Quantum Field Theory I, 6th problem sheet

Discussion of solutions in the tutorials of Dec 6th

Problem 1: Bose-Einstein and Fermi-Dirac statistics

Consider the Hamiltonian of a real free scalar field (discarding the zero point energy)

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}.$$

1. Show that for any c-number β

$$e^{-\beta H} a^\dagger_{\mathbf{p}} = a^\dagger_{\mathbf{p}} e^{-\beta(H+E_{\mathbf{p}})}.$$

2. Recall that in a mixed quantum mechanical state described by a density matrix ρ, the expectation value of any operator A is given by

$$\langle A \rangle = \mathrm{Tr}\,(\rho \mathrm{A}),$$

and that the canonical density matrix for a thermal state of temperature $1/\beta$ is

$$\rho = \frac{e^{-\beta H}}{\mathrm{Tr}\, \mathrm{e}^{-\beta \mathrm{H}}}.$$

Show that the thermal expectation value of $a^\dagger_{\mathbf{p}} a_{\mathbf{q}}$ is

$$\langle a^\dagger_{\mathbf{p}} a_{\mathbf{q}} \rangle = \frac{(2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q})}{e^{\beta E_{\mathbf{p}}} - 1}$$

and therefore, in a finite volume V,

$$\langle a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \rangle = \frac{V}{e^{\beta E_{\mathbf{p}}} - 1}.$$

This shows that $a^\dagger_{\mathbf{p}}$ creates quanta that obey Bose-Einstein statistics.

3. Repeat this exercise for fermionic creation and annihilation operators $c^\dagger_{\mathbf{p}}, c_{\mathbf{q}}$, which obey the canonical anticommutation relations (cf. problem 4.2)

$$\{c^\dagger_{\mathbf{p}}, c_{\mathbf{q}}\} = (2\pi)^3 \delta(\mathbf{p}-\mathbf{q}), \qquad \{c^\dagger_{\mathbf{p}}, c^\dagger_{\mathbf{q}}\} = \{c_{\mathbf{p}}, c_{\mathbf{q}}\} = 0.$$

Problem 2: The S-matrix from a generating functional

In the lecture you were given the following formula for the S-matrix of a field theory of interacting scalar fields:

$$S = :\exp\left(\int d^4x \left(\phi_{\rm in}(x)(\partial_x^2+m^2)\frac{1}{\sqrt{Z}}\frac{\delta}{\delta J(x)}\right)\right):\ Z[J]\Bigg|_{J=0},$$

where the generating functional $Z[J]$ is

$$Z[J] = \left\langle 0\middle| T\left(\exp\left(i\int d^4x\,\phi(x)J(x)\right)\right)\middle|0\right\rangle,$$

and the "normal ordering" prescription $:\phi_1\ldots\phi_n:$ amounts to placing all annihilation operators within the normal ordering symbol $: :$ to the right. Consider a matrix element with an incoming 2-particle state and an outgoing 2-particle state. For such a 2-2 scattering process, show that the above formula is equivalent to the LSZ formula,

$$\begin{aligned}\langle p_1,\ldots p_n,\ \text{out}\,|q_1,\ldots q_m,\ \text{in}\rangle =\ &(\text{disconnected pieces})\\ +\left(\frac{i}{\sqrt{Z}}\right)^{n+m}\int\prod_{i=1}^{n}d^4x_i\int\prod_{j=1}^{m}d^4y_j\,&e^{-ip_ix_i}e^{iq_jy_j}(\partial_{x_i}^2+m^2)(\partial_{y_j}^2+m^2)\times\\ &\times\langle 0|T\left(\phi(y_1)\ldots\phi(y_m)\phi(x_1)\ldots\phi(x_n)\right)|0\rangle.\end{aligned}$$

Problem 3: The time evolution operator

1. Show that, as stated in the lecture, the differential equation for the operator $U(t,t')\equiv U(t)U^{-1}(t')$ (where $U(t)$ is the operator that transforms free fields into interacting fields),

$$\frac{\partial U(t,t')}{\partial t} = -iH_{\rm int}(\phi_{\rm in}(t),\pi_{\rm in}(t))\,U(t,t'),\qquad (☺)$$

   is solved by

$$U(t,t') = T\,\exp\left(-i\int_{t'}^{t}dt''\,H_{\rm int}(\phi_{\rm in}(t''),\pi_{\rm in}(t''))\right).$$

   Hint: Consider the integral equation

$$U(t,t') = 1 - i\int_{t'}^{t}dt''\,H_{\rm int}(\phi_{\rm in}(t''),\pi_{\rm in}(t''))\,U(t'',t').$$

2. Show that $U(t,t')U(t',t'') = U(t,t'')$, and check that this relation leads to (☺) in the limit of infinitesimal $t-t'$ and $t'-t''$.