# Quantum Field Theory I, 11th problem sheet

Discussion of solutions in the tutorials of Jan 24th

## Problem 1: Decay of a scalar particle

*(Shamelessly stolen from Peskin and Schroeder.)*

Consider the following Lagrangian, involving two real scalar fields $\phi$ and $\chi$:

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}\partial^\mu\chi\partial_\mu\chi - \frac{1}{2}m^2\phi^2 - \frac{1}{2}M^2\chi^2 - \mu\chi\phi^2.$$

Assume that $M > 2m$, so that the decay $\chi \to 2\phi$ is kinematically possible. Calculate the lifetime of $\chi$ to leading order.
*Hint:* You may want to use the formula for the differential decay width,

$$d\Gamma = \frac{1}{2M}\prod_f d\tilde{p}_f \left|\mathcal{M}(p_i \to \{p_f\})\right|^2 (2\pi)^4\delta^{(4)}\left(p_i - \sum p_f\right),$$

where $\mathcal{M}(p_i \to \{p_f\})$ is the matrix element for a transition from an initial state with momentum $p_i = (M, \mathbf{0})$ to a final state with momenta $\{p_f\}$.

## Problem 2: The Lorentz and Poincaré algebras

Recall that the Lorentz algebra commutation relations are

$$[M_{\mu\nu}, M_{\kappa\lambda}] = -i(g_{\mu\kappa}M_{\nu\lambda} - g_{\nu\kappa}M_{\mu\lambda} + g_{\mu\lambda}M_{\kappa\nu} - g_{\nu\lambda}M_{\kappa\mu})$$

Here $M_{\mu\nu}$ is an antisymmetric matrix whose entries generate the Lorentz group. They can be represented as matrices themselves, acting on some representation space.

1. As shown in the lecture, a change of basis for the Lorentz algebra, introducing the generators of rotations and boosts,

   $$M_i \equiv \frac{1}{2}\epsilon_{ijk}M_{jk}, \quad N_i \equiv M_{i0}, \qquad i,j,k \in \{1,2,3\},$$

   leads to the commutation relations

   $$[M_i, M_j] = i\epsilon_{ijk}M_k, \quad [N_i, N_j] = -i\epsilon_{ijk}M_k, \quad [M_i, N_j] = i\epsilon_{ijk}N_k.$$

   Now consider another change of basis: Derive the commutation relations for $J_k \equiv \frac{1}{2}(M_k + iN_k)$ and $K_k \equiv \frac{1}{2}(M_k - iN_k)$.

2. Consider the matrix

$$A(\xi) = \begin{pmatrix} 0 & \xi & 0 & 0 \\ -\xi^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

When acting on four-vectors, what physical transformation are represented by a) $\exp(A(\xi))$ with real $\xi$, b) $\exp(iA(\xi))$ with imaginary $\xi$?

Now recall that the Lorentz algebra may be extended to the Poincaré algebra, by adding the translation generators $P_\mu$:

$$[M_{\mu\nu}, P_\kappa] = i(g_{\nu\kappa}P_\mu - g_{\mu\kappa}P_\nu),$$
$$[P_\mu, P_\nu] = 0.$$

3. In the lecture the Pauli-Ljubanski vector was defined as

$$W_\sigma = \frac{1}{2}\epsilon_{\sigma\mu\nu\lambda}M^{\mu\nu}P^\lambda \qquad (\epsilon_{0123} = 1).$$

Check the commutation relations

$$[M_{\mu\nu}, W_\rho] = -i(g_{\nu\rho}W_\mu - g_{\mu\rho}W_\nu), \quad [W_\mu, P_\nu] = 0, \quad [W_\mu, W_\nu] = i\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma.$$

**Problem 3: Dirac matrices**

Reminder: the Dirac matrices $(\gamma^\mu_{\alpha\beta})$ are a set of four ($\mu \in \{0, 1, 2, 3\}$) $4 \times 4$ matrices ($\alpha, \beta \in \{1, 2, 3, 4\}$) which satisfy the Clifford algebra,

$$(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = 2g^{\mu\nu}\,\mathbb{1}.$$

1. In the lecture, the matrices $\lambda^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ were introduced. Prove that they form a representation of the Lorentz algebra:

$$[\lambda^{\mu\nu}, \lambda^{\kappa\lambda}] = -i(g^{\mu\kappa}\lambda^{\nu\lambda} - g^{\nu\kappa}\lambda^{\mu\lambda} + g^{\mu\lambda}\lambda^{\kappa\nu} - g^{\nu\lambda}\lambda^{\kappa\mu}).$$

2. Given a Lorentz transformation in this representation, $S = \exp(\frac{i}{2}\lambda^{\mu\nu}\omega_{\mu\nu})$ (where $\omega_{\mu\nu}$ is the real antisymmetric matrix of transformation parameters), show that

$$S^{-1}\gamma^\mu S = \Lambda^\mu_\nu\gamma^\nu.$$

Here $\Lambda^\mu_\nu$ is the same Lorentz transformation in the vector representation, which is generated by the matrices $(M^{\mu\nu})_{\kappa\lambda} = i\delta^\mu_\kappa\delta^\nu_\lambda - i\delta^\mu_\lambda\delta^\nu_\kappa$.

You may want to check only the infinitesimal case, i.e. to drop any terms of quadratic or higher order in $||\omega||$.

3. Verify that the Dirac Lagrangian

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

is a Lorentz scalar.