Quantum Field Theory I, 12th problem sheet

Discussion of solutions in the tutorials of Jan 31th

Problem 1: More on the Poincaré algebra

1. Given the 5×5 matrix $\mathcal{M}$,

$$\mathcal{M} = \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix}$$

   where $\Lambda \in \mathrm{SO}(1,3)$ represents a proper Lorentz transformation and $a \in \mathbb{R}^{1,3}$ is a four-vector, write down its action on $(x,1)^T$ (x is also a four-vector).

2. Find a set of ten 5×5 matrices $\{M^{\mu\nu}, P^\lambda\}$ (with $\mu, \nu, \lambda = 0 \dots 3$ and $M^{\mu\nu} = -M^{\nu\mu}$) in terms of which any such $\mathcal{M}$ can be written as

$$\mathcal{M} = \exp\left(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + i\lambda_\mu P^\mu\right),$$

   with real parameters $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and λ_μ. Normalize them properly, and verify as many of the Poincaré algebra commutation relations for $M^{\mu\nu}$ and P^μ as you wish.

Problem 2: More on spinor representations of the Lorentz algebra

1. Show that

$$\mathbf{M} = \frac{1}{2}\gamma^5\gamma^0\boldsymbol{\gamma}, \qquad \mathbf{N} = \frac{i}{2}\gamma^0\boldsymbol{\gamma}$$

   satisfy the Lorentz algebra commutation relations in the basis of boosts and rotations as given in exercise 11.2,

$$[M_i, M_j] = i\epsilon_{ijk}M_k, \quad [N_i, N_j] = -i\epsilon_{ijk}M_k, \quad [M_i, N_j] = i\epsilon_{ijk}N_k.$$

2. Prove that, as claimed in the lecture, for any $M \in \mathrm{SL}(2,\mathbb{C})$

$$M^{-1} = \sigma^2 M^T \sigma^2.$$

   Hint: Write $M = a\mathbb{1} + \sum_i b_i\sigma^i$ with complex a and b_i satisfying $a^2 - \sum_i b_i^2 = 1$ (after convincing yourself that all $\mathrm{SL}(2,\mathbb{C})$ matrices can be written this way).

3. Consider the Lorentz boost which takes a four-vector $(m, \mathbf{0})^T \to (p^0, \mathbf{p})^T$, where of course $(p^0)^2 = \mathbf{p}^2 + m^2$. Show that it is represented, in Weyl spinor space, by the $\mathrm{SL}(2, \mathbb{C})$ matrix

$$M = \frac{(p^0 + m)\mathbb{1} - \mathbf{p} \cdot \boldsymbol{\sigma}}{\sqrt{2m(m + p^0)}}.$$

That is, show that for $\hat{p} = m\mathbb{1}$, the transformed $\hat{p}' = p^0\mathbb{1} - \mathbf{p} \cdot \boldsymbol{\sigma}$ is given by $\hat{p}' = M\hat{p}M^\dagger$. Show also that

$$(p^0\mathbb{1} + \mathbf{p} \cdot \boldsymbol{\sigma})M = mM^{-1}, \qquad (p^0\mathbb{1} - \mathbf{p} \cdot \boldsymbol{\sigma})M^{-1} = mM,$$

and

$$M = \exp\left(-u\,\frac{\mathbf{p}}{|\mathbf{p}|} \cdot \frac{\boldsymbol{\sigma}}{2}\right), \qquad \text{where } \tanh u = \frac{|\mathbf{p}|}{p^0}.$$

Hint: For the last identity, the following formulas may be useful:

$$\cosh\frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}, \quad \sinh\frac{x}{2} = \mathrm{sign}\,(x)\sqrt{\frac{\cosh x - 1}{2}},$$

$$\cosh(\mathrm{arctanh}\,(x)) = \frac{1}{\sqrt{1 - x^2}}.$$

Problem 3: Representations of the Clifford algebra

Show that the Dirac-Pauli representation (here written in terms of 2×2 block matrices)

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}$$

is indeed a representation of the Clifford algebra, and find the unitary transformation that takes it into the usual Weyl representation:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}.$$