Quantum Field Theory I, 13th problem sheet

Discussion of solutions in the tutorials of Feb 7th

Problem 1: More on Dirac matrices

The charge conjugation operator C and time reversal operator T were shown to act on γ matrices as

$$C\gamma^{\mu T}C^{-1} = -\gamma^{\mu}, \qquad T\gamma^{\mu}T^{-1} = \eta_{\mu\mu}\gamma^{\mu *} \quad \text{(no summation)}.$$

Check that they can be written, in the Weyl representation, as

$$C = i\gamma^0\gamma^2, \qquad T = -i\gamma^1\gamma^3.$$

Problem 2: Dirac spinors

1. Let $u^s_{\mathbf{p}}$ and $v^s_{\mathbf{p}}$ be a set of solutions of the Dirac equation in the chiral basis, normalized as in the lecture:

   $$u^s_{\mathbf{p}} = \frac{1}{\sqrt{2(p^0+m)}}\begin{pmatrix} p^0 + m - \boldsymbol{\sigma}\cdot\mathbf{p} \\ p^0 + m + \boldsymbol{\sigma}\cdot\mathbf{p} \end{pmatrix} \otimes \chi^s, \quad v^s_{\mathbf{p}} = C\bar{u}^{sT}_{\mathbf{p}}.$$

   Show that

   $$\bar{u}^s_{\mathbf{p}}u^{s'}_{\mathbf{p}} = 2m\delta^{ss'}, \quad \bar{v}^s_{\mathbf{p}}v^{s'}_{\mathbf{p}} = -2m\delta^{ss'}, \quad \bar{u}^s_{\mathbf{p}}\gamma^0 u^{s'}_{\mathbf{p}} = 2p^0\delta^{ss'}, \quad \bar{v}^s_{\mathbf{p}}\gamma^0 v^{s'}_{\mathbf{p}} = 2p^0\delta^{ss'}.$$

2. With the general solution of the Dirac equation given by

   $$\psi(x) = \int d\tilde{p} \sum_{s=1,2}\left(a^s(\mathbf{p})u^s_{\mathbf{p}}e^{-ipx} + b^s(\mathbf{p})v^s_{\mathbf{p}}e^{ipx}\right),$$

   show that the Dirac Hamiltonian can be written

   $$H = \int d\tilde{p}\, p^0 \sum_{s=1,2}\left(a^{s^*}(\mathbf{p})a^s(\mathbf{p}) - b^s(\mathbf{p})b^{s^*}(\mathbf{p})\right).$$

Problem 3: The Dirac propagator

Check explicitly that the two-point function

$$S_F(x-y)_{\alpha\beta} = \langle 0|T(\psi_\alpha(x)\overline{\psi}_\beta(y))|0\rangle,$$

is a Green function for the Dirac operator $D_{\alpha\beta} = i\gamma^{\mu}_{\alpha\beta}\partial_\mu - m\delta_{\alpha\beta}$, by showing

$$D_{\alpha\beta}\, S_F(x-y)_{\beta\gamma} = i\delta^4(x-y)\delta_{\alpha\gamma}.$$

Problem 4: Neutron-antineutron annihilation in Yukawa theory

A toy model for the strong interactions between neutrons at low energies is given by the Lagrangian

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - m_N)\psi + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m_{\pi^0}^2\phi^2 - g\phi\overline{\psi}\gamma^5\psi,$$

where ψ is the neutron field, ϕ is the pseudoscalar neutral π^0 meson, m_N and m_{π^0} MeV are their respective masses, and g is a coupling constant. Compute the tree-level cross section for $N\overline{N} \to \pi^0\pi^0$ scattering, assuming the incoming particles are unpolarized (you can average over incoming spins).

Particle physics bonus question: Why can this calculation not give a good approximation to the observed physical result?

Problem 5: Fierz identities

1. As you have learned, there are 16 independent Lorentz-covariant combinations of Dirac matrices, call them Γ^A. Up to scalar factors these are $\mathbb{1}$, the four Dirac matrices γ^μ, the six independent components of $\gamma^{\mu\nu} \equiv [\gamma^\mu, \gamma^\nu]$, the four $\gamma^\mu\gamma^5$, and γ^5. Find the normalization factors such that

$$\mathrm{tr}\,(\Gamma^A\Gamma^B) = 4\delta^{AB}.$$

   Hint: You may want to use

$$\mathrm{tr}\,(\text{any odd number of } \gamma^\mu\text{s}) = \mathrm{tr}\,\gamma^5 = \mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^5) = 0$$

   to check that indeed $\mathrm{tr}\,(\Gamma^A\Gamma^B) = 0$ if $A \neq B$.

2. Let u_i, $i = 1 \ldots 4$, be four Dirac spinors. The general *Fierz identity* reads

$$(\bar{u}_1\Gamma^A u_2)(\bar{u}_3\Gamma^B u_4) = C^{AB}{}_{CD}(\bar{u}_1\Gamma^C u_4)(\bar{u}_3\Gamma^D u_2),$$

   with some coefficients $C^{AB}{}_{CD}$. Show that these are given by

$$C^{AB}{}_{CD} = \frac{1}{16}\mathrm{tr}\,(\Gamma^C\Gamma^A\Gamma^D\Gamma^B).$$

   Hint: Use the completeness of the Γ^A matrices: every 4×4 matrix M^α_β has an expansion $M^\alpha_\beta = c_A\,\Gamma^{A\,\alpha}{}_\beta$. Try expanding $u_4^\alpha\bar{u}_{1\beta}$ and $u_2^\alpha\bar{u}_{3\beta}$ in this manner.

3. Work out explicitly the Fierz identity for the products $(\bar{u}_1 u_2)(\bar{u}_3 u_4)$ and $(\bar{u}_1\gamma^\mu u_2)(\bar{u}_3\gamma_\mu u_4)$.