Quantum Field Theory I, 1st problem sheet

Discussion of solutions in the tutorials of Oct 25, 2006

Problem 1: Chain of oscillators

1. Consider a one-dimensional classical harmonic oscillator whose Hamiltonian is
$$H = \frac{p^2}{2m} + \frac{m\Omega^2}{2} q^2.$$
Write down the Hamiltonian in terms of a and a^*, where a is given by
$$a = \sqrt{\frac{m\Omega}{2}}\, q + \frac{i}{\sqrt{2m\Omega}}\, p,$$
and solve the resulting equations of motion. By explicitly computing the Poisson bracket $\{a, ia^*\}$, show that the transformation $q, p \leftrightarrow a, ia^*$ is canonical.

2. The Hamiltonian for a circular chain of N coupled oscillators is
$$H = \sum_{j=1}^{N} \left(\frac{p_j^2}{2m} + \frac{m\Omega^2}{2} (q_j - q_{j-1})^2 \right).$$
We identify the 0-th and the N-th oscillator. Show that the Hamiltonian may be written as
$$H = \sum_{n=-N/2}^{N/2-1} \omega_n\, a_n a_n^*,$$
where
$$\begin{aligned} q_j =& \frac{1}{\sqrt{N}} \sum_{n=-N/2}^{N/2-1} \frac{1}{\sqrt{2m\omega_n}} \left(a_n e^{2\pi i\, nj/N} + a_n^* e^{-2\pi i\, nj/N} \right), \\ p_k =& -\frac{i}{\sqrt{N}} \sum_{n=-N/2}^{N/2-1} \sqrt{\frac{m\omega_n}{2}} \left(a_n e^{2\pi i\, nk/N} - a_n^* e^{-2\pi i\, nk/N} \right), \\ \omega_n =& 2\Omega \sin \frac{\pi n}{N}. \end{aligned}$$
Show also that the transformation $q_j, p_k \leftrightarrow a_m, ia_n^*$ is canonical, by computing the Poisson bracket $\{p_k, q_j\}$ with respect to the new coordinates a_m and momenta ia_n^*.

Hint: It may be helpful to write
$$q_j = \frac{1}{\sqrt{N}} \sum_{n=-N/2}^{N/2-1} \frac{1}{\sqrt{2m\omega_n}} e^{2\pi i\, nj/N} \left(a_n + a_{-n}^* \right)$$
and accordingly for p_k.

Problem 2: Coherent states and the classical limit of quantum mechanics

Consider a one-dimensional quantum mechanical harmonic oscillator with frequency ω in the Schrödinger picture.

1. Show that the sum of squared uncertainties $u \equiv (\Delta X)^2 + (\Delta P)^2$ is extremal if the state $|\psi\rangle$ of the system is a coherent state.

   Hint: Consider the change of u under a small variation variation $|\psi\rangle \to |\psi\rangle + |\delta\psi\rangle$, and use a Lagrange multiplier to enforce the constraint $\langle\psi|\psi\rangle = 1$. Recall also that coherent states are precisely the eigenstates of the annihilation operator a.

2. Show that a coherent state $|\lambda\rangle = e^{-\frac{1}{2}|\lambda|^2} e^{\lambda a^\dagger(0)}|0\rangle$ solves the Schrödinger equation if $\lambda(t) = \lambda_0 e^{-i\omega t}$.

3. Calculate the position-space probability density $|\langle x|\lambda\rangle|^2$, where $|\lambda\rangle = e^{-\frac{1}{2}|\lambda|^2} e^{\lambda a^\dagger(0)}|0\rangle$ is a coherent state and $|x\rangle$ is an eigenstate of the position operator, $X|x\rangle = x|x\rangle$. Compare the time dependence of the maximum in x for $\lambda(t) = \lambda_0 e^{-i\omega t}$ with the time evolution of a classical harmonic oscillator.

   Hint: Rewrite $e^{\lambda a^\dagger}$ in terms of X acting to the left and a acting to the right, using the Baker-Campbell-Hausdorff formula

$$\exp(A)\exp(B) = \exp\left(A + B + \frac{1}{2}[A,B] + (\text{commutators involving } [A,B])\right).$$