Chapter 3

The Schrödinger equation in the language of QFT

3.1 Second Quantization

For radiative transitions, the electromagnetic field is described by an operator in Fock space, whereas the atom retains its 'old' QM description, i.e. with an $e^{i\vec{k}\vec{\mathbf{X}}}$, where the position is the operator $\vec{\mathbf{X}}$.

Now, the following question comes up: can we treat the Schrödinger equation, written down according to the rules of the correspondence principle, in its x-space form, like an ordinary field equation and *quantize* it? This sounds odd, since it is already quantized; why do it again? The point is that we want QM to be a special case of QFT, as mentioned in chapter 1, and therefore, it should be possible to express it in the language of QFT. This is called *second quantization*.

Note that we are not looking for new physics: QM has been tested thoroughly, and looks far too nice to throw away.

3.1.1 Schrödinger equation

In classical mechanics, we have $E = \vec{p}^2/2m + V(\vec{x})$; following the correspondence principle, this gives

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{(x)},t) = \underbrace{\left[\left(\frac{-\hbar^2\vec{\nabla}^2}{2m}\right) + V(\vec{x})\right]}_{H_{Schr}}\psi(\vec{x},t) \tag{3.1}$$

This has stationary solutions $\psi_n = \phi_n(\vec{x})e^{-iE_nt/\hbar}$; we will also use $E_n/\hbar = \omega_n$. General solutions, obtained from a generalized Fourier transform, are

$$\psi(\vec{x},t) = \sum_n a_n(t)\phi_n(\vec{x}) \tag{3.2}$$

where $a_n(t) \propto e^{-i\omega_n t}$ and the $\{\phi_n\}$ form a complete set of orthonormal functions.

In the QFT quantization, the functions $a_n(t)$ are promoted to operators $\mathbf{a}_n(t)$; as usual, we are in the Heisenberg picture. The final result of this second quantization will be:

$$\boxed{\mathbf{H} = \int d^3x \psi^\dagger_{op}(\vec{x},t)\mathcal{H}_{Schr}\psi_{op}(\vec{x},t)} \tag{3.3}$$

which is an operator in Fock space. In terms of $\mathbf{a}_n$, we have

$$\mathbf{H} = \sum_n E_n \mathbf{a}^\dagger_n \mathbf{a}_n \tag{3.4}$$

where the $\mathbf{a}_n$ and $i\mathbf{a}^\dagger_n$ are canonically conjugate operators in the Hamiltonian formalism. Let us now consider in detail how we get this result.

3.1.2 Lagrange formalism for the Schrödinger equation

In order to quantize the Schrödinger equation, we will treat it as an ordinary field equation, for which we will first develop the Lagrange formalism. The Lagrangian density is:

$$\boxed{\mathcal{L} = i\hbar\psi^*\dot{\psi} - \frac{\hbar^2\vec{\nabla}\psi^*\cdot\vec{\nabla}\psi}{2m} - V(\vec{x})\psi^*\psi} \tag{3.5}$$

Following the usual procedure, we obtain the momentum fields:

$$\Pi_\psi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\hbar\psi^*; \quad \Pi_{\psi^*} = 0 \tag{3.6}$$

We could obtain our field equation by varying $\mathcal{L}$ with respect to the real and imaginary parts of ψ, but we can also vary with respect to ψ^* and ψ, since these are complex conjugates. This gives:

$$i\hbar\dot{\psi} + \frac{\vec{\nabla}\cdot\vec{\nabla}\psi}{2m} - V(\vec{x})\psi = 0 \tag{3.7}$$

which is the Schrödinger equation, as we had hoped. Now, our Hamiltonian density (as used in eq. (3.3)) is given by

$$\mathcal{H} = i\hbar\psi^*\dot{\psi} - \mathcal{L} = \hbar^2\frac{\vec{\nabla}\psi^*\cdot\vec{\nabla}\psi}{2m} + V(\vec{x})\psi^*\psi \tag{3.8}$$

Here, ψ and $i\hbar\psi^*$ are canonical conjugates. In this equation, we could also introduce electromagnetic coupling.

Note that we could also have taken a more symmetric Lagrangian density, $\frac{i\hbar}{2}(\psi^*\dot{\psi} - \dot{\psi^*}\psi)$ to obtain this result.

3.1.3 Canonical quantization

The canonical quantization relation, where the ψ and ψ^* are promoted to ψ_{op} and $\psi^\dagger_{op}$, is given by

$$\boxed{[\psi_{op}(\vec{x},t), i\hbar\psi^\dagger_{op}(\vec{x}',t)] = i\hbar\delta^3(\vec{x}-\vec{x}')} \tag{3.9}$$

Note that the factor $i\hbar$ appears on both sides of the equation, and hence cancels. In terms of the $\mathbf{a}^{(\dagger)}_n$, we have

$$\sum_{n,n'}[\mathbf{a}_n, \mathbf{a}^\dagger_{n'}]\phi_n(\vec{x})\phi_{n'}(\vec{x}') = \delta^3(\vec{x}-\vec{x}') \tag{3.10}$$

where the generation and annihilation operators fulfill their usual commutation rule

$$[\mathbf{a}_n, \mathbf{a}^\dagger_{n'}] = \delta_{n,n'} \tag{3.11}$$

This gives the completeness relation: the $\phi_n(\vec{x})$ form a complete basis.

The Hamiltonian operator $\mathbf{H} = \int d^3x\mathcal{H}$ is, as discussed in chapter 1, given by

$$\mathbf{H} = \sum_n E_n\mathbf{a}^\dagger_n\mathbf{a}_n \tag{3.12}$$

Using eq. (3.11), we find the Heisenberg equations for the $\mathbf{a}_n$ and $\mathbf{a}^\dagger_n$:

$$\frac{d}{dt}\mathbf{a}_n(t) = \frac{i}{\hbar}[\mathbf{H}, \mathbf{a}_n] = -i\omega_n\mathbf{a}_n(t) \tag{3.13}$$

$$\frac{d}{dt}\mathbf{a}^\dagger_n(t) = \frac{-i}{\hbar}[\mathbf{H}, \mathbf{a}^\dagger_n] = i\omega_n\mathbf{a}^\dagger_n(t) \tag{3.14}$$

For the ψ_{op} and $\psi^\dagger_{op}$, we have:

$$i\hbar\frac{d}{dt}\psi_{op}(\vec{x},t) = -[\mathbf{H}, \psi_{op}(\vec{x},t)] \tag{3.15}$$

$$i\hbar\frac{d}{dt}\psi^\dagger_{op}(\vec{x},t) = [\mathbf{H}, \psi^\dagger_{op}(\vec{x},t)] \tag{3.16}$$

The energy states in Fock space are obtained as usually, by applying $\mathbf{a}^\dagger_n$ to the vacuum. The action of $\mathbf{a}_n$ and $\mathbf{a}^\dagger_n$ on a state is given by

$$\begin{aligned}\mathbf{a}_n\,|\ldots, N_n, \ldots\rangle &= \sqrt{N_n}\,|\ldots, N_n-1, \ldots\rangle \\ \mathbf{a}^\dagger_n\,|\ldots, N_n, \ldots\rangle &= \sqrt{N_n+1}\,|\ldots, N_n+1, \ldots\rangle\end{aligned}$$

Letting the $\psi^{(\dagger)}_{op}(\vec{x},t)$ act on the vacuum gives:

$$\psi_{op}(\vec{x},t)\,|0\rangle = 0, \tag{3.17}$$

since ψ_{op} contains only annihilators, and

$$\boxed{\psi^\dagger_{op}(\vec{x},t)\,|0\rangle = |\vec{X}\rangle_H} \tag{3.18}$$

which is a state with sharp position. The position $\vec{X}$ is fixed, but described by time-varying Hilbert space vectors, which is where the time dependence comes into the picture.

Exercise: prove eq. (3.18). Hint: use the Schrödinger equation for $\psi(\vec{x},t) = \langle X|\psi_H\rangle$, or write $\vec{X}_H(t) = \int d^3x'\psi^\dagger_{op}(\vec{x}',t)\vec{x}'\psi_{op}(\vec{x},t)$ and show that $\vec{X}_H|\vec{X}\rangle_H = \vec{X}\,|X\rangle_H$.

For the particle number density, we have:

$$\mathbf{n}(\vec{x},t) = \psi^\dagger_{op}(\vec{x},t)\psi_{op}(\vec{x},t) \tag{3.19}$$

$$\mathbf{n}(\vec{x}',t)\psi^\dagger_{op}(\vec{x}',t)\,|0\rangle = \delta(\vec{x}-\vec{x}')\underbrace{\psi^\dagger_{op}(\vec{x},t)\,|0\rangle}_{localized} \tag{3.20}$$

The total particle number N should be conserved, since we are dealing with normal QM systems, where particle production is forbidden. Indeed, using eq. (3.19) and the Heisenberg equation, we find that

$$\int d^3x\mathbf{n}(\vec{x},t) = N$$

is conserved.

Quite generally, for any Schrödinger operator $\mathbf{A}_S(\mathbf{X}_S,\mathbf{P}_S)$, we have an associated (Heisenberg) operator $\mathbf{A}_{F(ock)}$ acting on the Fock space of the one-particle state:

$$\boxed{\mathbf{A}_F(t) = \int d^3x\psi^\dagger_{op}(\vec{x},t)\mathbf{A}_S\psi_{op}(\vec{x},t)} \tag{3.21}$$

Using the commutator $[\psi_{op},\psi^\dagger_{op}] = \delta^3$, the fact that double annihilation on a one-particle state gives zero ($\psi_{op}\psi_{op}\,|1\text{ part.}\rangle = 0$), and the definition above, we see that commutation relations of the form

$$[\mathbf{A}_S,\mathbf{B}_S] = \mathbf{C}_S$$

translate to

$$[\mathbf{A}_F,\mathbf{B}_F] = \mathbf{C}_F$$

(the notation $\mathbf{A}_H$ is also used for $\mathbf{A}_F$).

An example of the above correspondence is the momentum operator, which generates translations. It is defined by $\mathbf{U}_{\vec{a}} = e^{-i\vec{\mathbf{P}}_F\cdot\vec{a}/\hbar}$, where now $\vec{\mathbf{P}}_F$ acts on states in the Fock space. Applying it to a state $|\vec{x}\rangle$ gives

$$\mathbf{U}_{\vec{a}} \underbrace{\psi^\dagger_{op}(\vec{x},t)\,|0\rangle}_{=|\vec{x}\rangle} = \mathbf{U}_{\vec{a}}\psi^\dagger_{op}(\vec{x},t)\mathbf{U}^\dagger_{\vec{a}} \underbrace{\mathbf{U}_{\vec{a}}\,|0\rangle}_{=|0\rangle}$$

Using the fact that $|0\rangle$ is invariant under translations, this becomes:

$$|\vec{x}+\vec{a}\rangle = \psi^\dagger_{op}(\vec{x}+\vec{a},t)\,|0\rangle$$

For infinitesimal translations, we have

$$\left[\vec{\mathbf{P}}_F, \psi^{(\dagger)}_{op}\right] = \frac{\hbar}{i}\vec{\nabla}\psi^{(\dagger)}_{op}$$

For rotations, generated by the angular momentum operator $\vec{\mathbf{J}}_F$, we have the same procedure: $\mathbf{U}_{\vec{\omega}} = e^{-i\vec{\mathbf{J}}_F\cdot\vec{\omega}/\hbar}$, where $\vec{\mathbf{J}}_F$ is given by

$$\vec{\mathbf{J}}_F = \int d^3x \psi^\dagger_{op}(\vec{x},t)\left(\vec{\mathbf{X}}_S \times \frac{\hbar}{i}\vec{\nabla}\right)\psi_{op}(\vec{x},t)$$

And again, for infinitesimal rotations:

$$\left[\vec{\mathbf{J}}_F, \psi^{(\dagger)}_{op}\right] = \left(\vec{\mathbf{X}}_S \times \frac{\hbar}{i}\vec{\nabla}\right)\psi^{(\dagger)}_{op}$$

3.2 Multiparticle Schrödinger equation

3.2.1 Bosonic multiparticle space

The description of systems consisting of more than one identical particles is a nice application of the new QFT-inspired formalism obtained from the second quantization. Note again that we are still talking about QM as we know it; only the language has changed.

Consider a system of N identical particles, which can be in states like

$$\left|n_{\vec{k}_1} n_{\vec{k}_2} \dots n_{\vec{k}_l}\right\rangle = \left|n_{\vec{k}_1}\right\rangle \otimes \left|n_{\vec{k}_2}\right\rangle \otimes \cdots \otimes \left|n_{\vec{k}_l}\right\rangle$$

where the possibility of polarization has been left out to avoid drowning in indices. The states $\left|n_{\vec{k}}\right\rangle$ are given by

$$\left|n_{\vec{k}}\right\rangle = \frac{(\mathbf{a}^\dagger_{\vec{k}})^{n_{\vec{k}}}}{\sqrt{n_{\vec{k}}!}}\,|0\rangle \tag{3.22}$$

General states (still without polarization) look like this:

$$\sum_{l=1}^{\infty}\sum_{\{\vec{k}_1\dots\vec{k}_l\}} f(n_{\vec{k}_1}\vec{k}_1,\dots,n_{\vec{k}_l}\vec{k}_l)\left|n_{\vec{k}_1}\dots n_{\vec{k}_l}\right\rangle \tag{3.23}$$

Here, the order of the $\vec{k}_i$ is arbitrary, since the $\mathbf{a}^\dagger_{\vec{k}_i}$ commute. The total particle number is given by $N = \sum_{i=1}^{l} n_{\vec{k}_i}$. These states live in a Fock space obtained by a direct product of harmonic oscillator Hilbert spaces:

$$\mathfrak{F} = \prod_{l=1} \otimes \mathfrak{H}^{osc}_{\vec{k}_l}$$

They obviously inherit the linear structure of their constituents. Similarly, the inner product between two such states is obtained by taking the inner product in each $\mathfrak{H}_{\vec{k}}$ separately.

Another way of obtaining $\mathfrak{F}$ is summing spaces with fixed particle numbers:

$$\mathfrak{F} = \mathfrak{H}^{(0)} \oplus \mathfrak{H}^{(1)} \oplus \cdots \oplus \mathfrak{H}^{(l)} = \sum_{N} \oplus \mathfrak{H}^{(n)}$$

where

$$\mathfrak{H}^{(i)} = \prod_{i} \otimes \mathfrak{H}^{(1)} \quad \text{symmetrized}$$

is the Hilbert space for i identical particles.

$$\boxed{|\vec{x}_i \dots \vec{x}_l\rangle_H = N_s \psi^\dagger_{op}(\vec{x}_1, t) \dots \psi^\dagger_{op}(\vec{x}_l, t) \, |0\rangle} \tag{3.24}$$

The $\psi^\dagger_{op}$ generate *identical* particles, and hence commute. They are defined as follows:

$$\psi^\dagger_{op}(\vec{x}_j, t) = \begin{cases} \sum\limits_{\vec{k}} N_{\vec{k}} \mathbf{a}^\dagger_{\vec{k}}(t) e^{-i\vec{k}\cdot\vec{x}} & \text{for free particles} \\ \sum\limits_{\vec{k}} \mathbf{a}^\dagger_n(t) \phi^*_n(\vec{x}_j) \quad j = 1, \dots, l & \text{for bound particles} \end{cases} \tag{3.25}$$

Because of the fact that the $\psi^\dagger_{op}$ commute, we obtain $l!$ terms with the same set of $\vec{k}_i$ in the free particles-case. This gives rise to a factor of $l!$ in the inner product of a state with itself. This factor is always the same (exercise: show this), even if some of the $\vec{k}$ are identical. Examples are the case where all $\vec{k}$ are different and the case of m identical $\vec{k}$: in the first case, the norm squared of the state gets a factor $l!$, as just mentioned, and in the second, it is a factor

$$\begin{pmatrix} l \\ m \end{pmatrix} \cdot m!(l-m)! = l!$$

So, we set the normalization coefficient in eq. (3.24) $N_s = 1/\sqrt{l!}$.

Exercise: show, using the normalization coefficient derived above, that the following identity holds for a function f_{sym} that is symmetric in the x_l:

$$\int d^3x'_1 \dots d^3x'_l \, \langle \vec{x}_1 \dots \vec{x}_l | \vec{x}'_1 \dots \vec{x}'_l \rangle \, f_{sym}(\vec{x}'_1 \dots \vec{x}'_l) = f_{sym}(\vec{x}_1 \dots \vec{x}_l)$$

3.2.2 Interactions

The total Hamiltonian can be split up into the parts concerning the individual particles, parts resulting from interactions between two particles, etc.:

$$H = H^{(1)} + H^{(2)} + \dots \tag{3.26}$$

The parts $H^{(i)}$ are defined as follows:

$$H^{(1)} = \int d^3x \psi^\dagger_{op}(\vec{x},t)\left(\frac{-\hbar\nabla^2}{2m} + V(\vec{x})\right)\psi_{op}(\vec{x},t) \tag{3.27}$$

$$H^{(2)} = \int d^3x d^3x' \psi^\dagger_{op}(\vec{x},t)\psi^\dagger_{op}(\vec{x'},t)V(\vec{x},\vec{x'})\psi_{op}(\vec{x},t)\psi_{op}(\vec{x'},t) \tag{3.28}$$

The interaction potential $V(\vec{x},\vec{x}') = V^*(\vec{x},\vec{x}')$ is self-adjoint, e.g. $e^2/|\vec{x}-\vec{x}'|$ for the interaction between two electrons.

Exercise: using $[\psi_{op}(\vec{x},t), \psi^\dagger_{op}(\vec{x}',t)] = \delta^3(\vec{x}-\vec{x}')$, show that

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\underbrace{\psi(\vec{x}_1,\dots,\vec{x}_l,t)}_{=_H\langle\vec{x}_1\dots\vec{x}_l|\psi\rangle_H} &= i\hbar\frac{\partial}{\partial t}\langle 0|\frac{\psi_{op}(\vec{x}_1,t)\dots\psi_{op}(\vec{x}_l,1)}{\sqrt{l!}}|\psi\rangle_H \\
&= \langle 0| - [H^{(1)}+H^{(2)}, \frac{\psi_{op}(\vec{x}_1,t)\dots\psi_{op}(\vec{x}_l,t)}{\sqrt{l!}}|\psi\rangle \\
&\stackrel{!}{=} \langle 0|\sum_{i=1}^{l}\left(H^{(1)}_S(\vec{x}_i,\frac{\hbar}{i}\vec{\nabla}_i) + \frac{1}{2}\sum_{i,j=1,i\neq j}^{l} V(\vec{x}_i,\vec{x}_j)\right) \\
&\quad \times\frac{\psi_{op}(\vec{x}_1,t)\dots\psi_{op}(\vec{x}_1,t)}{\sqrt{l!}}|\psi\rangle \\
&= (H^{(1)}_S + H^{(2)}_S)\psi(\vec{x}_1,\dots,\vec{x}_l,t)
\end{aligned}$$

Given that we are dealing with two-body interactions here, instead of the single particles discussed so far, one could think that the commutation relation between ψ_{op} and $\psi^\dagger_{op}$ might not be the same as before. However, since the canonical momentum remains unchanged, we know that the old relation must still hold.

We started with the *linear* Schrödinger equation, but we could also have started with a nonlinear equation, such as the Gross-Pitaevsky equation for Bose-gasses. In this case, we would have to give up the superposition principle for probability amplitudes. Since we are doing nonrelativistic QM here, we prefer taking into account nonlinear parts only in $H^{(2)}$ and higher orders. Of course, one can also apply ordinary perturbative QFT, as will be discussed later on.

Remark

With the results obtained above, we can write any interaction completely in quantized fields. The electromagnetic interaction, with $\vec{A}(x)$, is, as we will see later on, obtained by minimal gauge coupling:

$$\frac{\hbar}{i}\vec{\nabla} \to \frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{\mathbf{A}}$$

$$H_0 = \int d^3x \psi^\dagger_{op}(\vec{x},t)\left\{\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla}\right)^2 + V(\vec{x})\right\}\psi_{op}(\vec{x},t) + \frac{1}{8\pi}\int d^3x(\vec{E}^2+\vec{B}^2)$$

$$H_I = H^{(2)} + \int d^3x \psi^\dagger_{op}(\vec{x},t)\left[-\frac{e\hbar}{imc}\vec{\mathbf{A}}\cdot\vec{\nabla} + \frac{e^2}{2mc^2}\vec{\mathbf{A}}^2\right]\psi_{op}(\vec{x},t)$$

where ψ, $\psi^\dagger$ and $\vec{\mathbf{A}}$ are now quantum fields.

3.2.3 Fermions

We know from QM that we need the Pauli exclusion principle for electrons, protons, and other particles with half-integer spin. Later on, this will appear as a necessary consequence of QFT; we will see this in our discussion of the Dirac equation. Here, only the results will be given: fermionic fields are quantized by substituting anti-commutators, rather than commutators, for the Poisson bracket. We postulate:

$$[\mathbf{c}_n, \mathbf{c}^\dagger_{n'}]_+ = \mathbf{c}_n\mathbf{c}^\dagger_{n'} + \mathbf{c}^\dagger_{n'}\mathbf{c}_n = \delta_{nn'} \tag{3.29}$$

$$[\mathbf{c}_n, \mathbf{c}_{n'}]_+ = [\mathbf{c}^\dagger_n, \mathbf{c}^\dagger_{n'}]_+ = 0 \tag{3.30}$$

(the notation $\{\cdot,\cdot\}$ is often used as an alternative for $[\cdot,\cdot]_+$.)

Let us investigate the consequences of postulating anticommutating creation and annihilation operators. A striking result is that is immediately obvious is that $\mathbf{c}_n^2 = \mathbf{c}_n^{\dagger 2} = 0$. Now, consider the action of the occupation number operator $\mathbf{N} = \mathbf{c}^\dagger\mathbf{c}$, $\mathbf{N}|n\rangle = n|n\rangle$. Let it act on a state $\mathbf{c}|n\rangle$:

$$\begin{aligned}\mathbf{N}\mathbf{c}|n\rangle &= \mathbf{c}^\dagger \underbrace{\mathbf{c}\mathbf{c}}_{=0}|n\rangle = 0 = \\ &= (-\mathbf{c}\mathbf{c}^\dagger + 1)\mathbf{c}|n\rangle = (-\mathbf{c}\mathbf{N} + \mathbf{c})|n\rangle = (-n+1)\mathbf{c}|n\rangle = 0\end{aligned}$$

So, either $n = 1$ or $\mathbf{c}|n\rangle = 0$. Now, let it act on a state $\mathbf{c}^\dagger$:

$$\mathbf{N}\mathbf{c}^\dagger|n\rangle = \mathbf{c}^\dagger\mathbf{c}\mathbf{c}^\dagger|n\rangle = \underbrace{\mathbf{c}^\dagger(-\mathbf{c}^\dagger}_{=0}\mathbf{c} + 1)|n\rangle = \mathbf{c}^\dagger|n\rangle = (-n+1)\mathbf{c}^\dagger|n\rangle$$

So, either $n = 0$ or $\mathbf{c}^\dagger |n\rangle = 0$, giving the following relations:

$$\mathbf{c}^\dagger |1\rangle = 0, \quad \mathbf{c}|0\rangle = 0 \tag{3.31}$$
$$\mathbf{c}^\dagger |0\rangle = |1\rangle, \quad \mathbf{c}|1\rangle = |0\rangle \tag{3.32}$$

In other words, the only occupation numbers are 0 and 1, as we wanted for fermions. The normalization is as follows:

$$\langle 0|0\rangle = 1 \tag{3.33}$$
$$\langle 1|1\rangle = \langle 0|\mathbf{c}^\dagger\mathbf{c}|0\rangle = \langle 0|-\mathbf{c}\mathbf{c}^\dagger + 1|0\rangle = \langle 0|0\rangle = 1 \tag{3.34}$$

A general fermion state is:

$$\sum_{\{\vec{k}_1 \ldots \vec{k}_l\}} \prod_{i=1}^{l} f(\vec{k}_1 \ldots \vec{k}_l)\mathbf{c}^\dagger_{\vec{k}_i}|0\rangle \tag{3.35}$$

with $n_{\vec{k}_i} = 1$. Since the $\mathbf{c}^\dagger_{\vec{k}_i}$ anticommute, $\prod \mathbf{c}^\dagger_{\vec{k}_i}$ is totally antisymmetric, which means that only the totally antisymmetric part of f is interesting.

In x-space, states look like this:

$$|\vec{x}_1(t)\ldots\vec{x}_l(t)\rangle = \frac{1}{\sqrt{l!}}\psi^\dagger_{op}(\vec{x}_1,t)\ldots\psi^\dagger_{op}(\vec{x}_1,t)|0\rangle \tag{3.36}$$

where the $\psi^\dagger_{op}$ are, as usual, general solutions of the field equations:

$$\boxed{\psi^\dagger_{op}(\vec{x},t) = \sum_n \mathbf{c}^\dagger_n \phi^*_n(\vec{x})} \tag{3.37}$$

A state with l different oscillators occupied is given by:

$$\frac{1}{\sqrt{l!}}\sum_{\{1,\ldots,l\}_P}(-1)^P \phi^*_{k_{p_1}}(\vec{x}_1)\ldots\phi^*_{k_{p_l}}(\vec{x}_l)\mathbf{c}^\dagger_{k_1}\ldots\mathbf{c}^\dagger_{k_l}|0\rangle =$$
$$\frac{1}{\sqrt{l!}}\begin{vmatrix} \phi^*_{k_1}(\vec{x}_1) & \ldots & \phi^*_{k_l}(\vec{x}_1) \\ \phi^*_{k_1}(\vec{x}_2) & \ldots & \phi^*_{k_l}(\vec{x}_2) \\ \vdots & & \vdots \\ \phi^*_{k_1}(\vec{x}_l) & \ldots & \phi^*_{k_l}(\vec{x}_l) \end{vmatrix} \times \mathbf{c}^\dagger_{k_1}\ldots\mathbf{c}^\dagger_{k_l}|0\rangle \tag{3.38}$$

The determinant is called *Slater*-determinant, and gives rise to antisymmetric wave functions. It arises from the permutations which are summed over with changing sign (the part $\sum_{\{1,\ldots,l\}_P}(-1)^P$ in the left hand side).