Quantum Field Theory I, 7th problem sheet

Discussion of solutions in the tutorials of Dec 13th

Problem 1: Particle creation from a classical source

Consider a massive real scalar field coupled to an external classical source $J(x)$ via an interaction Hamiltonian

$$H_{\text{int}}(t) = \int d^3x \; (-\phi_{\text{in}}(t, \mathbf{x}) J(t, \mathbf{x}))$$

(assuming that the source is switched on at some time t_i and switched off again at some later time t_f). Note that now $|0\rangle_{\text{in}} = S^\dagger |0\rangle_{\text{out}}$ with the S-matrix given by

$$S = T \exp\left(-i \int_{\infty}^{\infty} dt' \, H_{\text{int}}(t')\right).$$

1. Write down the amplitude for the source to produce an n-particle state from the vacuum.

2. Evaluate this formula to compute the probability for producing n particles. You should find a Poisson distribution:

   $$P(n) = \frac{1}{n!} \lambda^n \exp(-\lambda),$$

   where

   $$\lambda = \int d\tilde{p} \, |\tilde{J}(p)|^2,$$

   and $\tilde{J}$ is the Fourier transform of J.

   Hints: To find a suitable form for the S-matrix, divide the time interval during which the source is switched on into N pieces of length Δt. Then show with the help of the BCH formula as given in exercise 1.2 that

   $$T \exp\left(-i \int_{t_i}^{t_f} dt' \, H_{\text{int}}(t')\right)$$
   $$= \lim_{\substack{N\to\infty \\ \Delta t \to 0}} \exp\left(-i\Delta t \sum_{k=1}^{N} H_{\text{int}}(t_k) - \frac{1}{2}\Delta t^2 \sum_{1\le k<l\le N} [H_{\text{int}}(t_k), H_{\text{int}}(t_l)]\right),$$

   where t_k is the midpoint of the k-th time interval, $t_k = t_i + (k - 1/2)\Delta t$. After evaluating the $N \to \infty, \Delta t \to 0$ limit, you can express the commutators in terms of retarded Green's functions.

Problem 2: The time evolution operator II

Complete the proof from the lecture that the vacuum is an eigenstate of the $U(t)$ operator as $t \to \pm\infty$:

$$\lim_{t\to\infty} U(\pm t)|0\rangle = \lambda_\pm|0\rangle.$$

It then follows by unitarity that the $\lambda_\pm$ are pure phase factors.

Hints: Show that

$$\lim_{t\to\infty} \langle\beta, \text{in}|U(-t)|0\rangle = 0,$$

if $\langle\beta, \text{in}| = \langle\alpha, \text{in}|a(p)$ contains some one-particle state with momentum p. You can do this by LSZ reduction with respect to p and using the asymptotic condition. Eventually you will need to show that $\dot{U}\phi + U\phi\dot{U}^{-1}U = 0$. (If you get stuck, you might also have a look at Bjorken/Drell II p. 168.)

Problem 3: Vacuum graphs

For a theory of an interacting scalar field, show that the vacuum-vacuum transition amplitude is given by the exponentiated sum of connected vacuum Feynman diagrams:

$$\langle 0|T\ \exp\left(-i\int_{-\infty}^{\infty} dt\, H_{\text{int}}(t)\right)|0\rangle = \exp\left(\sum(\text{connected vacuum diagrams})\right).$$

You may want to only check the first few orders, to see that the combinatorial factors work out correctly.