Chapter 8

Path integral formulation of QFT

So far, we have studied QFT in the canonical formalism with operator-valued fields, in the Heisenberg representation. Richard Feynman has formulated another representation of both QM and QFT, which is very intuitive and does not use operators. Its mathematical status, however, is still in development. This method is particularly powerful if one wants to quantize gauge theories - this is why it is necessary to discuss it - and it also allows one to derive the Feynman rules very easily, and to discuss problems beyond perturbation theory, although we will not go into the latter here.

8.1 Path integrals in QM

The path integral formulation of QM centers around the transition amplitude for a QM particle from a position $x(t)$ at a time t to a position $x'(t')$ at time t'. It starts from the Heisenberg picture, where the time dependent operators $\mathbf{X}(t)$ and $\mathbf{P}(t)$ have their respective eigenvectors $|x(t)\rangle$ and $|p(t)\rangle$, with time developments

$$\mathbf{X}(t) = e^{i\mathbf{H}(t-t_0)/\hbar}\mathbf{X}(t_0)e^{-i\mathbf{H}(t-t_0)/\hbar} \tag{8.1}$$

$$|x(t)\rangle = e^{i\mathbf{H}(t-t_0)/\hbar}\,|x(t_0)\rangle \tag{8.2}$$

and similarly for $\mathbf{P}$ and p. The factors of $\hbar$ have been reinserted here for clarity. Note that the sign in eq. (8.2) is opposite to that of the Schrödinger equation. Note also that this equation describes a transition in time, while the position does not change. Finally, in eq. (8.1), $\mathbf{X}(t_0) = \mathbf{X}_{\mathrm{S}}$; in the Schrödinger picture, t_0 is usually taken to be 0.

Let us start at the end of our discussion of the path integral formulation, with the result:

$$\boxed{\langle x'(t')|x(t)\rangle = \int \mathcal{D}x\mathcal{D}p \, \exp\left[i\int_t^{t'} d\tau \left\{p(\tau)\frac{dx}{d\tau} - \mathbf{H}(p,x)\right\}/\hbar\right]} \tag{8.3}$$

Here, $\int \mathcal{D}x$ is a *path integral*, in mathematical circles know as *functional integral*, an integral over all possible paths $x(\tau)$ connecting x and x', with $x(t) = x$ and $x(t') = x'$.

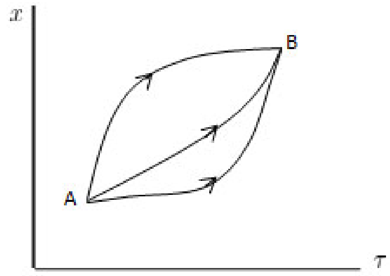


Figure 8.1: Integrate over all possible paths

$\int \mathcal{D}p$ does not have boundary conditions, since the problem asks for the transition amplitude between positions, but not between momenta. (As a small aside: note that the exponent is just the classical action times $i/\hbar$.)

After discretization of the time integral, it becomes a product of integrals at $\tau_1, \tau_2, \ldots$ over $x(\tau_1), x(\tau_2), \ldots$:

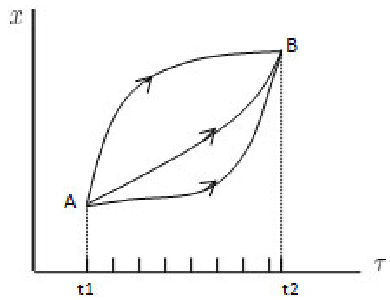


Figure 8.2: Discretization of the time integral

$$\int \mathcal{D}x \exp\left(\int d\tau \ldots\right) \rightarrow \prod_i \int_{-\infty}^{\infty} dx(\tau_i)$$

Defining $\delta\tau := \tau_i - \tau_{i-1}$, one eventually has to take the limit $\delta\tau \rightarrow 0$, which is mathematically demanding. Obviously, the integral over $\mathcal{D}x$ does not need to be evaluated at the endpoints, since these are fixed.

Now, let us derive this result. From the canonical formalism, we have the following formula:

$$\langle x'(t')|x(t)\rangle = \langle x'|\, e^{-i\mathbf{H}(t'-t)/\hbar}\,|x\rangle \tag{8.4}$$

where $|x\rangle$ is a Schrödinger state. Decomposing the interval into N bits, $t'-t=N\epsilon$, gives

$$\langle x_N|\, e^{-i\mathbf{H}\epsilon/\hbar}\,|x_{N-1}\rangle\,\langle x_{N-1}|\, e^{-i\mathbf{H}\epsilon/\hbar}\,|x_{N-2}\rangle\,\langle x_{N-2}|\ldots$$
$$\ldots|x_1\rangle\,\langle x_1|\, e^{-i\mathbf{H}\epsilon/\hbar}\,|x_0\rangle$$

where $x_N = x'$ and $x_0 = x$, and $\mathbf{H} = \frac{\mathbf{P}^2}{2m} + V(\mathbf{X})$. Consider one of these matrix elements, to order ϵ:

$$\begin{aligned}
&\langle x_{k+1}|\, e^{-i\mathbf{H}\epsilon/\hbar}\,|x_k\rangle = \langle x_{k+1}|\, 1 - \frac{i\mathbf{H}}{\hbar}\epsilon + \ldots |x_k\rangle = \\
&\quad \langle x_{k+1}|x_k\rangle - i\epsilon\int \frac{dp_k}{2\pi\hbar}\left\{V\left(\frac{x_{k+1}+x_k}{2}\right)\langle x_{k+1}|p_k\rangle\,\langle p_k|x_k\rangle + \right.\\
&\qquad \left.\langle x_{k+1}|p_k\rangle\left\langle p_k\left|\frac{\mathbf{P}^2}{2m}\right|x_k\right\rangle\right\} + \cdots = \\
&\quad \int\frac{dp_k}{2\pi\hbar}\left[1 - \frac{i\epsilon}{\hbar}\left(\frac{p_k^2}{2m} + V\left(\frac{x_{k+1}+x_k}{2}\right)\right)\right]\exp\left(\frac{ip_k}{\hbar}(x_{k+1}-x_k)\right) + \ldots
\end{aligned} \tag{8.5}$$

Note: for more complicated $\mathbf{X}/\mathbf{P}$-mixed operators one needs *Weyl ordering*, a symmetrization of the operator sequence in $\mathbf{X}/\mathbf{P}$; see Peskin & Schröder, p. 281 for more on this topic. Note that the argument in $V(\frac{x_{n+1}+x_n}{2})$ is written like this for cosmetic reasons; we could just as well have written x_n, since in the end, the limit $N\to\infty$ will be taken. Continuing our derivation, let us define

$$\theta(\epsilon) = \int\frac{dp_k}{2\pi\hbar}\exp\left(-\frac{i\epsilon}{\hbar}\left(\frac{p_k^2}{2m} + V\left(\frac{x_k + x_{k+1}}{2}\right) - p_k\frac{x_{k+1}-x_k}{\epsilon}\right)\right) \tag{8.6}$$

which is the right hand side of eq. (8.5) to order $\mathcal{O}(\epsilon)$. Multiplying all θ's and taking the limit $\epsilon\to 0$, we have

$$\langle x'(t')|x(t)\rangle = \int \mathcal{D}x\mathcal{D}p\,\exp\left[i\int_t^{t'} dt\left(\frac{p\dot{x} - H(p,q)}{\hbar}\right)\right] \tag{8.7}$$

with $x(t)=x$ and $x(t')=x'$. This limit is of course accompanied by some higher-level mathematics. The naïve expression, however, has to be based on the discretized version we started from. Concretely, for physicists, this means that in QFT, numerical lattice calculations are an adequate way to approach this integral. Note that the continuous and differentiable functions are a dense set of measure zero in the functional integral.

The dp_k-integration in eq. (8.6) can be performed: it is just a Gaussian integral, here restricted to one dimension for simplicity. It is solved by completing the square:

$$
\begin{aligned}
&-\frac{p_k^2\epsilon}{2m}+p_k(x_k-x_{k+1})=\\
&-\frac{1}{2}\left[\frac{p_k^2}{m}\epsilon-2p_k(x_k-x_{k+1})+\frac{(x_k-x_{k-1})^2}{\epsilon}m\right]+\frac{1}{2}m\frac{(x_k-x_{k-1})^2}{\epsilon}=\\
&-\frac{{p'_k}^2}{2}\frac{\epsilon}{m}+\frac{1}{2}m\frac{(x_k-x_{k-1})^2}{\epsilon}
\end{aligned}
$$

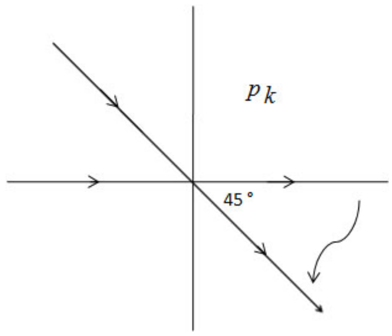
with $p'_k := \frac{p_k}{m}-\frac{x_k-x_{k+1}}{\epsilon}$. Using the standard Gaussian integral,

$$\int_{-\infty}^{\infty} dx e^{-\alpha x^2}=\left(\frac{\pi}{\alpha}\right)^{1/2} \tag{8.8}$$

we can perform the p'_k-integral. The first part becomes

$$\int_{-\infty}^{\infty}\frac{dp'_k}{2\pi\hbar}\exp\left(-\frac{1}{2}\frac{i\epsilon}{\hbar m}p_k'^2\right)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dy e^{-y^2/2}=\frac{1}{2\pi\hbar}\sqrt{\frac{2\pi\hbar m}{i\epsilon}}$$

where $y=p'_k\sqrt{i\epsilon/\hbar m}$. Note that due to the presence of $\sqrt{i}$ in the conversion from p'_k to y, this substitution constitutes a 45-degree rotation of the integration path:



So, the final result, the product of all the separate integrals, is

$$\prod_{j=1}^{N-1}\int dx_j\left(\sqrt{\frac{m}{i\epsilon\hbar 2\pi}}\right)^N \exp\left(i\sum_{i=1}^{N}\epsilon\left\{\frac{m}{2}\frac{(x_i-x_{i-1})^2}{\epsilon^2}-V(x_i)\right\}/\hbar\right)$$

which, after taking ϵ to zero and N to infinity, becomes:

$$\boxed{\langle x(t)|x'(t')\rangle=\int\mathcal{D}x\exp\left(i\int_t^{t'}d\tau\mathcal{L}(x,\dot{x})/\hbar\right)} \tag{8.9}$$

with singular integration measure ($\sim \epsilon^{-N/2}$) and Lagrangian density $\mathcal{L} = \frac{m\dot{x}^2}{2} - V(x)$.

The x-space path integral is less general (namely only for H quadratic in p) than the first version, but for our purpose, we can settle for this one.

Remarks

- For $\langle x'(t')| \, \mathrm{T}(\mathbf{O}_1(t_1)\mathbf{O}_2(t_2)\dots) \, |x(t)\rangle$, the operators $\mathbf{O}_1, \mathbf{O}_2, \dots$ act in the time slices around $t_1, t_2, \dots$. Then, $\langle x(t_1)| \, \mathbf{O}_1(t_1) \, |x(t_1')\rangle = O_1(x(t_1))\delta(t_1 - t_1')$ where O_1 is a function. Thus, we obtain time-ordering in the path integral, which is decomposed into time slices. This remark is also important if the potential has the shape of a matrix in more complicated settings.
- The oscillating behaviour of the Feynman exponential, which makes the convergence of the integral a more subtle affair, can be avoided if we go to imaginary, or Euclidean, time ($t = x^0 = -ix^4 = -it_{\mathrm{E}}$). This so-called *Wick rotation* helps us to define certain expressions properly. Of course, one has to rotate back at the end of the calculation .

8.1.1 Vacuum expectation values

When going to QFT, we will be interested in vacuum expectation values (cf. correlation functions in statistical physics). Let us briefly investigate them here:

$$\langle 0| \, \mathrm{T}(\mathbf{x}(t_1)\dots\mathbf{x}(t_n)) \, |0\rangle = ?$$

Let us start from

$$\langle x_T(T)| \, \mathrm{T}(\mathbf{x}(t_1)\dots\mathbf{x}(t_n)) \, |x_{-T}(-T)\rangle = \int_{x_{-T}}^{x_T} \mathcal{D}x \, x(t_1)\dots x(t_n)\times \exp\left(-\int d\tau_e \left(\frac{m\dot{x}^2}{2} + V(x)\right)\right) \tag{8.10}$$

where τ_e stands for Euclidean (Wick-rotated) time. Now,

$$|x_{-T}(-T)\rangle = e^{(-T)H} \, |x_{-T}(0)\rangle = \sum_n |n\rangle \, \langle n|x_{-T}\rangle \, e^{-E_n T}$$

where $|x_{-T}(0)\rangle$ is also written $|x_{-T}\rangle$, and is a Schrödinger vector, which at $t = 0$ coincides with the corresponding Heisenberg vector. When T is large, only the ground state contributes. Then, eq. (8.10) becomes

$$= \langle x_T(0)|0\rangle \, \langle 0| \, \mathrm{T}(\mathbf{x}(t_1)\dots\mathbf{x}(t_n)) \, |0\rangle \, \langle 0|x_{-T}(0)\rangle \, e^{-2E_0 T}$$

Dividing by $\langle x_T(T)|x_{-T}(-T)\rangle$, like in the Gell-Mann-Low formula, removes the outer parts, leaving

$$\langle 0| \, \mathrm{T}(\mathbf{x}(t_1)\dots\mathbf{x}(t_n)) \, |0\rangle = \lim_{T\to\infty} Z_T \int_{x_{-T}}^{x_T} \mathcal{D}x \, x(t_1)\dots x(t_n) e^{-S} \tag{8.11}$$

where $Z_T = \int_{x_{-T}}^{x_T} \mathcal{D}x\, e^{-S}$ and $S = \int d\tau_e(\frac{m\dot{x}^2}{2} + V(x))$.

Remarks

- Note the similarity with the thermal average in statistical mechanics with the Boltzmann weight $e^{-\beta H}$.
- $x_{\pm\infty}$ is in general not a fixed value; taking the Gell-Mann-Low quotient, the final result should not depend on it.
- The *Feynman-Kac formula*: $\lim_{T\to\infty} \log \langle x_T(T)|x_{-T}(-T)\rangle / (-2T) = E_0$. **Exercise**: derive this result.

8.2 Path integrals in QFT

8.2.1 Framework

Consider real scalar field theory, with $\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - V(\Phi)$, where $V(\Phi)$ could for example be $\frac{m^2}{2}\Phi^2 + \frac{\lambda}{4}\Phi^4$. Now, $\Phi(\vec{x},t)$ substitutes $x_i(t)$ as we go to infinitely many degrees of freedom. The following correspondences hold:

QM	QFT
• $\mathbf{X}_i(t)$ (Heisenberg picture)	• $\Phi(\vec{x},t)$ Heisenberg operator
• $\|x_i(t)\rangle$ state vectors with $\mathbf{X}_i(t)\|x_i(t)\rangle = x_i(t)\|x_i(t)\rangle$	• Fock space (Φ-eigenvector states; coherent states)
• boundary conditions	• vacuum state $\|0\rangle$ for $t \to \pm\infty$

$|0\rangle$ is unique only without outer fields ("currents"). *With* outer fields (or "currents"), as we have seen before in the generating functional $Z(j)$, we have an extra term $\mathcal{L}_j$ in the Lagrangian density: $\mathcal{L}_j = j(x)\Phi(x)$, where $j \to 0$ for large $\vec{x}$ and t. This implies that $|\Omega_{t\to-\infty}\rangle \neq |\Omega_{t\to\infty}\rangle$ and $|0\rangle_{\text{out}} \neq |0\rangle_{\text{in}}$.

Going from x_i to Φ, our path integral will run over fields Φ, and momentum fields Π, where the latter are given by

$$\Pi(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\Phi)} = \partial_0\Phi = \partial_t\Phi \tag{8.12}$$

Now, the Φ- and Π-integrals become:

$$\int \mathcal{D}\Phi \to \lim \int \prod_{i,j} d\Phi(\vec{x}_i,\tau_j) = \lim \int \prod_k d\Phi_k$$

$$\int \mathcal{D}\Pi \to \lim \int \prod_k d\Pi_k$$

where the index k represents a 4-dimensional lattice (the limit $\epsilon \to 0$ will be taken in the end). Of these, $\int \mathcal{D}\Pi$ can be performed like in the QM case, and starting at the end again, we obtain:

$$\langle 0 | \, \mathrm{T}(\Phi(x_1)\dots\Phi(x_n)) \, |0\rangle = \frac{\int \mathcal{D}\Phi \exp\left\{ i \int d^4x \left(\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - V(\Phi(x))\right) \Phi(x_1)\dots\Phi(x_n) \right\}}{\int \mathcal{D}\Phi \exp(\dots)} \tag{8.13}$$

($\hbar$ has been set equal to 1 again).

Now, we can sum over all fields with $\Phi(\vec{x},t) \to 0$ as $|\vec{x}|, |t| \to \infty$. (In the quotient, this condition should drop out; strictly speaking, one has to calculate the Fock space vacuum in x-space. See the exercises.) The integrals are, like in QM, well defined after Wick rotation ($x_0 \to -ix_4$ substitution and 90-degree rotation in the x_4-plane) to imaginary time; another option is inserting a term $\frac{1}{2}(m^2 - i\epsilon)\Phi^2(x)$ in the potential. The Wick rotation results in:

$$\partial_\mu\Phi\partial^\mu\Phi - V(\Phi(x)) \to -\partial_\mu\Phi\partial_\mu\Phi - V(\Phi(x_\mathrm{E})) \tag{8.14}$$

$$\int d^4x \to -i \int d^4x_\mathrm{E} \tag{8.15}$$

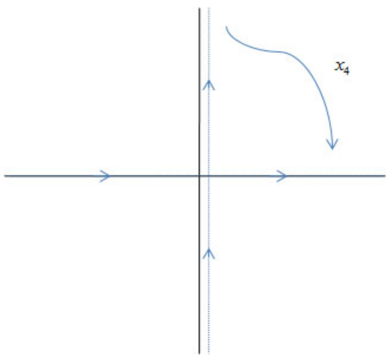


Figure 8.3: Wick rotation

Eq. (8.13) is very similar to the Gell-Mann-Low formula. It can also be written with the help of a source $j(x)$,

$$Z(j) = \int \mathcal{D}\Phi \exp\left(i \int d^4x (\mathcal{L} + j\Phi) \right) \tag{8.16}$$

which gives

$$\langle 0|\, \mathrm{T}(\Phi(x_1)\dots\Phi(x_n))\, |0\rangle = \frac{1}{(i)^n Z(0)} \left. \frac{\delta^n Z(j)}{\delta j(x_1)\dots\delta j(x_n)}\right|_{j=0} \tag{8.17}$$

where the variational derivative provides factors $i\Phi(x_i)$. Note: the normalization of $\int \mathcal{D}\Phi$ is often chosen to give $Z(0) = 1$.

8.2.2 QFT path integral calculations

Deriving eq. (8.13) in a discretized version, like we did for the QM case, involves a multi-component version of the standard Gaussian integral, eq. (8.8):

$$I(A,b) = \prod_{i=1}^{N} \int_{-\infty}^{\infty} d\Phi_i \exp\left(-\frac{1}{2}\Phi_i A_{ik} \Phi_k + b_i \Phi_i\right) \tag{8.18}$$

We diagonalize A by an orthogonal transformation $\mathbf{O}$, whose Jacobi-determinant is 1. Then, we have

$$I(a,b') = \prod_{i=1}^{N} \int_{-\infty}^{\infty} d\Phi'_i \exp\left(-\frac{1}{2}\Phi'_i a_{ii} \Phi'_i + b'_i \Phi'_i\right)$$

Completing the square, we rewrite $-\frac{1}{2}\Phi'_i a_i \Phi'_i + b'_i \Phi'_i$ into $-\frac{1}{2}a_i(\Phi'_i - \frac{b'_i}{a_i})^2 + \frac{1}{2}b'_i \frac{1}{a_i} b'_i$, which gives

$$I(a,b') = \prod_{i=1}^{N} \sqrt{\frac{2\pi}{a_i}} \exp\left(\frac{1}{2} b'_i \frac{1}{a_i} b'_i\right) = (2\pi)^{N/2} \frac{1}{(\det A)^{1/2}} \exp\left(\frac{1}{2} b^T A^{-1} b\right)$$

where the second step is the result of rotating back. So, in the end, we are left with

$$\boxed{I(A,b) = \frac{(2\pi)^{N/2}}{(\det A)^{1/2}} \exp\left(\frac{1}{2} b^T A^{-1} b\right) = I(A,0) \exp\left(\frac{1}{2} b^T A^{-1} b\right)} \tag{8.19}$$

Observe that $\frac{(2\pi)^{N/2}}{(\det A)^{1/2}} = I(A,0)$.

Differentiation with respect to b_i produces vacuum expectation values:

$$\left. \frac{\partial}{\partial b_i} \frac{\partial}{\partial b_k} I(A,b) \right|_{b=0} = I(A,0) \times \underbrace{(A^{-1})_{ik}}_{\text{2-point function}} \tag{8.20}$$

Let us apply this to the $Z(j)$ from above, at first without interaction:

$$\mathcal{L} = \partial_\mu \Phi \partial_\mu \Phi + \frac{m^2}{2} \Phi^2 \tag{8.21}$$

Note that this is the Euclidean, i.e. rotated, form. With this $\mathcal{L}$,

$$Z_0^{\mathrm{E}}(j) = Z_0^{\mathrm{E}}(0) \exp\left(\frac{1}{2}\int j(x)(-\partial_{\mathrm{E}}^2 + m^2)^{-1}_{xx'} j(x') d^4x^{\mathrm{E}} d^4x'^{\mathrm{E}}\right) \tag{8.22}$$

For comparison: in the Minkowski metric, it looks like this:

$$\begin{aligned} Z_0(j) =& Z_0(0) \exp\left(-\frac{1}{2i}\int j(x)(\partial^2 + m^2 - i\epsilon)^{-1}_{xx'} j(x') d^4x d^4x'\right) = \\ & Z_0(0) \exp\left(-\frac{1}{2}\int j(x) D_{\mathrm{F}}(x - x') j(x') d^4x d^4x'\right) \end{aligned}$$

In the Euclidean metric, the calculation is easier due to the absence of the i-factors. It is easy to check that

$$\langle 0|\, \mathrm{T}(\Phi(x_1)\Phi(x_2))\, |0\rangle = \frac{1}{i^2} \left.\frac{\delta^2 Z(j)}{\delta j(x_1)\delta j(x_2)}\right|_{Z(0)} = D_{\mathrm{F}}(x_1 - x_2)$$

Remarks

- This can also be performed in the Fourier transformed form:

  $$S_0^{\mathrm{E}} = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\left(\widetilde{\Phi}(p)(p^2 + m^2)\widetilde{\Phi}(-p) - \widetilde{j}(p)\widetilde{\Phi}(-p) - \widetilde{j}(-p)\widetilde{\Phi}(p)\right)$$

  with $\widetilde{j}(-p) = j^*(p)$ and $\widetilde{\Phi}(-p) = \widetilde{\Phi^*}(p)$, since $\widetilde{j}$ and Φ are real. Redefining

  $$\widetilde{\Phi}(p) = \widetilde{\Phi}'(p) + (p^2 + m^2)^{-1}\widetilde{j}(p)$$

  (i.e., completing the square) gives

  $$Z_0^{\mathrm{E}}(j) = Z_0^{\mathrm{E}}(0) \exp\left(\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\frac{\widetilde{j}(p)\widetilde{j}(-p)}{p^2 + m^2}\right)$$

- With

  $$W_0(j) = \frac{1}{2}\int dx dx' j(x)(iD_{\mathrm{F}}) j(x')$$

  we obtain the following relation for Z and W:

  $$\frac{Z_0(j)}{Z_0(0)} = e^{iW_0(j)} \qquad \left(= e^{-W^{\mathrm{E}}(j)}\right)$$

  where the factor iD_{F} generates the connected 2-point functions; this will be generalized later on. This relation resembles the one between the partition function and the free energy in thermodynamics. Note, by the way, that Z and W have been exchanged in the text by Ramond (see literature list).

8.2.3 Perturbation theory

The perturbative approach to path integrals will, again, be introduced in the Euclidean notation. We will be dealing with a potential $V(\Phi)$, which can be written as

$$\boxed{V_{\mathrm{E}}(\Phi) \to V_{\mathrm{E}}\left(\frac{\delta}{\delta j}\right)} \tag{8.23}$$

where the differentiation is acting on the generating functional $Z_0^{\mathrm{E}}(j)$. (For example, $\frac{\lambda}{4!}\Phi^4 \to \frac{\lambda}{4!}\left(\frac{\delta}{\delta j(x)}\right)^4$.) Let us start with

$$\langle 0|\, \mathrm{T}(\Phi(x_1)\ldots\Phi(x_n))\, |0\rangle = \frac{1}{Z(0)} \frac{\delta^n Z(j)}{\delta j(x_1)\ldots\delta j(x_n)}\bigg|_{j=0} \tag{8.24}$$

Now, expand the exponential in the path integral for $Z(j)$ in powers of $V(\Phi)$ and act on $Z_0(j)$ with $V(\frac{\delta}{\delta j})$ as discussed before. Note that also the $\frac{\delta}{\delta j(x_i)}$ in eq. (8.24) act on $Z_0(j)$. The $\frac{\delta}{\delta j}$ of the outer Φ and vertex Φ pairwise remove the j-legs of $\exp(\frac{1}{2}\int jD_{\mathrm{F}}j)$. Division by $Z(0)$ eliminates the vacuum graphs (n.b.: $Z_0(0)$ is not the same as $Z(0)$). For example, a vertex with $V = \frac{\lambda}{4!}\Phi^4$ requires 4 propagators (permutation gives the factor of 4!). Also, several vertices, each with their factor of $\frac{1}{n!}$, can be permuted and require a combinatorial factor. From all this, the usual rules for Feynman graphs emerge again:

(i) only graphs with outer lines (no vacuum graphs)

(ii) D_{F} for inner and outer propagators

(iii) λ and $\int d^4y$ for each vertex

(iv) combinatorial factors

Exercise: complex scalar fields:

$$Z(j,\bar{j}) = \int \mathcal{D}\Phi \exp(-\Phi^\dagger \mathbf{A}\Phi + V(\Phi) - \bar{j}\Phi - \Phi^\dagger j)$$

Here, Φ is a complex vector (with index x). Rewriting it as $\Phi = (\Phi_1 + i\Phi_2)/\sqrt{2}$ reduces the problem to two real fields $\Phi_{1,2}$. $\mathcal{D}\Phi = \mathcal{D}\Phi_1\mathcal{D}\Phi_2$, or, more elegantly, $\mathcal{D}\Phi\mathcal{D}\Phi^\dagger$ with independent $\Phi, \Phi^\dagger$. $\mathbf{A} = -\partial_{\mathrm{E}}^2 + m^2$ in case of the free complex Klein-Gordon field, and $Z_0(j,\bar{j}) = \exp -\bar{j}\mathbf{A}^{-1}j)$. Derive the Feynman rules for $V(\Phi) = \lambda\frac{(\Phi^\dagger\Phi)^2}{4}$.

Recapitulation

In the path integral formulation of QFT, vacuum expectation values of time ordered products of operators are calculated with the following formula:

$$\langle 0| \mathrm{T}(\Phi(x_1)\dots\Phi(x_n))\,|0\rangle = \frac{1}{Z(0)}\,\frac{\delta^n Z(j)}{\delta j(x_1)\dots\delta j(x_n)}\bigg|_{j=0} \tag{8.25}$$

where

$$Z(j) = e^{-V(\frac{\delta}{\delta j})} Z_0(j) \quad \text{and}$$
$$Z_0(j) = Z_0(0)\exp\left(\frac{1}{2}\int dx dx' j(x) D_{\mathrm{F}}(x-x') j(x')\right) \tag{8.26}$$

- $Z_0(0)$ drops out in eq. (8.25).
- Vacuum diagrams are cancelled by the division by $Z(0)$.
- The Feynman rules are the same as those obtained in the framework of canonical quantization.