Quantum Field Theory I, 9th problem sheet

Discussion of solutions in the tutorials of Jan 10th

Problem 1: Feynman diagrams for ϕ^3 theory

Consider a real scalar field with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{3!}\phi^3.$$

Draw all connected Feynman diagrams that contribute to the 3-point function up to 3-loop order.

Problem 2: Wick's theorem applied to generalized Gaussian integrals

Calculate the integral

$$I = \int_{-\infty}^{\infty} dx\, x^{2n} e^{-ax^2},$$

where $a \in \mathbb{R}_+$ and $n \in \mathbb{N}$,

1. by repeated differentiation of the Gaussian integral $\int dx\, e^{-ax^2}$ with respect to a,
2. using Wick's theorem.

Problem 3: The path integral in quantum mechanics

For a free, nonrelativistic quantum mechanical particle in one dimension, calculate the amplitude to propagate from x_i at time t_i to x_f at time t_f, $\langle x_f(t_f)|x_i(t_i)\rangle$, using the path integral.

Hints: Divide $t_f - t_i$ into N pieces of length ϵ, and write the classical action as

$$S = \lim_{\substack{N\to\infty \\ \epsilon\to 0}} \sum_{k=1}^{N} \epsilon \left(\frac{m}{2} \frac{(x_k - x_{k-1})^2}{\epsilon^2} \right).$$

Now you can turn the path integral into a Gaussian integral and evaluate it with the usual formula

$$\prod_{k=1}^{N-1} \int_{-\infty}^{\infty} dy_j \, \exp\left(-\mathbf{y}^T A\, \mathbf{y} + \mathbf{b}\cdot\mathbf{y}\right) = \pi^{\frac{N-1}{2}} (\det A)^{-\frac{1}{2}} \exp\left(\frac{1}{4}\mathbf{b}^T A^{-1}\, \mathbf{b}\right).$$

Problem 4: The $i\epsilon$ prescription for path integrals

This exercise is rather technical, but don't let that put you off — we will discuss it at length in the tutorials.

Consider a real scalar field $\phi(\mathbf{x}, t)$, and let $|\Phi; T\rangle$ be an eigenstate of the $\phi(\mathbf{x}, t)$ operator at time T:

$$\phi(\mathbf{x}, T)|\Phi; T\rangle = \Phi(\mathbf{x}, T)|\Phi; T\rangle,$$

with $\Phi(\mathbf{x}, t)$ a c-number function. Show that

$$\begin{aligned}&\langle 0|\Phi; T \to +\infty\rangle\langle\Phi; T \to -\infty|0\rangle \\ &\quad = \lim_{\epsilon\to 0^+} \exp\left(-\frac{1}{2}\epsilon \int d^3x \int d^3y \int_{-\infty}^{\infty} d\tau\, \tilde{E}(\mathbf{x}, \mathbf{y})\Phi(\mathbf{x}, \tau)\Phi(\mathbf{y}, \tau)e^{-\epsilon|\tau|}\right),\end{aligned}$$

where $\tilde{E}(\mathbf{x}, \mathbf{y})$ is the Fourier transform of the energy $E(\mathbf{p})$,

$$\tilde{E}(\mathbf{x}, \mathbf{y}) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \sqrt{\mathbf{p}^2 + m^2}.$$

Hints: In the following, we denote by $\tilde{\Phi}$ the spatial Fourier transform of Φ, and $\tilde{\Phi}_{\text{in/out}}(\mathbf{p}) \equiv \tilde{\Phi}(\mathbf{p}, T \to \mp\infty)$. Consider first $\langle\Phi; T \to -\infty)|0\rangle$. The vacuum $|0\rangle$ is defined such that $a_{\text{in}}(p)|0\rangle = 0$. Using the (time-independent!) expression for a_{in} from exercise 4.3,

$$a_{\text{in}}(p) = i \int d^3x\, e^{ip_\mu x^\mu} \overleftrightarrow{\partial_0}\, \phi_{\text{in}}(x),$$

show that

$$\left(E(\mathbf{p})\tilde{\Phi}_{\text{in}}(\mathbf{p}) + \frac{\delta}{\delta\tilde{\Phi}_{\text{in}}(-\mathbf{p})}\right) \langle\Phi; T \to -\infty|0\rangle = 0.$$

Note the analogies to quantum mechanics: In "field space", $\partial_0\phi(x) = \pi(x)$ acts as a functional derivative, just as the momentum operator of quantum mechanics is represented by a derivative in position space. For a quantum mechanical harmonic oscillator, one would determine the wave function $\psi_0(x)$ of the ground state by solving $a\,\psi_0(x) = 0$ with $a = \frac{1}{\sqrt{2}}(x/c + c\,\partial_x)$, $c = 1/\sqrt{m\omega}$. The above equation is a functional version of this. Show that it is also solved by a Gaussian. Finally, evaluate $\langle 0|\Phi; T \to +\infty\rangle$ in the same manner, rewrite the $\tilde{\Phi}$s in terms of Φs, and use the identity (which you may want to prove first)

$$f(+\infty) + f(-\infty) = \lim_{\epsilon\to 0^+} \epsilon \int_{-\infty}^{\infty} dt\, f(t)\, e^{-\epsilon|t|}.$$

Comments: You have just derived the $i\epsilon$ prescription for evaluating correlation functions in the path integral formalism: The product of matrix elements you calculated appears in the path integral to fix the boundary conditions. It can be combined with the exponentiated action to give an $i\epsilon$ term — compare Weinberg I chapter 9.2.