, \partial^{\alpha_0}\dot{C}\; \partial^{\alpha_1} H_1\; \partial^{\alpha_2} H_2 \tag{60}$$

as well as $\|\partial^{\alpha_0}\partial_\Lambda D_\Lambda\|_\infty \le \kappa_{|\alpha_0|}\Lambda^{-3-|\alpha_0|}$ This gives

$$A_{2\Lambda,\eta}(\partial^\alpha R^{\Lambda\Lambda_0}_{mr}) \le \left\langle \frac{\kappa_{|\alpha_0|}}{\Lambda^{3+|\alpha_0|}} \|D_\Lambda\|_1^l A_{2\Lambda,\eta}(\partial^{\alpha_1} H^{\Lambda,\Lambda_0}_{m_1,r_1}) A_{2\Lambda,\eta}(\partial^{\alpha_2} H^{\Lambda,\Lambda_0}_{m_2,r_2}) \right\rangle \tag{61}$$

Here, by abuse of notation, I have included the sum over the α_i weighted with the multinomial factors in the definition of $\langle\cdot\rangle$.

The dependence on the dimension d enters through the properties of D_Λ — the propagator determines the power counting of the theory.

$$\|D_\Lambda\|_1 = \int D_\Lambda(p) \mathrm{d}^d p \le \int_{|p|\le 2\Lambda} \frac{\mathrm{d}^d p}{p^2} \le \tilde{\kappa}\; \Lambda^{d-2} \tag{62}$$

for $d \ge 3$. For $d = 2$, the dependence on Λ is logarithmic; also the mass can not be set equal to zero in two dimensions.

The estimate for the integrated RGDE

$$\partial^\alpha H^{\Lambda\Lambda_0}_{mr} = \partial^\alpha H^{\Lambda_0\Lambda_0}_{mr} - \int_\Lambda^{\Lambda_0} \mathrm{d}\ell\; \partial^\alpha R^{\ell\Lambda_0}_{mr} \tag{63}$$

reads

$$A_{2\Lambda,\eta}(\partial^\alpha H^{\Lambda\Lambda_0}_{mr}) \le A_{2\Lambda,\eta}(\partial^\alpha H^{\Lambda_0\Lambda_0}_{mr}) + \int_\Lambda^{\Lambda_0} \mathrm{d}\ell\; A_{2\Lambda,\eta}(\partial^\alpha R^{\ell\Lambda_0}_{mr}) \tag{64}$$

$$\le A_{2\Lambda,\eta}(\partial^\alpha H^{\Lambda_0\Lambda_0}_{mr}) + \int_\Lambda^{\Lambda_0} \mathrm{d}\ell\; A_{2\ell,\eta}(\partial^\alpha R^{\ell\Lambda_0}_{mr}). \tag{65}$$

In the second inequality, I used that $A_{\xi,\eta}$ is increasing in ξ.

3.9 Removing the lattice regulator

I now take the limits $L \to \infty$ and $\varepsilon \to 0$ and show that the RGDE (34) remains valid in these limits and that the Euclidian symmetries are restored. All this is at fixed Λ_0; the renormalization problem of convergence in the limit $\Lambda_0 \to \infty$ is dealt with in the next section.

Theorem 1 *Let $m \geq 2$. If the action at the cutoff, $H_{mr}^{\Lambda_0\Lambda_0}$, is continuous, the limits $L \to \infty$ and $\varepsilon \to 0$ of $H_{mr}^{\Lambda_0\Lambda_0}$ exist and are continuous functions of momentum. They still satisfy the RGDE (34) (where the momentum integrals now mean integrals, not scaled finite sums). If the action at the cutoff, $H_{mr}^{\Lambda_0\Lambda_0}$, is Euclidian invariant and C^∞ in the momenta p_i, the same holds for $H_{mr}^{\Lambda\Lambda_0}$ for all $\Lambda \in [0, \Lambda_0]$. Moreover, $H_{mr}^{\Lambda\Lambda_0}$ is C^∞ in Λ for all $\Lambda \in [0, \Lambda_0]$, and $\partial_\Lambda H_{mr}^{\Lambda\Lambda_0} = 0$ for $\Lambda < \mu/2$. Because every connected graph must contain a tree graph, $H_{mr}^{\Lambda\Lambda_0} = 0$ for $m > r + 2$.*

The case $m = 1$ is special: by translation invariance, the expectation value of $\phi(x)$ is independent of x, hence

$$H_{1r}^{\Lambda\Lambda_0}(p) = \begin{cases} H_{1,r}^{\Lambda\Lambda_0} & \text{for } p = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{66}$$

This theorem contains statements about derivatives, but the momentum space of the finite lattice is discrete and finite. This can be understood as follows. Momentum space for the infinite lattice is the continuous torus $T_d = \mathbb{R}^d/2\pi\mathbb{Z}^d$. The statement that the initial action is C^∞ is that its components in the sense of the expansion in the fields have Fourier transforms defined as functions on T^d (up to removal of the overall momentum conservation delta function discussed above) and are C^∞. The right hand side of the RGDE (34) defines the functions $\partial_\Lambda H_{m,r}^{\Lambda,\Lambda_0}$ also for momenta in T^d.

The proof of this theorem is obtained by taking the statements of the theorem as inductive hypotheses in r and following the same simple inductive strategy as in the proof of Lemma 1. Using this theorem, one can then take the limits $L \to \infty$ and $\varepsilon \to 0$ by an easy dominated convergence argument (for a variant of this argument, see [15]).

A crucial ingredient in the theorem is that the propagators are C^∞ and all integrals converge, in particular the one over the intermediate scale parameter ℓ in (64). This is only true in presence of an ultraviolet cutoff Λ_0, and therefore the bounds for the functions and their derivatives depend on Λ_0. For the unrenormalized model, they diverge for $\Lambda_0 \to \infty$. For the renormalized model, they converge, as proven in the following section.

4 Renormalizability of ϕ_6^3

For the unrenormalized model, the initial condition is

$$\mathcal{V}_{\Lambda_0}(\phi) = g \int \phi^3 \quad \text{so that} \quad H_{mr}^{\Lambda_0 \Lambda_0} = \delta_{m,3}\delta_{r,1} \tag{67}$$

corresponding to a bare action without counterterms. For the renormalized model,

$$\mathcal{V}_{\Lambda_0}(\phi) = g_{\Lambda_0} \int \phi^3 + \int \frac{1}{2}\phi(\zeta_{\Lambda_0}(-\Delta) + \nu_{\Lambda_0})\phi + \xi_{\Lambda_0}\phi. \tag{68}$$

In the formal power series expansion in g

$$g_{\Lambda_0} = g + \sum_{r=2}^{\infty} a_r(\Lambda_0)\, g^r \tag{69}$$

and

$$\zeta_{\Lambda_0} = \sum_{r=2}^{\infty} b_r(\Lambda_0)\, g^r, \quad \nu_{\Lambda_0} = \sum_{r=2}^{\infty} c_r(\Lambda_0)\, g^r, \quad \xi_{\Lambda_0} = \sum_{r=2}^{\infty} d_r(\Lambda_0)\, g^r. \tag{70}$$

This corresponds to an initial condition

$$H_{mr}^{\Lambda_0 \Lambda_0} = \delta_{m,3} a_r(\Lambda_0) \; + \frac{1}{2}(b_r(\Lambda_0)\, p^2 + c_r(\Lambda_0))\delta_{m,2} + d_r(\Lambda_0)\, \delta_{m,1} 1(p=0) \tag{71}$$

for the RG flow. That the b_r, c_r and d_r are at least second order in g is due to Wick ordering. The L^1 norm of the propagator is in $d = 6$

$$\|D_\Lambda\|_1 \leq \tilde{\kappa}\, \Lambda^4. \tag{72}$$

Theorem 2 *Let $\mathcal{V}$ be given by* (67)*, with the coefficients given by the formal power series* (69)*,* (70)*.*

(a) For any sequence of real numbers $(a_r^{(R)}, b_r^{(R)}, c_r^{(R)}, d_r^{(R)})_{r\geq 2}$ the renormalization conditions *(R.C.)*

$$\begin{aligned} &H_{3,r}^{0\Lambda_0}(0,0) = a_r^{(R)},\ (-\Delta H_{2,r}^{0\Lambda_0})(0) = b_r^{(R)}, \\ &H_{2,r}^{0\Lambda_0}(0) = c_r^{(R)},\ H_{1,r}^{0\Lambda_0} = d_r^{(R)}\ 1(p=0) \end{aligned} \tag{73}$$

can be imposed consistently. They uniquely fix the counterterms $g_{\Lambda_0}, \zeta_{\Lambda_0}, \nu_{\Lambda_0}$ *and* ξ_{Λ_0}*: the coefficients* $a_r(\Lambda_0), b_r(\Lambda_0), c_r(\Lambda_0), d_r(\Lambda_0)$ *are determined by* Λ_0 *and* $(a_s^{(R)}, b_s^{(R)}, c_s^{(R)}, d_s^{(R)})_{s\le r}$.

(b) With the R.C. of (a), all $H_{mr}^{\Lambda\Lambda_0}$ *are at fixed* Λ *bounded uniformly in* Λ_0*, and their limit* $\Lambda_0 \to \infty$ *exists and satisfies the RGDE.*
Specifically, $\forall m, r \exists E_{mr}\ \forall \alpha\ \forall \eta \exists h_{m,r,|\alpha|}$ *such that* $\forall \Lambda \in [\mu/2, \Lambda_0]$

$$A_{2\Lambda,\eta}(\partial^\alpha H_{mr}^{\Lambda\Lambda_0}) \le h_{m,r,|\alpha|}\ \Lambda^{6-2m-|\alpha|}\ \left(\log\frac{2\Lambda}{\mu}\mathrm{e}\right)^{E_{mr}} \tag{74}$$

(here e *denotes Euler's constant). Because* $H_{mr}^{0,\Lambda_0} = H_{mr}^{\mu/2,\Lambda_0}$, (74) *implies*

$$A_{\mu,\eta}(\partial^\alpha H_{mr}^{0\Lambda_0}) \le h_{m,r,|\alpha|}\ \left(\frac{\mu}{2}\right)^{6-2m-|\alpha|} \tag{75}$$

Thus the connected amputated Green functions have a finite limit as $\Lambda_0 \to \infty$*, and the theory is perturbatively renormalizable.*

Proof: Induction on r, with the statement of the theorem as inductive hypothesis. The case $r=1$ is trivial. Let $r \ge 2$ and assume the statement of Theorem 2 to hold for all $r' < r$. Because all terms entering the definition of $R_{mr}^{\ell\Lambda_0}$ are of order $r_i < r$, the inductive hypothesis can be used in the expression for $A_{2\ell,\eta}(\partial^\alpha R_{mr}^{\ell\Lambda_0})$. This gives

$$\begin{aligned} A_{2\ell,\eta}(\partial^\alpha R_{mr}^{\ell\Lambda_0}) &\le \langle \tilde\kappa^l \ell^{4l-3-|\alpha_0|}\kappa_{|\alpha_0|} h_{m_1,r_1,|\alpha_1|}\ell^{6-2m_1-|\alpha_1|} \\ &\qquad h_{m_2,r_2,|\alpha_2|}\ell^{6-2m_2-|\alpha_2|}\ \left(\log\frac{2\ell}{\mu}\mathrm{e}\right)^{E_{m_1r_1}+E_{m_2r_2}} \rangle \\ &\le \ell^{5-2m-|\alpha|}\ \left(\log\frac{2\ell}{\mu}\mathrm{e}\right)^{\tilde E_{mr}} \tilde h_{m,r,|\alpha|} \end{aligned} \tag{76}$$

where the constant

$$\tilde h_{m,r,|\alpha|} = \left\langle \tilde\kappa^l\ \kappa_{|\alpha_0|}\ h_{m_1,r_1,|\alpha_1|} h_{m_2,r_2,|\alpha_2|} \right\rangle \tag{77}$$

is given in terms of lower order constants and

$$\tilde E_{mr} = \sup\{E_{m_1r_1} + E_{m_2r_2} : m_1 + m_2 \ge m+2, r_1 + r_2 = r, m_i \le \bar m_{r_i}\} \tag{78}$$

is finite because the supremum is over a finite set. Let $E_{mr} = 1 + \tilde E_{mr}$.

Case 1. Irrelevant terms $2m + |\alpha| > 6$.

That is, $m \geq 4$ and $|\alpha| \geq 0$, or $m = 3$ and $|\alpha| \geq 1$, or $m = 2$ and $|\alpha| \geq 3$, or $m = 1$ and $|\alpha| \geq 5$

In all cases, $\partial^\alpha H_{mr}^{\Lambda_0\Lambda_0} = 0$, so the initial interaction does not contribute to the RHS of (64). Thus

$$A_{2\Lambda,\eta}(\partial^\alpha H_{mr}^{\Lambda\Lambda_0}) \leq \tilde{h}_{m,r,|\alpha|} \int_\Lambda^{\Lambda_0} \mathrm{d}\ell \; \ell^{5-2m-|\alpha|} \left(\log \frac{2\ell}{\mu} \mathrm{e}\right)^{\tilde{E}_{mr}} \tag{79}$$

Because $5 - 2m - |\alpha| \leq -2$, this is bounded by

$$A_{2\Lambda,\eta}(\partial^\alpha H_{mr}^{\Lambda\Lambda_0}) \leq h_{m,r,|\alpha|} \Lambda^{6-2m-|\alpha|} \left(\log \frac{2\Lambda}{\mu} \mathrm{e}\right)^{\tilde{E}_{mr}} \tag{80}$$

with a new constant $h_{m,r,|\alpha|}$.

The remaining cases. There remain only the cases in the following list. All but four involve functions that are zero by translational symmetry or evenness in p. The remaining four relevant and marginal terms have to be renormalized. This is the reason why there are four constants and renormalization conditions.

$2m + \alpha = 6$	$m = 3$	$\alpha = 0$	**coupling renormalization**
	$m = 2$	$\|\alpha\| = 2$	**field renormalization**
	$m = 1$	$\|\alpha\| = 4$	vanishes (no momentum dependence)
$2m + \alpha = 5$	$m = 2$	$\|\alpha\| = 1$	vanishes by evenness in p
	$m = 1$	$\|\alpha\| = 3$	vanishes (no momentum dependence)
$2m + \alpha = 4$	$m = 2$	$\|\alpha\| = 0$	**mass renormalization**
	$m = 1$	$\|\alpha\| = 2$	vanishes (no momentum dependence)
$2m + \alpha = 3$	$m = 1$	$\|\alpha\| = 1$	vanishes (no momentum dependence)
$2m + \alpha = 2$	$m = 1$	$\|\alpha\| = 0$	**vacuum expectation value**

Coupling renormalization. $m = 3$, $\alpha = 0$. By Case 1, for all $|p_1|, |p_2| \leq \eta$ and $|\alpha| = 1$

$$|\partial^\alpha H_{3,r}^{\Lambda\Lambda_0}(p_1, p_2)| \leq h_{3,r,1} \Lambda^{-1} \left(\log \frac{2\Lambda}{\mu} \mathrm{e}\right)^{E_{3,r}} \tag{81}$$

Thus by Taylor expansion

$$H_{3,r}^{\Lambda\Lambda_0}(p_1,p_2) = H_{3,r}^{\Lambda\Lambda_0}(0,0) + \left((p_1\cdot\nabla_1 + p_2\cdot\nabla_2)H_{3,r}^{\Lambda\Lambda_0}\right)(\pi_1,\pi_2) \tag{82}$$

with $|\pi_i| \le |p_i|$ and hence

$$\sup_{p_1,p_2:|p_i|\le\eta} |H_{3,r}^{\Lambda\Lambda_0}(p_1,p_2) - H_{3,r}^{\Lambda\Lambda_0}(0,0)| \le 2\eta\; h_{3,r,1}\Lambda^{-1}\left(\log\frac{2\Lambda}{\mu}\mathrm{e}\right)^{E_{3,r}} \tag{83}$$

This bound is uniform in Λ_0, so all that is needed now is a bound for $H_{3,r}^{\Lambda\Lambda_0}$ independently of Λ_0 at the single point $(0,0)$. This is shown by integrating the RGDE from 0 to Λ and using the renormalization condition

$$H_{3,r}^{\Lambda\Lambda_0}(0,0) = H_{3,r}^{0\Lambda_0}(0,0) + \int_{\mu/2}^{\Lambda}\mathrm{d}\ell\; R_{3,r}^{\ell\Lambda_0}(0,0) \tag{84}$$

The integration starts at $\mu/2$ because the derivative with respect to Λ vanishes for $\Lambda < \mu/2$. By the renormalization condition, $H_{3,r}^{0\Lambda_0}(0,0) = a_r^{(R)}$. By (64),

$$|R_{3r}^{\ell\Lambda_0}(0,0)| \le A_{2\ell,\eta}(R_{3,r}^{\ell\Lambda_0}) \le \ell^{-1}\;\left(\log\frac{2\ell}{\mu}\mathrm{e}\right)^{\tilde{E}_{3,r}} \tag{85}$$

so

$$\int_{\mu/2}^{\Lambda}\mathrm{d}\ell\;|R_{3,r}^{\ell\Lambda_0}(0,0)| \le \int_{\mu/2}^{\Lambda}\frac{\mathrm{d}\ell}{\ell}\left(\log\frac{2\ell}{\mu}\mathrm{e}\right)^{\tilde{E}_{3,r}} = \frac{1}{\tilde{E}_{3,r}+1}\;\left(\log\frac{2\Lambda}{\mu}\mathrm{e}\right)^{\tilde{E}_{3,r}+1} \tag{86}$$

Thus

$$H_{3,r}^{\Lambda\Lambda_0}(0,0) = a_r^{(R)} + \int_{\mu/2}^{\Lambda}\mathrm{d}\ell\; R_{3,r}^{\ell\Lambda_0}(0,0) \tag{87}$$

is well–defined and

$$|H_{3,r}^{\Lambda\Lambda_0}(0,0) - a_r^{(R)}| \le \tilde{h}_{3,r,0}\;\left(\log\frac{2\Lambda}{\mu}\mathrm{e}\right)^{E_{3,r}} \tag{88}$$

The $O(g^r)$ bare coupling constant

$$a_r(\Lambda_0) = H_{3,r}^{\Lambda_0\Lambda_0}(0,0) = a_r^{(R)} + \int_{\mu/2}^{\Lambda_0}\mathrm{d}\ell\; R_{3,r}^{\ell\Lambda_0}(0,0) \tag{89}$$

is uniquely fixed by the R.C. because $R_{3,r}$ has already been fixed by the inductive hypothesis. By (88), it satisfies

$$|a_r(\Lambda_0)| \leq \tilde{h}_{3,r,0} \left(\log \frac{2\Lambda_0}{\mu}\right)^{E_{3,r}}, \tag{90}$$

thus it diverges at most logarithmically as $\Lambda_0 \to \infty$ (that it indeed diverges when calculated perturbatively can be verified in low orders of perturbation theory).

Convergence as $\Lambda_0 \to \infty$ follows by an application of the dominated convergence theorem. One can also get a rate of convergence as a function of Λ_0 by a simple extension of the method [16, 2].

All remaining cases are done by simple repetitions of this argument: Any third derivative of the two–point function ($m = 2$, $|\alpha| = 3$) satisfies $2m + |\alpha| = 7$, so it is convergent. As above, Taylor expansion implies that it suffices to fix the second derivative of the two–point function at zero to get it finite everywhere. This is done using the second renormalization condition (involving $b_r^{(R)}$) in (73), and it uniquely fixes the $O(g^r)$ part b_r of the counterterm ζ_{Λ_0}. By Taylor expansion and evenness, this implies that the two–point function itself is finite uniformly in Λ_0 if this holds at zero momentum. This is guaranteed by the third renormalization condition in (73), which then also fixes the $O(g^r)$ part c_r of the counterterm ν_{Λ_0}. Finally, the one–point function is just a constant due to translation invariance, and it is fixed to be finite by the fourth renormalization condition in (73), which also fixes the $O(g^r)$ part d_r of the counterterm ξ_{Λ_0}. ■

Remarks.

1. Because η occurs in the Taylor expansion bound, $h_{m,r,0}$ depends on η. This η–dependence can be studied in more detail by an appropriate inductive ansatz.

2. The particular choice of RC

$$\begin{aligned} &H_{3,r}^{0\Lambda_0}(0,0) = \delta_{r,1},\ (-\Delta H_{2,r}^{0\Lambda_0})(0) = 0, \\ &H_{2,r}^{0\Lambda_0}(0) = 0,\ H_{1,r}^{0\Lambda_0} = v_r\ 1(p = 0) \end{aligned} \tag{91}$$

   means that g is the renormalized coupling constant, defined as the value of the three–point function at zero momenta, that μ is the mass of the particle and the renormalized field strength is one, and that the order r vacuum expectation value is v_r.

3. The method is natural to study the flow of coupling constants and to calculate beta functions. Doing this, one sees that the divergence of the bare coupling as $\Lambda_0 \to \infty$ need not carry over to the nonperturbative theory. The flow of the coupling constant can also tend to zero (asymptotic freedom). This property is used in the construction of infrared ϕ_4^4 [7, 8, 9].

4. Clearly, the details of the cutoff function were inessential. One can also relax the condition that the function K should be a strict cutoff function; a rapid enough decay at infinity would also be sufficient. In that case, the norms also have to be adjusted.

5. The proof required no discussion of graphs or subgraphs.

6. The behaviour of $h_{m,r,|\alpha|}$ on m and r can be studied in detail by an appropriate inductive ansatz.

7. A rate of convergence as $\Lambda_0 \to \infty$ can be proven by applying ∂_{Λ_0} to the RGDE and making an appropriate inductive ansatz.

8. For $m = 4$, Theorem 2 implies that

$$H_{4,r}^{\Lambda\Lambda_0} \le h_{4,r,0}\Lambda^{-2}(\log\Lambda)_{4,r}^{E} \tag{92}$$

A glance at (79) shows that the condition that there is no ϕ^4 term in the initial action can be relaxed: one can put a term

$$\frac{g^2}{{\Lambda_0}^2}\int \phi^4$$

into the action. The only effect of this term is that $A_{2\Lambda,\eta}(H_{4,2}^{\Lambda\Lambda_0}) \le const.{\Lambda_0}^{-2}$ appears as an additional summand in (79) fpr $m = 4$, $\alpha = 0$. This changes the constant $h_{4,2,0}$ in the bound, and hence the other constants, but everything else remains the same. Similarly, one can add higher powers of ϕ, scaled with appropriate inverse powers of Λ_0, to the interaction and the same bounds hold, with the same proof. This shows that cutoff ϕ^3 theory can after all be defined by a convergent integral if a stabilizing small ϕ^4 term is included. Moreover, it shows the robustness of the method. Adding additional terms would change a graphical analysis but here the proof gets only trivial modifications.

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