Quantum Field Theory I, 2nd problem sheet

Discussion of solutions in the tutorials of Nov 8th

Problem 1: The electromagnetic field as a collection of harmonic oscillators

As shown in the lecture, the energy of a classical electromagnetic field may be written in terms of the vector potential $\mathbf{A}$ (in Coulomb gauge, with $c = 1$) as

$$E = \frac{1}{8\pi}\int d^3x \left(\left(\frac{\partial \mathbf{A}}{\partial t}\right)^2 + \sum_{i,j=1}^{3}\left(\frac{\partial A^j}{\partial x^i}\right)^2\right).$$

1. Show that, with $\mathbf{A}$ solving the wave equation and with the Fourier decomposition of $\mathbf{A}$ given by

$$\mathbf{A}(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} N_{\mathbf{k}} \left(\mathbf{a}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} + \mathbf{a}^*_{\mathbf{k}}(t)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\right),$$

(where $\mathbf{k} \perp \mathbf{a}_{\mathbf{k}}$: only transversal modes are physical) the energy becomes

$$E = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\omega_{\mathbf{k}} \left(\mathbf{a}^*_{\mathbf{k}}(t)\cdot\mathbf{a}_{\mathbf{k}}(t) + \mathbf{a}_{\mathbf{k}}(t)\cdot\mathbf{a}^*_{\mathbf{k}}(t)\right).$$

Here $\omega_{\mathbf{k}} = |\mathbf{k}|$ and the normalization $N_{\mathbf{k}}$ needs to be chosen properly (how?)

Following the lecture, let us now regard the above system as a quantum system with the Fourier coefficients $a^i_{\mathbf{k}}(t)$ and $a^{i*}_{\mathbf{k}}(t)$ promoted to operators $\hat{a}^i_{\mathbf{k}}(t), \hat{a}^{i\dagger}_{\mathbf{k}}(t)$ (where $i = 1, 2$ is an index for the transversal spatial direction spanned by a unit vector $\mathbf{e}^i$, such that $\mathbf{e}^1 \perp \mathbf{e}^2 \perp \mathbf{k}$). We take these operators to satisfy the usual harmonic oscillator commutation relations. Field operators $\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$ are derived as in the classical source-free case from the vector potential, now also an operator. A classical electromagnetic plane wave is approximated by the expectation values of $\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$ in a coherent state

$$|\lambda_{\mathbf{k},i}\rangle = e^{-\frac{1}{2}|\lambda_{\mathbf{k},i}|^2} e^{\lambda_{\mathbf{k},i}\,\hat{a}^{i\,\dagger}_{k}(0)}|0_{\mathbf{k},i}\rangle.$$

2. Compute the expectation value $\langle\lambda_{\mathbf{k},i}|\hat{\mathbf{E}}|\lambda_{\mathbf{k},i}\rangle$, and show that $\lambda_{\mathbf{k},i}$ corresponds to the wave's amplitude.

3. Compute $\langle\hat{\mathbf{E}}^2\rangle - |\langle\hat{\mathbf{E}}\rangle|^2$ in a coherent state and verify that this quantity vanishes in the classical limit $\hbar \to 0$.

Hint: Recall that, by construction of the Fock space, coherent states corresponding to different oscillators are always orthogonal:

$$\langle\lambda_{\mathbf{k},i}|\lambda'_{\mathbf{p},j}\rangle \sim \delta^{(3)}(\mathbf{k} - \mathbf{p})\,\delta_{ij}.$$

Problem 2: Lagrangian formalism for classical electrodynamics

The action of classical electrodynamics is (in Heaviside-Lorentz units with $c = 1$)

$$S[A] = \int d^4x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A_\mu j^\mu \right) .$$

1. Show that this action leads to the correct equations of motion, namely, the inhomogenous Maxwell equations. (Recall from your electrodynamics course that the homogenous equations are identically satisfied, by definition of $F^{\mu\nu}$.)

2. Consider a gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$ with Λ and all its derivatives vanishing at infinity. Show that the above action is invariant under this transformation if the current j^μ obeys the continuity equation $\partial_\mu j^\mu = 0$.

3. Show that, if we add a "mass term" $-\frac{1}{2} m^2 A_\mu A^\mu$ for the gauge field to the Lagrangian, gauge symmetry is broken. Calculate the equations of motion and the electrostatic potential for a point source in this case.

   Hint: To find the potential, first find its Fourier transform, and then apply the residue theorem to evaluate the inverse transform.