Universität Heidelberg WS 06/07

Quantum Field Theory I, 5th problem sheet

Discussion of solutions in the tutorials of Nov 29th

Problem 1: Green's functions for the scalar field

The aim of this exercise is to repeat the construction of the various Green's functions for the Klein-Gordon operator $\partial^2 + m^2$. We will work in Minkowski space, so we use the notation $\partial^2 \equiv \partial_\mu \partial^\mu$, $px \equiv p_\mu x^\mu$ etc.

1. Show that, given some differential operator $\mathcal{D}$, the inhomogenous differential equation $\mathcal{D}\phi(x) = j(x)$ is solved by

$$\phi(x) = \phi_h(x) + \int d^4x\, G(x,x')j(x').$$

Here $\phi_h(x)$ is a solution of the homogenous equation, $\mathcal{D}\phi_h(x) = 0$, and $G(x,x')$ is a Green's function associated with $\mathcal{D}$, i.e.

$$\mathcal{D}G(x,x') = \delta^4(x-x'). \qquad (☺)$$

2. We now set $\mathcal{D} = \partial^2 + m^2$. Since $G(x,x')$ can only depend on the combination $x - x'$ (why?), let us write $\Delta(x-x') \equiv G(x,x')$. Using the Fourier transform of equation (☺), show that

$$-i\Delta(x-x') = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2} e^{-ip(x-x')}. \qquad (☺☺)$$

As discussed in the lecture, equation (☺☺) is ambiguous, and should be supplemented with a prescription on how to perform the integration. We define the p^0 integral as a complex contour integral over the real axis, with the contour closed in a semicircle at infinity, in the upper or lower half-plane depending on the sign of $x^0 - x'^0$. The integrand has poles at $p^0 = \pm\sqrt{\mathbf{p}^2 + m^2}$, which may be bypassed in either the upper or lower half-plane.

3. Show that the prescription to bypass both poles in the upper half-plane (positive imaginary part) leads to the *retarded Green's function* Δ_R:

$$-i\Delta_R(x-x') = \theta(x^0 - x'^0)\langle 0|[\phi(x),\phi(x')]|0\rangle.$$

You can use the causal Green's function from the lecture:

$$\langle 0|[\phi(x),\phi(x')]|0\rangle = \int d\tilde{p}\left(e^{-ip(x-x')} - e^{ip(x-x')}\right).$$

Here, as always, $d\tilde{p}$ denotes the Lorentz invariant measure $\frac{d^3p}{(2\pi)^3}\frac{1}{2p^0_+}$.

4. Show that the prescription to bypass both poles in the lower half-plane (negative imaginary part) leads to the *advanced Green's function* Δ_A:

$$-i\Delta_A(x-x') = -\theta(x'^0 - x^0)\langle 0|[\phi(x),\phi(x')]|0\rangle.$$

5. Check explicitly that Δ_A and Δ_R satisfy equation (☺).

   Hint: Recall that the action of differential operators on distributions, such as $\theta(x-x')$ and $\delta(x-x')$, is defined by their action on a test function after integrating by parts.

6. Reproducing the calculation from the lecture, show that the "$i\epsilon$" prescription

   $$D_F(x-x') \equiv -i\Delta_F(x-x') = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip(x-x')}$$

   (which means that the integration contour runs below the pole at $p^0 = -\sqrt{\mathbf{p}^2+m^2}$ and above the one at $p^0 = \sqrt{\mathbf{p}^2+m^2}$) gives the *Feynman propagator*,

   $$D_F(x-x') = \langle 0|T(\phi(x)\phi(x'))|0\rangle.$$

   Recall that time-ordering is defined as

   $$T(\phi(x)\phi(x')) \equiv \begin{cases} \phi(x)\phi(x') & \text{if } x^0 > x'^0, \\ \phi(x')\phi(x) & \text{otherwise.} \end{cases}$$

Problem 2: Charge conjugation

The charge conjugation operator acting on complex scalar fields was given in the lecture as

$$U_c = \exp\left(-\frac{i\pi}{2}\int d\tilde{k}\, \left(a^\dagger(k)a(k) + b^\dagger(k)b(k) - a^\dagger(k)b(k) - b^\dagger(k)a(k)\right)\right).$$

1. Show that U_c is unitary (by proving that $U = e^{iH}$ is unitary if H is Hermitian).

2. Show that U_c is really the charge conjugation operator, i.e.

   $$\phi^\dagger(x) = U_c\,\phi(x)\,U_c^\dagger.$$

   Hint: Show first that $U_c\,a(p)\,U_c^\dagger = b(p)$, using the BCH formula

   $$e^{-A}\,B\,e^{A} = \sum_{k=0}^{\infty} \frac{1}{k!}\Big[\dots[[B,\underbrace{A],A],\dots,A}_{k\text{ times}}\Big].$$

Problem 3: Kaluza-Klein modes

Consider a five-dimensional spacetime with topology $\mathbb{R}^4 \times S^1$. That is, we add an additional space coordinate y and identify $y \sim y + 2\pi R$, so the fifth dimension is compact. A free scalar field in this spacetime satisfies the 5d Klein-Gordon equation,

$$\left(\partial_t^2 - \nabla^2 - \partial_y^2 + m^2\right)\phi(t,\mathbf{x},y) = 0.$$

Show that, from the point of view of a four-dimensional observer, this equation describes an infinite set of scalar particles ("Kaluza-Klein modes"), and compute their masses.

Note: The original idea of Kaluza and Klein was to regard 5d general relativity linearized about $\mathbb{R}^4 \times S^1$ as a unified theory of 4d gravity and electromagnetism. Using similar methods as above, it can be shown that the resulting effective 4d theory contains 4d gravity, a Maxwell field, a massless scalar, and infinitely many massive fields. It was then abandoned, but has undergone a revival in modified form in the last two decades to compactify effective field theories obtained from superstring theories, because these predict a number of extra spacetime dimensions.