Quantum Field Theory I, 4th problem sheet

Discussion of solutions in the tutorials of Nov 22th

Problem 1: Second quantization with interacting fields

Consider a Fock-space Hamiltonian $\hat{H}$ of the form $\hat{H} = \hat{H}^{(1)} + \hat{H}^{(2)}$ with a "single-particle" piece

$$\hat{H}^{(1)} = \int d^3x\, \hat{\psi}^\dagger(\mathbf{x},t) \left(-\frac{\Delta}{2m} + V(\mathbf{x})\right) \hat{\psi}(\mathbf{x},t)$$

and an "interaction" piece

$$\hat{H}^{(2)} = \frac{1}{2} \int d^3x \int d^3x'\; \hat{\psi}^\dagger(\mathbf{x},t)\, \hat{\psi}^\dagger(\mathbf{x}',t)\, \tilde{V}(\mathbf{x},\mathbf{x}')\, \hat{\psi}(\mathbf{x},t)\, \hat{\psi}(\mathbf{x}',t).$$

Derive the position-space Schrödinger equation from the Fock-space Heisenberg equation for $\hat{\psi}(\mathbf{x}_1,t)\hat{\psi}(\mathbf{x}_2,t)\cdots\hat{\psi}(\mathbf{x}_n,t)$.

Problem 2: Fermionic oscillators

In the lecture fermionic annihilation and creation operators $c_k, c_k^\dagger$ were introduced. These are taken to satisfy the anticommutation relations

$$\{c_k, c_{k'}^\dagger\} = \delta_{kk'}, \qquad \{c_k, c_{k'}\} = \{c_k^\dagger, c_{k'}^\dagger\} = 0,$$

with the notation $\{a,b\} \equiv ab + ba$. The occupation number operator N (for fixed k, now dropping the index) was defined as in the bosonic case,

$$N = c^\dagger c.$$

Show:

1. If $|1\rangle$ has occupation number one, $N|1\rangle = |1\rangle$, then $c^\dagger|1\rangle = 0$.
2. If $|0\rangle$ has occupation number zero, then $c|0\rangle = 0$.
3. N has eigenvalues 0 and 1: each state can be occupied at most once.
4. If $|0\rangle$ is properly normalized, then so is $|1\rangle \equiv c^\dagger|0\rangle$.

Problem 3: The free complex scalar field revisited

In Problem 3.1 you were given the following expresssion for the Fourier expansion of a complex scalar field of mass m:

$$\phi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_{\mathbf{p}}(t) e^{i\mathbf{p}\cdot\mathbf{x}} + b^\dagger_{\mathbf{p}}(t) e^{-i\mathbf{p}\cdot\mathbf{x}} \right).$$

1. Find the definition of $a(p)$ and $b(p)$ such that

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0_+} \left(a(p) e^{-ip_\mu x^\mu} + b^\dagger(p) e^{ip_\mu x^\mu} \right),$$

   where $x = (t, \mathbf{x})$ and $p = (p^0_+, \mathbf{p})$ are four-vectors, and $p^0_+ \equiv \omega_p$ satisfies $(p^0_+)^2 - \mathbf{p}^2 = m^2$.

2. Work out the commutation relations for $a(p)$ and $b(p)$, using the commutation relations for $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ from exercise 3.1.

3. Convince yourself that $a(p)$ and $b^\dagger(p)$ can be written as

$$a(p) = i \int d^3x \, e^{ip_\mu x^\mu} \overleftrightarrow{\partial_0} \phi(x), \quad b^\dagger(p) = -i \int d^3x \, e^{-ip_\mu x^\mu} \overleftrightarrow{\partial_0} \phi(x).$$

4. Show that $\delta(p_\mu p^\mu - m^2)\,\theta(p^0)$ is invariant under proper ortochronous Lorentz transformations. (As shown in the lecture, this implies that the measure

$$d\tilde{p} \equiv \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0_+} = \frac{d^4p}{(2\pi)^4} (2\pi)\,\delta(p_\mu p^\mu - m^2)\,\theta(p^0)$$

   is invariant.)