Quantum Field Theory I, 10th problem sheet

Discussion of solutions in the tutorials of Jan 17th

Problem 1: Path integrals for complex fields

Consider a free complex scalar field ϕ. Derive the formal identity

$$\begin{aligned} Z[J,J^\dagger] &\equiv \int D\phi D\phi^\dagger \, \exp\left(i\int d^4x\,(\partial_\mu\phi^\dagger\partial^\mu\phi - m^2|\phi|^2 + J^\dagger\phi + \phi^\dagger J)\right) \\ &= Z[0,0]\exp\left(-i\int d^4x\,d^4x'\,J^\dagger(x)\,D_F(x-x')\,J(x')\right). \end{aligned}$$

Problem 2: Cubic interactions

Consider a real scalar field ϕ with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\mu}{3!}m\phi^3 - \frac{\lambda}{4!}\phi^4.$$

Write down the Feynman rules. Find the tree-level diagrams which contribute to 2-2 scattering and discuss their crossing symmetries. Compute the tree-level amplitude for such a scattering process.

Problem 3: The Casimir effect

The Casimir force in electrodynamics is a force between conducting surfaces. It is caused by the change of the vacuum energy when boundary conditions for the electromagnetic field are introduced. Here we use a toy model to calculate a similar effect in a simplified setting, for a treatment of a realistic setup in electrodynamics see e.g. Huang's book.

Consider a real scalar field in $\mathbb{R}^{1,1}$, i.e. in a two-dimensional spacetime, with coordinates (t,x). Introduce four "conducting points" at $x=-L,-a,a,L$, where $a \ll L$ (so that the field is required to vanish at $x=\pm L$ and $x=\pm a$). Calculate the Casimir force between the two inner boundary points at $\pm a$: Find a formal expression for the vacuum energy between $-L$ and L, by summing over the zero mode energies of the field in the intervals $[-L,-a]$, $[-a,a]$ and $[a,L]$ (introduce an exponential cutoff function for the high-frequency modes to avoid infinities). Differentiation with respect to a gives the force, which should not depend on the cutoff.