Quantum Field Theory I, 3rd problem sheet

Discussion of solutions in the tutorials of Nov 15th

Problem 1: Free complex scalar field

Given the Lagrange density of a relativistic, complex-valued, free scalar field ϕ,

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi)^* - m^2 |\phi|^2,$$

find the canonical momenta π and π^* (treat the fields ϕ and its complex conjugate ϕ^* as independent degrees of freedom). With the Fourier mode expansion of ϕ and ϕ^* given by

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_{\mathbf{p}}(t) e^{i\mathbf{p}\cdot\mathbf{x}} + b^*_{\mathbf{p}}(t) e^{-i\mathbf{p}\cdot\mathbf{x}} \right),$$

$$\phi^*(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(b_{\mathbf{p}}(t) e^{i\mathbf{p}\cdot\mathbf{x}} + a^*_{\mathbf{p}}(t) e^{-i\mathbf{p}\cdot\mathbf{x}} \right),$$

express π and π^* in terms of the Fourier coefficients $a_{\mathbf{p}}, a^*_{\mathbf{p}}, b_{\mathbf{p}}, b^*_{\mathbf{p}}$. Invert the Fourier transformations to express these coefficients in terms of the fields and momenta. Imposing the canonical equal-time commutation relations

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = [\phi^*(\mathbf{x}, t), \pi^*(\mathbf{y}, t)] = i\hbar\, \delta^{(3)}(\mathbf{x} - \mathbf{y}),$$

$$(\text{all other independent commutators involving } \phi, \pi, \phi^*, \pi^*) = 0,$$

derive the commutation relations

$$[a_{\mathbf{p}}(t), a^*_{\mathbf{q}}(t)] = [b_{\mathbf{p}}(t), b^*_{\mathbf{q}}(t)] = (2\pi)^3 \hbar\, \delta^{(3)}(\mathbf{p} - \mathbf{q}).$$

Problem 2: Second quantization of the Schrödinger equation

As shown in the lecture, it is possible to obtain the Schrödinger equation from a Lagrangian, to define a canonical momentum for the wave function ψ via this Lagrangian, and finally to promote ψ to a field operator $\hat{\psi}$ on Fock space by imposing canonical commutation relations.

1. Show that $\hat{\psi}^\dagger(\mathbf{x}, t)$ creates a position eigenstate from the Fock space vacuum:

$$\hat{\mathbf{X}}(t) \hat{\psi}^\dagger(\mathbf{x}, t) |0\rangle = \mathbf{x}(t) \hat{\psi}^\dagger(\mathbf{x}, t) |0\rangle.$$

Hint: Recall that the $\hat{\mathbf{X}}$ operator is defined as

$$\hat{\mathbf{X}}(t) = \int \hat{\psi}^\dagger(\mathbf{x}', t)\, \mathbf{x}'\, \hat{\psi}(\mathbf{x}', t)\, d^3x'$$

and that the operator conjugate to $\hat{\psi}$ is $i\hbar \hat{\psi}^\dagger$.

2. Show that the position eigenstates $|\mathbf{x}(t)\rangle \equiv \hat{\psi}^\dagger(\mathbf{x}, t)|0\rangle$ are properly normalized:
$$\langle \mathbf{x}(t)|\mathbf{x}'(t)\rangle = \delta^{(3)}(\mathbf{x} - \mathbf{x}').$$

3. Show that the Fock-space Hamiltonian $\hat{H}$,
$$\hat{H}(t) = \int \hat{\psi}^\dagger(\mathbf{x}', t)\, H\, \hat{\psi}(\mathbf{x}', t)\, d^3x',$$
acts on a single-particle Fock state as the quantum mechanical position-space Hamiltonian H on the wave function.

4. The probability density operator is given by
$$\hat{n}(\mathbf{x}, t) = \hat{\psi}^\dagger(\mathbf{x}, t)\hat{\psi}(\mathbf{x}, t)\, d^3x.$$
Show that the overall probability
$$\hat{N}(t) = \int d^3x\, \hat{n}(\mathbf{x}, t)$$
is conserved, $d\hat{N}/dt = 0$, if H is Hermitian.