Contents

Chapter 1

Introduction

1.1 Particle field duality

From classical physics we know both particles and fields/waves. These are two different concepts with different characteristics. But some experiments show, that there exists a duality between both.

1. electromagnetic waves
   waves $\leftrightarrow$ photons (photoelectric effect, $E = h\nu = \hbar\omega$)
   fields are quantized, consisting of particles called photons.

2. particles (e.g. electrons) may exhibit interference phenomena, like waves. Thus, particles must be described by a wavefunction ψ. However, this has a probabilistic interpretation, it is *not* like an electromagnetic field.

The latter leads to *quantum mechanics* (QM), the former to *quantum field theory* (QFT). QM is nonrelativistic, and describes systems with fixed particle number. The quantization of the electromagnetic field requires quantum field theory, but is based on the same principles as quantum mechanics.

1.2 Short repetition of QM

QM cannot be derived from mechanics; rather, mechanics should follow from QM. But in obtaining appropriate Hamiltonians in QM, the *correspondence principle*, which substitutes quantities from mechanics by quantum mechanical operators, plays a key role.

1.2.1 Mechanics

In the Lagrangian formulation of mechanics, we substitute the equations of motion by an extremal postulate for an action functional

$$S[L] = \int_{t_0}^{t} dt L \tag{1.1}$$

of the Lagrange function $L(q_i, \dot{q}_i)$, where the q_i are the (finitely many) generalized coordinates in the specific problem. Postulating $\delta S = 0$ for variations in the $q_i(t)$ and keeping the endpoints fixed, we obtain the Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \tag{1.2}$$

Here, $\partial L/\partial \dot{q}_i = p_i$ are the generalized canonical momenta. The Hamiltonian $H(p_i, q_i)$ is the Legendre transform of L:

$$H = \sum_i \dot{q}_i p_i - L \tag{1.3}$$

and the Hamiltonian equations in phase space follow:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \dot{q}_i = \frac{\partial H}{\partial p_i} \tag{1.4}$$

The Poisson bracket of two functions $f(p_i, q_i), g(p_i, q_i)$ in phase space is defined as

$$\{f(p_i, q_i), g(p_i, q_i)\}_{Poisson} = \sum_i \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right) \tag{1.5}$$

We have

$$\{p_i, q_j\}_{Poisson} = \delta_{ij} \tag{1.6}$$

and the Hamiltonian equations can be generalized to

$$\frac{d}{dt} f(p_i, q_i) = \{H, f\}_{Poisson} \tag{1.7}$$

In case of explicitly time-dependent f we have $\dot{f}(p_i, q_i) = \{H, f\}_{Poisson} + \partial f/\partial t$. For these formal aspects see e.g. F. Scheck, "Mechanik".

Continuum mechanics can be obtained by taking the number of coordinates N to infinity, as will be seen in a specific example in QM.

1.2.2 QM States

States are described by Hilbert space (ket) vectors $|\psi\rangle \in \mathfrak{H}$ (or by the density matrix ρ; see below) with the following properties:

1. representation space: $\psi(\vec{x}) = \langle\vec{x}|\psi\rangle$ are functions in the L_2 Hilbert space $\mathfrak{H}$. These are coordinates in the $\langle\vec{x}|$-basis.

2. probabilistic interpretation:

   - $|\psi(\vec{x})|^2 \, d^3x$ is the probability to find the particle in the state $|\psi\rangle$ in volume element d^3x.

   - $\langle\varphi|\psi\rangle = \int d^3x \; \varphi^*(\vec{x}) \; \psi(\vec{x}) = \int d^3p \; \varphi^*(\vec{p}) \; \psi(\vec{p})$ is called the inner product of ψ and φ

   - $|\langle\varphi|\psi\rangle|^2$ is the probability to find the state $|\psi\rangle$ in $|\varphi\rangle$, and vice versa.

1.2.3 Observables

Observables are described by self-adjoint linear operators $\mathbf{A}$ in $\mathfrak{H}$: $\mathbf{A} = \mathbf{A}^\dagger$ and $\mathrm{def}(\mathbf{A}) = \mathrm{def}(\mathbf{A}^\dagger)$.

The eigenstates of $\mathbf{A}$ are orthogonal and form a complete basis, and the eigenvalues are real. The expectation value of an observable $\mathbf{A}$ in a state $|\psi\rangle$ is given by

$$\langle\mathbf{A}\rangle = \langle\psi|\mathbf{A}|\psi\rangle$$

More generally, one can introduce a *density matrix* ρ, and obtain

$$\langle\mathbf{A}\rangle = \mathrm{Tr}(\mathbf{A}\rho)$$

with

$$\rho: \quad \rho^+ = \rho, \quad \rho \geq 0, \quad \mathrm{Tr}(\rho) = 1$$

for general mixed states, and

$$\rho = P_\psi = |\psi\rangle\langle\psi|$$

for pure states.

1.2.4 Position and momentum

Position $\vec{\mathbf{X}}$ and momentum $\vec{\mathbf{P}}$ fulfill the canonical (Heisenberg) commutation relations

$$\left[\vec{\mathbf{X}_\mathbf{k}}, \vec{\mathbf{P}_\mathbf{l}}\right] = i\hbar\delta_{kl} \tag{1.8}$$

in accordance with the general quantization rule

$$-i\hbar\{A, B\}_{Poisson} \Rightarrow [\mathbf{A}, \mathbf{B}] \tag{1.9}$$

where the Poisson bracket for $f(x,p), g(x,p)$ is defined in eq. (1.5).

1.2.5 Hamilton-Operator

The correspondence principle relates mechanics to QM:

$$\text{mechanics}\left\{\begin{array}{ccc} \vec{p} & \Rightarrow & \frac{\hbar}{i}\vec{\nabla} \\ E & \Rightarrow & i\hbar\frac{\partial}{\partial t} \\ \vec{x} & \Rightarrow & \vec{x} \end{array}\right\}\text{QM in x-space}$$

So:

$$H = \frac{\vec{p}^2}{2m} + V(\vec{x}) \quad \Rightarrow \quad \mathbf{H} = -\frac{\hbar^2\vec{\nabla}^2}{2m} + V(\vec{x}),$$

and similarly for other operators corresponding to observables.

Time development (without measurement!)

$$|\psi(t)\rangle = \underbrace{\exp\left(-i\frac{\mathbf{H}}{\hbar}t\right)}_{=:\mathbf{U}}|\psi(0)\rangle \tag{1.10}$$

This is the representation of time development in the Schrödinger picture. The exponential function is a unitary operator ($\mathbf{U}^\dagger = \mathbf{U}^{-1}$).

In the *Schrödinger picture*, the states are time-dependent, and operators are time-independent (except when explicitly time-dependent). In contrast, in the *Heisenberg picture* the states are time-independent, and the operators are time-dependent. The expectation values are the same:

$$\langle\psi(t)|\mathbf{A}_S|\psi(t)\rangle = \left\langle\psi(0)|\mathbf{U}^\dagger\mathbf{A}_S\mathbf{U}|\psi(0)\right\rangle = \langle\psi(0)|\mathbf{A}_H|\psi(0)\rangle = \langle\psi_H|\mathbf{A}_H|\psi_H\rangle$$

where the Hamilton operator $\mathbf{H}$ is the same as above. The (Heisenberg) equation for time development is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}_H(t) = \frac{i}{\hbar}\left[\mathbf{H}, \mathbf{A}_H(t)\right]$$

(with an additional term $\partial\mathbf{A}/\partial t$ in case of explicit time dependence in $\mathbf{A}$).

In the Heisenberg picture,

$$\psi(\vec{x}, t) = \left\langle\vec{\mathbf{X}}_S|\psi_S(t)\right\rangle = \left\langle\vec{\mathbf{X}}_S|\mathbf{U}\psi(0)\right\rangle \stackrel{!}{=} \left\langle\vec{\mathbf{X}}_H|\psi_H(0)\right\rangle$$

The actions of the operators $\vec{\mathbf{X}}_S$ and $\vec{\mathbf{X}}_H$ are as follows:

$$\begin{aligned} \vec{\mathbf{X}}_S\,|x_S\rangle &= \vec{x}_S\,|x_S\rangle \\ \underbrace{\mathbf{U}^\dagger\vec{\mathbf{X}}_S\mathbf{U}}_{\vec{\mathbf{X}}_H}\underbrace{\mathbf{U}^\dagger\,|x_S\rangle}_{|x\rangle_H} &= \vec{x}\,\underbrace{\mathbf{U}^\dagger\,|x_S\rangle}_{|x_H\rangle} \end{aligned} \tag{1.11}$$

Remark

In QM, multiparticle states can be represented, in spaces like

$$\mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \ldots \otimes \mathfrak{H}_N$$

which describe the whole space as a tensor product of the individual spaces of each of the N particles. This representation is used to describe e.g. atomic structure, nuclear shells, or solid state physics, but *the particle number N is always fixed!*

1.3 The need for QFT

QFT is the quantum theory of fields, the main difference to QM being the huge number of degrees of freedom ($\to \infty$). The principles, however, are the same as those of QM. There is a (multiparticle) Hilbert space, called Fock space, and a probability interpretation, all as we know them from QM. So don't worry!

The electromagnetic wave equation is the prototype of a relativistic field equation:

$$\left(\vec{\nabla}^2 - \frac{1}{c}\frac{\partial^2}{\partial t^2}\right) A(\vec{x}, t) = 0 \tag{1.12}$$

It can be solved by a wave ansatz, which leads to:

$$\left(k^2 - \frac{\omega^2}{c^2}\right) A_k e^{-i(\omega t - \vec{k}\vec{x})} = 0 \tag{1.13}$$

With $\vec{p} = \hbar\vec{k}$ and $E = \hbar\omega$ we get the *dispersion relation for photons* in the particle language:

$$\left(k^2 - \frac{\omega^2}{c^2}\right) \to p^2 - \frac{E^2}{c^2} = 0 \tag{1.14}$$

Note that the wave equation is *not* a kind of Schrödinger equation for the probability amplitude of photons. In this case it would be an equation for the probability amplitude of a *single* photon, which would lead to contradictions. Consider the physics:

- It is very easy to produce "soft" or "colinear" quanta
  "soft": $E \sim p \sim$ small
  "colinear": $\xrightarrow{\vec{p}}$ $\overset{\vec{p}_1}{\rightarrow}\atop\overset{\vec{p}_2}{\rightarrow}$ with $\vec{p} = \vec{p}_1 + \vec{p}_2$ and $\vec{p} \,||\, \vec{p}_1 \,||\, \vec{p}_2$

- We want to measure x with precision Δx. Assume $\Delta x \leq \lambda_{DeBroglie} = \hbar/p$. Multiplying with Δp and using the uncertainity relation we obtain $\Delta x\ \Delta p \geq \hbar \rightarrow \Delta p \geq p$. This means that new particles may be produced, because $E^2 = p^2c^2$ for massless (highly relativistic) particles.

More formally: the wave equation contains second derivatives with respect to time, which means that a probability interpretation like the one for the Schrödinger equation fails ($\sqrt{m^2c^2 + p^2}$ is nonlocal).

Remarks

- The same problems arise for *massive* relativistic particles (e.g. pion $\pi^{\pm,0}$, e): we want to measure x with precision $\Delta x \leq \frac{\hbar}{mc} = \lambda_{Compton}$ Again using $\Delta x\ \Delta p \geq \hbar$, we now obtain

$$\Delta p \geq mc$$

With the relativistic relation

$$E^2 = p^2c^2 + m^2c^4 \quad \rightarrow \quad 2E\Delta E = 2p\Delta pc^2$$

we see that

$$\Delta E = v\Delta p \gtrsim mcv$$

This allows particle production for $v \rightarrow c$. Thus, a particle can *not* be localized without allowing for the production of further particles.

- $\psi(\vec{x}, t)$ assumes that one can measure $\vec{x}$ arbitrarily exactly, but we have just seen that then the particle number is not conserved. QM emerges for *non-relativistic* massive particles in the limit of neglegible particle production (i.e., it is a special case of QFT). However, also for very slow massive particles there are quantum field theoretical corrections: an example is the uncertainity relation $\Delta E \Delta t \geq \hbar$, which leads to tunneling and particle production. For small times Δt a very high energy ΔE is possible, i.e. at small time scales we cannot exclude particle production.

Figure 1.1: Particle production

In short

Relativistic field equations require to be treated in the framework of QFT, where particles can be produced and annihilated. Their interpretation is different from that of the Schrödinger equation. The electromagnetic field has nothing to do with the localization probability of a single photon.

Still: The principles of QM remain true!

Other relativistic field equations are the Klein-Gordon equation and the Dirac equation. The Dirac equation, which contains only first order time derivatives, allows for a one particle interpretation in the nonrelativistic limit, although not without further ado, as will be seen later.

We will first discuss free particles, and later their interactions, almost exclusively in the context of *perturbation theory*. Particularly interesting are gauge theories (electrodynamics, chromodynamics, flavordynamics).

This might give the impression that QFT was made exclusively for fundamental theories, for elementary particle physics. However, it is also very important in statistical mechanics and solid state physics (see e.g. the role of path integrals and their relation to the partition function); and of course historically, the connection between QFT and relativity was very important!

1.4 History

1.5 Harmonic oscillator, coherent states

1.5.1 Classical mechanics

In classical mechanics, we know the Hamiltonian of the harmonic oscillator:

$$H = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}x^2$$

with the equations of motion

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x$$

(see Hamilton equations, sec. 1.2.1)

By a canonical transformation, we obtain the *holomorphic representation*:

$$\begin{aligned} \tilde{a} &= (\sqrt{m\omega}x + i\frac{p}{\sqrt{m\omega}})/\sqrt{2} \\ \tilde{a}^* &= (\sqrt{m\omega}x + i\frac{p}{\sqrt{m\omega}})/\sqrt{2} \qquad (1.15) \\ \tilde{x},\tilde{p} &\rightarrow \tilde{a}, i\tilde{a}^* \end{aligned}$$

In terms of these new variables, the Hamiltonian becomes:

$$H = \frac{\omega}{2}\left(\tilde{a}^*\tilde{a} + \tilde{a}\tilde{a}^*\right)$$

The equations of motion are:

$$\begin{aligned} i\dot{\tilde{a}}^* &= -\frac{\partial H}{\partial \tilde{a}} = -\omega\tilde{a}^* \qquad (1.16) \\ \dot{\tilde{a}} &= \frac{\partial H}{\partial (i\tilde{a}^*)} = -i\omega\tilde{a} \end{aligned}$$

These are first order differential equations, allowing us to solve for $\tilde{a}$, $\tilde{a}^*$:

$$\begin{aligned} \tilde{a}(t) &= e^{-i\omega t}\,\tilde{a}(0) \qquad (1.17) \\ \tilde{a}^*(t) &= e^{i\omega t}\,\tilde{a}^*(0) \end{aligned}$$

1.5.2 Quantization

We use the quantization rule known from QM: Poisson-Bracket $\rightarrow \frac{i}{\hbar}\times$ commutator

For example:

$$\{p,x\}_P = 1 \Rightarrow \frac{i}{\hbar}\left[\mathbf{P},\mathbf{X}\right] = 1$$

$$\dot{f}(x,p) = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial p}\dot{p} = \{H,f\}_P \Rightarrow \dot{f}(\mathbf{X},\mathbf{P}) = \frac{i}{\hbar}\left[\mathbf{H},f\right]$$

When we quantize the harmonic oscillator, we promote $\tilde{a}$ and $\tilde{a}^*$ to operators:

$$\tilde{a},\tilde{a}^* \quad \Rightarrow \quad \tilde{\mathbf{a}},\tilde{\mathbf{a}}^\dagger$$

which obey the usual commutation relation:

$$\frac{i}{\hbar}\left[i\tilde{\mathbf{a}}^\dagger,\tilde{\mathbf{a}}\right] = 1$$

In QM we will often find $\hbar$ included in $\tilde{a}$. So we can set $\hbar = 1$, or define a new $\mathbf{a}$, to make the equations look nicer:

$$\mathbf{a} := \frac{\tilde{\mathbf{a}}}{\sqrt{\hbar}} \quad \Rightarrow \quad \left[\mathbf{a}, \mathbf{a}^\dagger\right] = 1$$

$$\rightarrow \mathbf{H} = \hbar\omega \underbrace{\left(\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}\mathbf{a}^\dagger\right)}_{\mathbf{a}^\dagger\mathbf{a}+1} \quad \Rightarrow \quad \mathbf{H} = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right)$$

The occupation number operator $\mathbf{N}$ is defined as

$$\mathbf{N} := \mathbf{a}^\dagger\mathbf{a}$$

and has eigenvalues

$$\mathbf{N}\left|n\right\rangle = n\left|n\right\rangle \quad n = 0, 1, ...$$

$\left|0\right\rangle$ is the ground state. In this setting $\mathbf{a}$ and $\mathbf{a}^\dagger$ are the *annihilation* and *creation* operators, and have the following actions:

$$\mathbf{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle \tag{1.18}$$

$$\mathbf{a}^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle \tag{1.19}$$

1.5.3 Coherent states

The states $\left|n\right\rangle$ do *not* correspond to quasiclassical states. The closest approximations to classical states are called *coherent states*, which, in the Heisenberg picture, which we will use most frequently, are defined as follows:

$$\left|\lambda\right\rangle = Ne^{\lambda(t)\mathbf{a}^\dagger(t)}\left|0\right\rangle \tag{1.20}$$

From

$$i\hbar\frac{\partial}{\partial t}\mathbf{a}^\dagger = -\left[\mathbf{H}, \mathbf{a}^\dagger\right] = -\mathbf{a}^\dagger\hbar\omega$$

we obtain

$$\rightarrow \quad \mathbf{a}^\dagger(t) = e^{i\omega t}\mathbf{a}^\dagger(0) \rightarrow \lambda(t) = e^{-i\omega t}\lambda(0)$$

which, if plugged into the definition of a coherent state, gives:

$$\left|\lambda\right\rangle = N\sum_{n=0}^{\infty}\frac{\lambda^n\mathbf{a}^{+n}}{n!}\left|0\right\rangle = N\sum_{n=1}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\left|n\right\rangle$$

To determine the normalization N, we take the inner product of $\left|\lambda\right\rangle$ with itself:

$$\langle\lambda|\lambda\rangle = |N|^2 \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{n!} = |N|^2 e^{|\lambda|^2} \overset{!}{=} 1$$

$$\Rightarrow \quad N = e^{-|\lambda|^2/2} \tag{1.21}$$

If we now apply the annihilation operator **a** to such a coherent state, we obtain:

$$\mathbf{a}\,|\lambda\rangle = N \sum_{n=1}^{\infty} \frac{\lambda^n(0)\sqrt{n}}{\sqrt{n!}} |n-1\rangle = \lambda\,|\lambda\rangle \tag{1.22}$$

We can also use this as a *definition* of coherent states. From the above, it follows that the expectation values of **a** and $\mathbf{a}^\dagger$ are given by

$$\langle\lambda|\mathbf{a}|\lambda\rangle = \lambda \quad \left\langle\lambda|\mathbf{a}^\dagger|\lambda\right\rangle = \lambda^*$$

and

$$\left\langle\lambda|\mathbf{a}^\dagger\mathbf{a}|\lambda\right\rangle = |\lambda|^2$$

Analogously, we can calculate $\mathbf{a}^\dagger\,|\lambda\rangle$, which gives

$$\mathbf{a}^\dagger\,|\lambda\rangle = \frac{\mathrm{d}}{\mathrm{d}\lambda}\,|\lambda\rangle \tag{1.23}$$

i.e., **a** and $i\mathbf{a}^\dagger$ act on coherent states as λ and $i\frac{\mathrm{d}}{\mathrm{d}\lambda}$.

For coherent states, the sum of the variances of **X** and **P** is minimal, i.e. $|\lambda\rangle$ comes closest to classical motion.

For the variance in the occupation number N, we have:

$$\begin{aligned}
(\Delta N)^2 &= \langle\lambda|(\mathbf{N} - \langle\mathbf{N}\rangle)^2|\lambda\rangle \\
&= \langle\lambda|\mathbf{N}^2|\lambda\rangle - \langle\lambda|\mathbf{N}|\lambda\rangle^2 \\
\left\langle\lambda|\mathbf{a}^\dagger\mathbf{a}\mathbf{a}^\dagger\mathbf{a}|\lambda\right\rangle &= |\lambda|^4 + |\lambda|^2 \\
\left\langle\lambda|\mathbf{a}^\dagger\mathbf{a}|\lambda\right\rangle &= |\lambda|^2 \\
\Rightarrow (\Delta N)^2 &= |\lambda|^2 = \langle\mathbf{N}\rangle
\end{aligned}$$

$$\Rightarrow \quad \frac{\Delta N}{\langle N\rangle} = \frac{1}{\sqrt{\langle N\rangle}} \tag{1.24}$$

The relative variance goes to 0 for large N.

Remarks

- Introducing coherent states:

  In the Heisenberg picture, we have

$$\mathbf{a}\,|\lambda\rangle = \lambda\,|\lambda\rangle$$

$$|\lambda\rangle_H = e^{-|\lambda|^2/2}\,\underbrace{e^{\lambda(t)}}_{e^{-i\omega t}}{}^{\mathbf{a}^\dagger(t)}\,|0\rangle$$

  In the Schrödinger picture, this becomes:

$$\begin{aligned} |\lambda\rangle_S &= e^{-iHt/\hbar}\,|\lambda\rangle_H \\ &= Ne^{\lambda(t)\mathbf{a}^\dagger(0)}\,|0\rangle \end{aligned}$$

- Coherent states are not orthogonal:

$$\langle\lambda|\lambda'\rangle = e^{|\lambda-\lambda'|^2/2}e^{i\Im(\lambda^*\lambda')} \tag{1.25}$$

- The completeness relation holds for coherent states:

$$\int \frac{d\lambda d\lambda^*}{2\pi i}\,|\lambda\rangle\,\langle\lambda| = \mathbf{1} \tag{1.26}$$

  We can obtain this relation by expanding an inner product $\langle n|m\rangle$ in the λ-basis and integrating:

$$\int \frac{d\lambda d\lambda^*}{2\pi i}\,\langle n|\lambda\rangle\,\langle\lambda|m\rangle = \int \frac{d\lambda d\lambda^*}{2\pi i} NN^* \frac{\lambda^n}{\sqrt{n!}}\frac{\lambda^{*m}}{\sqrt{m!}} = \delta_{nm} = \langle n|m\rangle$$

  where in the second step we have used

$$\langle\lambda|m\rangle = \psi_m(\lambda^*) = N\frac{\lambda^{*m}}{\sqrt{m!}}$$

- The following normalization is often used in the literature:

$$\langle\lambda|\lambda\rangle = e^{|\lambda|^2}$$

  In this case, the completeness relation changes:

$$\int \frac{d\lambda d\lambda^*}{2\pi i} e^{-\lambda^*\lambda}\,|\lambda\rangle\,\langle\lambda| = \mathbf{1}$$

- The representation of an arbitrary operator $\mathbf{A}$ in the λ-basis is as follows:

$$
\begin{aligned}
\langle\lambda|\mathbf{A}f\rangle &= \frac{1}{2\pi i}\int \underbrace{\langle\lambda|\mathbf{A}|\lambda'\rangle}_{\mathbf{A}(\lambda^*,\lambda')}\langle\lambda'|f\rangle \, \mathrm{d}\lambda' \, \mathrm{d}\lambda'^* \\
\mathbf{A}(\lambda^*,\lambda') &= \langle\lambda|n\rangle\langle n|\mathbf{A}|m\rangle\langle m|\lambda'\rangle \\
&= \mathbf{A}_{mn}\frac{(\lambda^*)^n}{\sqrt{n!}}\frac{\lambda'^m}{\sqrt{m!}}e^{-|\lambda|^2/2}e^{-|\lambda'|^2/2} \quad \text{(with summing convention)} \\
\text{with} \quad \mathbf{A} &= K_{nm}\mathbf{a}^{\dagger n}\mathbf{a}^m \quad \text{(“normal ordered”)} \\
K_{nm}(\lambda^*,\lambda') &:= K_{nm}\lambda^{*n}\lambda'^m \\
\Rightarrow \quad \mathbf{A}(\lambda^*,\lambda') &= e^{-(|\lambda|^2+|\lambda'|^2)/2}K(\lambda^*,\lambda')
\end{aligned}
$$

- Applying this to $\mathbf{a}$ and $\mathbf{a}^\dagger$, we plug in $K_{nm} = 0$ except for $n = 1$, $m = 0$ for $\mathbf{a}^\dagger$, and $n = 0$, $m = 1$ for $\mathbf{a}$, in which cases $K = 1$. This gives $\mathbf{a}^\dagger(\lambda^*,\lambda') = e^{-(|\lambda|^2+|\lambda'|^2)/2}\lambda^*$ and $\mathbf{a}(\lambda^*,\lambda') = e^{-(|\lambda|^2+|\lambda'|^2)/2}\lambda'$

- Canonical transformations in phase space are *symplectic* (see e.g. F. Scheck, “Mechanik”). A general phase space vector

$$
\vec{z}_{Ph} = \begin{pmatrix} \vec{x} \\ \vec{p} \end{pmatrix} \qquad \begin{matrix} x_1,\ldots,x_N = z_1,\ldots,z_N \\ p_1,\ldots,p_N = z_{N+1},\ldots,z_{2N} \end{matrix}
$$

has the following Hamilton equation:

$$
\frac{d\vec{z}_{Ph}}{dt} = J\frac{\partial H}{\partial \vec{z}_{Ph}}
$$

with

$$
J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \qquad \big(\text{a “metric in phase space”}, J^{-1} = -J\big)
$$

A canonical transformation $\vec{z} \to \vec{z}'$ preserves the Hamilton equation:

$$
\begin{aligned}
M_{\alpha\beta} &:= \frac{\partial z_\alpha}{\partial z'_\beta} \quad \alpha,\beta = 1,\ldots 2N \\
M^{-1}_{\alpha\beta} &:= \frac{\partial z'_\alpha}{\partial z_\beta}
\end{aligned}
$$

$$\dot{z}'_\alpha = \frac{\partial z'_\alpha}{\partial t} = \frac{\partial z'_\alpha}{\partial z'_\beta}\frac{\partial z_\beta}{\partial t} = \underbrace{\frac{\partial z'_\alpha}{\partial z_\beta}}_{M^{-1}_{\alpha\beta}} J_{\beta\gamma}\frac{\partial \mathbf{H}}{\partial z_\gamma} = M^{-1}_{\alpha\beta} J_{\beta\gamma}\frac{\partial \mathbf{H}}{\partial z'_\delta}\underbrace{\frac{\partial z'_\delta}{\partial z_\gamma}}_{M^{-1}_{\delta\gamma}} \stackrel{!}{=} J_{\alpha\delta}\frac{\partial \mathbf{H}}{\partial z'_\delta} \tag{1.27}$$

$$\Rightarrow \quad M^{-1}_{\alpha\beta} J_{\beta\gamma} {M^{T}}^{-1}_{\gamma\delta} = J_{\alpha\delta}$$

$$\boxed{\Rightarrow \quad J = MJM^T} \tag{1.28}$$

M is a *symplectic matrix* (the M form a group)!

- The Poisson bracket

$$\{f, g\}_{Poisson}(z) = -\frac{\partial f}{\partial z_\alpha} J_{\alpha\beta}\frac{\partial g}{\partial z_\beta}$$

is invariant under canonical (symplectic) transformations.

1.6 The closed oscillator chain

1.6.1 The classical system

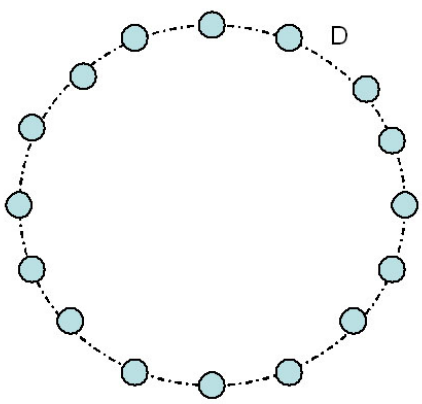

Figure 1.2: The closed oscillator chain

Let us consider a circular chain of radius R, with N masses m, connected by springs with spring constant D. Let the ith mass be at a displacement $q_i = R\Theta_i$ from some rest position, and let us impose the periodic boundary condition $q_0 \equiv q_N$ for convenience of writing. The distance between the masses when they are at rest is $a = 2\pi R/N$.

The Lagrangian for this system is:

$$L(q,\dot{q}) = \sum_{j=1}^{N} \left[\frac{m}{2}\dot{q}_j^2 - \frac{D}{2}\left(q_{j+1} - q_j\right)^2 \right]$$

Varying q_l, we get the equation of motion

$$m\ddot{q}_l + D\left(q_l - q_{l-1} - \left(q_{l+1} - q_l\right)\right) = 0$$

which is solved by the ansatz

$$q_l(t) = e^{i(lk - \omega t)} \quad (+vt)$$

leading to

$$m\omega^2 + \underbrace{D}_{m\bar{\omega}^2} \underbrace{\left(2 - e^{-ik} - e^{+ik}\right)}_{4\sin^2\frac{k}{2}} = 0$$

Because of the periodic boundary condition $q_N = q_0$, we get $Nk = 2\pi n$ with $n = 0, \ldots, N-1$ or $n = -N/2, \ldots, +N/2 - 1$ (for even N), and we obtain for the *oscillator modes*

$$\boxed{\omega_n^2 = \underbrace{\frac{4D}{m}}_{4\bar{\omega}^2} \sin^2 \frac{\pi n}{N}} \tag{1.29}$$

Superposition of modes gives the general solution for real $q_j(t)$:

$$q_j(t) = \frac{1}{\sqrt{N}} \sum_{n=-N/2}^{N/2-1} \left[c_n e^{i(2\pi nj/N - \omega_n t)} + c_n^* e^{-i(2\pi nj/N - \omega_n t)} \right] \tag{1.30}$$

(note that this is a Fourier expansion).

The Hamiltonian is then given by:

$$\mathbf{H}(q,p) = \sum_{j=1}^{N} \left[\frac{p_j^2}{2m} + \frac{D}{2}\left(q_j - q_{j-1}\right)^2 \right],$$

where $p_j = m\dot{q}_j$, and the corresponding equations of motion are

$$\dot{p}_j = -\frac{\partial \mathbf{H}}{\partial q_j}, \quad \dot{q}_j = \frac{\partial \mathbf{H}}{\partial p_j}.$$

To ensure that the the transformation $(q_j, p_j) \to (c_n, c_n^*)$ is canonical, we normalize c_n and c_n^* by defining the new canonical variables a_n and ia_n^*

through $a_n = \sqrt{2m\omega_n} c_n e^{-i\omega_n t}$ (cf. harmonic oscillator, sec. 1.5.1). These variables are canonical:

$$\{a_n, ia_m^*\}_{Poisson} = -\delta_{nm}, \quad \{a, a\} = \{a^*, a^*\} = 0$$

The new Hamiltonian is

$$H = \sum_n \frac{1}{2}\omega_n (a_n^* a_n + a_n a_n^*)$$

Note that this transformation is still classical. It is preferable, however, to discuss this in the already quantized version.

1.6.2 Quantization

The procedure is exactly the same as it was for the harmonic oscillator:

$$\begin{aligned} p_l, q_l &\rightarrow \mathbf{p}_l, \mathbf{q}_l \\ \{f(p,q), g(p,q)\}_{Poisson} &\rightarrow \frac{i}{\hbar}[\mathbf{f}, \mathbf{g}] \\ a_n, a_n^* &\rightarrow \bar{\mathbf{a}}_n, \bar{\mathbf{a}}_n^\dagger \end{aligned}$$

Then the $\bar{\mathbf{a}}_n$, $i\bar{\mathbf{a}}_n^\dagger$ fulfill the usual commutation relation

$$\left[\bar{\mathbf{a}}_n, \bar{\mathbf{a}}_m^\dagger\right] = i\hbar\delta_{nm}$$

and the Hamiltonian looks like this:

$$\mathbf{H} = \sum_n \omega_n \left(\bar{\mathbf{a}}_n^\dagger \bar{\mathbf{a}}_n + \frac{1}{2}\hbar\right)$$

Exercise: obtain the commutator $[p_j, q_{j'}]$ from $[\bar{\mathbf{a}}_n, \bar{\mathbf{a}}_m^\dagger] = i\hbar\delta_{nm}$ and derive the Hamiltonian above (use $\sum_{n=0}^{N-1} e^{2\pi i n(j-j')/N} = N\delta_{jj'}$).

In the following we will often use units where $\hbar = 1$. We will sometimes reinsert the factors of $\hbar$, in cases with experimental relevance; the same goes for setting $c = 1$.

Defining $\mathbf{a}_n^{(\dagger)} = \sqrt{\hbar}\bar{\mathbf{a}}_n^{(\dagger)}$, the Hamiltonian becomes:

$$\mathbf{H} = \sum_n \hbar\omega_n \left(\mathbf{a}_n^\dagger \mathbf{a}_n + \frac{1}{2}\right) \tag{1.31}$$

After quantization each mode n has an excitation number (the "occupation number" of some quasiparticle-mode), as we have already seen in the case of the harmonic oscillator. This begs the question whether, as a physicist, one can distiguish these modes (called *phonons* if they live in crystals) from "real" particles.

If one can see the "lattice" (e.g. a solid state crystal), the answer is yes, because a real particle should exist independently of the medium it lives in. One can not always see this medium, however, in particular when the continuum limit has been taken. Then, the "lattice" has become some sort of "ether", which is not necessarily visible to the physicist, and might not be needed anymore.

An interesting case is fundamental (super)string theory, where particles are modes of strings.

The states on which the operators defined above act live in a *Fock space* $\mathfrak{F} = \mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \ldots \otimes \mathfrak{H}_N$, where $\mathfrak{H}_i$ the Hilbert space of i-th mode. The "vacuum" state is denoted by $|0\rangle$, and has $a_m |0\rangle = 0$ for all m.

General states

$$|n_1, n_2, \ldots, n_N\rangle = \frac{(\mathbf{a}_1^\dagger)^{n_1}}{\sqrt{n_1}} \frac{(\mathbf{a}_2^\dagger)^{n_2}}{\sqrt{n_2}} \ldots \frac{(\mathbf{a}_N^\dagger)^{n_N}}{\sqrt{n_N}} |0\rangle$$

are normalized eigenstates of $\mathbf{H}$ and have energy

$$E = \sum_m \hbar\omega_m \left(n_m + \frac{1}{2}\right)$$

Note: Elastic binding to lattice, mimicking a crystal, at $x_j = j2\pi R/N$ gives a term $-(m/2)\Omega^2 q_j^2$ in L which shifts the ω_n^2 in eq. (1.29) by $+\Omega^2$; there is no zero mode.

One can rewrite eq. (1.30) by substituting $-n$ for n in the second term:

$$q_j = \sum_{n=-N/2}^{N/2-1} e^{2\pi inj/N} \left[\frac{\mathbf{a}_n + \mathbf{a}_{-n}^\dagger}{\sqrt{2m\omega_n}}\right] = \sum_{n=-N/2}^{N/2-1} e^{2\pi inj/N} \mathbf{Q}_n$$

Similarly, one can introduce

$$\mathbf{P}_n = \sqrt{\frac{m\omega_n}{2}} i(\mathbf{a}_n + \mathbf{a}_{-n}^\dagger)$$

Check the following relations:

$$[\mathbf{P}_{-n}, \mathbf{Q}_l] = -i\delta_{nl}$$

and

$$\mathbf{H} = \sum_n \frac{1}{2m} \mathbf{P}_{-n}\mathbf{P}_n + \frac{1}{2} m\omega_n^2 \mathbf{Q}_{-n}\mathbf{Q}_n$$

with $\mathbf{P}_n = \mathbf{P}_n^\dagger$ and $\mathbf{Q}_n = \mathbf{Q}_n^\dagger$

1.6.3 Continuum limit

In the continuum limit, we take N to infinity and a to zero while keeping $2\pi R = Na$ fixed. The index j becomes a continuous variable, leading to:

$$q_j(t) \to q(x,t), \quad x = a \cdot j \tag{1.32}$$

Note that in the "closed string" case discussed above, one does not have elastic binding to the lattice like one has for a crystal, and x is defined up to an overall translation.

Now, we make the following translations:

$$\begin{aligned} (q_j - q_{j-1})^2 &\to a^2 \left(\frac{\partial q(x,t)}{\partial x}\right) \\ \sum_j &\to \frac{1}{a} \int_0^{2\pi R} \mathrm{d}x \end{aligned}$$

Further, we postulate

$$\begin{aligned} m = \rho a & \qquad \text{with finite mass density } \rho \\ D = \frac{\sigma}{a} & \qquad \text{with finite string density } \sigma \end{aligned}$$

and obtain the following Lagrangian:

$$L^{cont} = \frac{1}{2} \int_0^{2\pi R} \mathrm{d}x \left[\rho \left(\frac{\partial q(x,t)}{\partial t}\right)^2 - \sigma \left(\frac{\partial q(x,t)}{\partial x}\right)^2\right] \tag{1.33}$$

The equations of motion

$$\rho \frac{\partial^2 q(x,t)}{\partial t^2} - \sigma \frac{\partial^2 q(x,t)}{\partial x^2} = 0$$

can be read off from the discontinous case. Later, we will obtain them directly from eq. (1.33).

Now, with $c = \sqrt{\sigma/\rho}$ we can rewrite the above equation as:

$$\frac{\partial^2 q}{\partial t^2} - c^2 \frac{\partial^2 q}{\partial x^2} = 0 \tag{1.34}$$

Note: introducing elastic binding (cf. note in section 1.6.2) gives an additional mass term $-\frac{1}{2}\int_0^{2\pi R} dx \rho \Omega^2 q^2(x,t)$ in eq. (1.33), and $-\Omega^2 q$ in eq. (1.34), which then has the form of a Klein-Gordon equation to be discussed in the next chapter.

Now, we can just translate our "discrete" solution to the continuum case, *or* we can make the following ansatz (from electrodynamics):

$$q(x,t) = Ae^{i(kx-\omega t)}$$

Plugging this into eq. (1.34) gives

$$\omega^2 = c^2k^2$$

Imposing periodic boundary conditions $q(0,t) = q(2\pi R, t)$ leads to

$$k_n = \frac{n}{R} \quad (n = 0, \pm 1, \pm 2, \ldots)$$

which, in combination with $\omega^2 = c^2k^2$, gives

$$\boxed{\omega_n = \frac{|n|c}{R}} \tag{1.35}$$

Note that in the finite N case, the continuum ω_n is only approximately valid. We have to use the formula from the discrete case:

$$\omega_n = \sqrt{\frac{4D}{m}} \sin\left(\frac{\pi|n|}{N}\right) \tag{1.36}$$

We expand this for small n, using the linear approximation for the sine:

$$\omega_n \sim \sqrt{\frac{4D}{m}} \frac{\pi|n|}{N} = \frac{2c\pi|n|}{aN} = \frac{c|n|}{R}$$

(remember $D = c^2m/a^2$ and $a = 2\pi R/N$). This approximation is only allowed when $\frac{\pi|n|}{N} < \frac{\pi}{4}$; then, we have:

$$\omega_n \leq \frac{c}{R}\frac{N}{4} = \omega_c \tag{1.37}$$

Note that in the above discussion, c does not always have to be the speed of light; it may also be a characteristic speed for the medium in which our phonons are living.

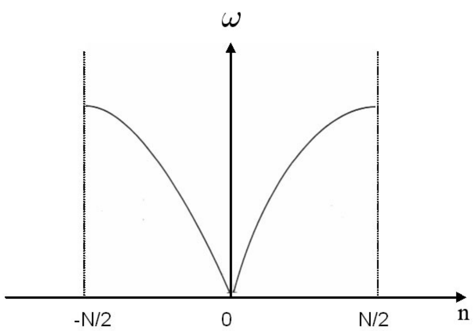

Figure 1.3: Validity range for the linear approximation

1.6.4 Ground state energy

$$E_0 = \sum_n \frac{\hbar\omega_n}{2} = \sum_{n=-\infty}^{+\infty} \hbar\frac{|n|c}{2R} \tag{1.38}$$

In the continuum limit the ground state energy is divergent! We will observe such divergences more often in QFT; they are related to the continuum limit and to interaction at a point, and require physical discussion.

In the finite N case:

$$\begin{aligned}
E_0 &= \frac{\hbar}{2}\sum_{n=-N/2}^{N/2-1}\frac{c}{R}\frac{N}{\pi}\sin\left(\frac{\pi|n|}{N}\right) \\
&\approx \frac{\hbar}{2}\int\limits_{-1/2}^{1/2}\mathrm{d}x\frac{c}{R}\frac{N^2}{\pi}\sin(\pi|x|) \quad (\text{use } x\sim\frac{n}{N}) \\
\frac{E_0}{2\pi R} &\approx \frac{\hbar}{2}\cdot 2\cdot\frac{1}{\pi}\cdot\frac{cN^2}{\pi R\cdot 2\pi R} \\
&= \frac{\hbar c}{2\pi^3}\frac{N^2}{R^2} = \frac{2\hbar c}{\pi a^2} \quad \text{with} \quad \frac{N^2}{R^2} = \left(\frac{2\pi}{a}\right)^2
\end{aligned}$$

The absolute value of E_0 does not have a physical meaning (except in general relativity, where it appears on the right hand side of the Einstein equation), but we can compare ground state energy in two *different* physical situations; in doing so, we can observe the Casimir effect.

Summary of chapter 1: