Chapter 11

Feyman rules for theories with fermions

11.1 Bilinear Covariants

The Lagrange density is a Lorentz scalar, which has to be constructed to include fermions. The algebra of Dirac matrices has 16 elements, which are linearly independent 4x4 matrices:

$$\mathbb{1};\ \gamma^\mu;\ \sigma^\mu\nu = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right];\ \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3;\ \gamma^5\gamma^\mu \tag{11.1}$$

(Check: there are $1 + 4 + 6 + 1 + 4 = 16$ of them and they are linearly independent.) Remember that a spinor transforms like

$$\Psi'_\alpha(x') = S_{\alpha\beta}(\Lambda)\Psi_\beta(x) \tag{11.2}$$

Its barred conjugate transforms like

$$\bar{\Psi}'_\alpha(x') = \bar{\Psi}_\beta(x)\bar{S}_{\beta\alpha} \qquad \text{with} \quad \bar{S} = \gamma^0 S^\dagger \gamma^0 \tag{11.3}$$

From $S^{-1}(\Lambda)\gamma^\nu S(\Lambda) = \Lambda^\nu{}_\mu\gamma^\mu$ it follows that $\bar{S} = S^{-1}$. Using this, we can see that $\bar{\Psi}\Psi$ is a Lorentz scalar:

$$\bar{\Psi}'_\alpha(x')\Psi'_\alpha(x') = \bar{\Psi}'_\alpha(x')\bar{S}S\Psi'_\alpha(x') = \bar{\Psi}_\beta(x)\Psi_\beta(x) \tag{11.4}$$

Similarly, we obtain

$$\bar{\Psi}'(x')\gamma^\nu\Psi'(x') = \bar{\Psi}\bar{S}\gamma^\nu S\Psi = \Lambda^\nu{}_\mu\bar{\Psi}(x)\gamma^\mu\Psi(x) \tag{11.5}$$

That is, $\bar{\Psi}(x)\gamma^\nu\Psi(x)$ is a Lorentz vector. More generally, if we denote the 16 Dirac matrices by $\Gamma^{(i)}$, then $\bar{\Psi}(x)\Gamma^{(i)}\Psi(x)$ is a tensor with the indices of the sandwiched Dirac matrix. Indeed, $\bar{\Psi}\sigma^{\mu\nu}\Psi$ is a symmetric, traceless tensor of rank 2.

With the transformation properties of γ^5 under proper Lorentz transformations and space inversion

$$S^{-1}(\Lambda)\gamma^5 S(\Lambda) = \gamma^5$$
$$S^{-1}(\Pi)\gamma^5 S(\Pi) = -\gamma^5 \qquad (S(\Pi) = \gamma^0)$$

(**exercise**: check this; remember that $\gamma^5 = \frac{i}{4!}\epsilon_{\lambda\mu\nu\sigma}\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\sigma$) we see that

- $\bar{\Psi}\gamma^5\Psi$ is a pseudoscalar
- $\bar{\Psi}\gamma^5\gamma^\mu\Psi$ is a pseudovector

This will be very important for constructing the coupling terms in the Lagrangian. For example, in the electromagnetic case we have

$$\mathcal{L}^{\text{elmg}} = \bar{\Psi}(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\Psi \tag{11.6}$$

(minimal coupling) with as a possible additional coupling

$$\mathcal{L}^{\text{int}} = \bar{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu}$$

Another possibility is Yukawa coupling of a Dirac field to scalars and pseudoscalars, respectively. The latter one can, for example, be found in $\pi N\bar{N}$ coupling.

$$\mathcal{L}^{\text{sc}}_{\text{int}} = -f\bar{\Psi}(x)\Psi(x)\phi(x) \tag{11.7}$$

$$\mathcal{L}^{\text{ps.sc.}}_{\text{int}} = -f\bar{\Psi}(x)\gamma^5\Psi(x)\phi(x) \tag{11.8}$$

Note that in both, ϕ may be real or complex valued.

11.2 LSZ reduction

To calculate S-matrix elements with fermions one has to apply the LSZ reduction again. To this end, similar to the scalar case, one can use:

$$a^s_{\text{in}}(k) = \int d^3x \bar{u}^s(k)\exp(ikx)\gamma^0\Psi_{\text{in}}(x)$$
$$b^s_{\text{in}}(k) = \int d^3x \bar{v}^s(k)\exp(-ikx)\gamma^0\Psi_{\text{in}}(x) \tag{11.9}$$

The reduction formula for n incoming particles ($u(k_i)$), n' incoming antiparticles ($\bar{v}(k'_i)$), m outgoing particles ($\bar{u}(q_i)$) and m' outgoing antiparticles ($v(q'_i)$) then reads

$$\begin{aligned}
&\langle 0|\, b_{\text{out}}(q'_1)\ldots a_{\text{out}}(q_1)\ldots a^\dagger_{\text{in}}(k_1)\ldots b^\dagger_{\text{in}}(k'_1)\ldots|0\rangle = \\
&\quad(-iZ_2^{-1/2})^{(n+m)}(-iZ_2^{-1/2})^{(n'+m')}\int d^4x_1\ldots d^4y'_1 \qquad (11.10)\\
&\quad\exp\left\{-\sum(k\cdot x + k'\cdot x' - q\cdot y - q'\cdot y')\right\}\bar{u}(q_1)(i\partial\!\!\!/_{y_1} - m)\ldots\bar{v}(k_1)(i\partial\!\!\!/_{x'_1} - m)\\
&\quad\langle 0|\,\mathrm{T}\left\{\bar{\Psi}(y'_1)\ldots\Psi(y_1)\bar{\Psi}(x_1)\ldots\Psi(x'_1)\right\}|0\rangle\,(-i\overleftarrow{\partial\!\!\!/}_{x_1} - m)u(k_1)\ldots(-i\overleftarrow{\partial\!\!\!/}_{y'_1} - m)v(q'_1)
\end{aligned}$$

where the disconnected part has been left out like before. Dirac and spin indices have been suppressed in this formula, to improve the legibility. **Be careful**: anticommutations in the T-product will produce minus signs!

Remark

For photons, we have

$$\langle \beta; k, \epsilon\, out | \alpha_{in} \rangle = (-i)(Z_3)^{-1/2} \int d^4x \exp(ikx) \langle \beta_{\text{out}} | \, \epsilon j(x) \, | \alpha_{\text{in}} \rangle \quad (11.11)$$

$$\text{with} \quad \partial^2 A^\mu = j^\mu \quad (11.12)$$

Here, further reduction requires a redefinition of the T-product (called T*-product), adding distributions ($\neq 0$ for $x = y$) in order to obtain covariant expressions.

11.3 Feynman rules

Here we can be rather short:

(i) One derives the Gell-Mann-Low formula again, this time formulated in Ψ^{in} and $\bar{\Psi}^{\text{in}}$ fields.

(ii) The analogous Wick theorem for fermionic fields leads to "contractions" of the spinor fields. Since the Dirac theory is similar to that of complex scalars, it is pretty obvious (**exercise**: check this) that only the propagator

$$\langle 0 | \, T \left(\Psi_\alpha(x) \bar{\Psi}_\beta(y) \right) | 0 \rangle$$

is not zero

Note

Fermions can also be treated in the path integral formalism very elegantly. We will come back to this in chapter 13.

11.3.1 The Dirac propagator

Let us start off by defining

$$S_{\text{F},\alpha\beta}(x - y) := \langle 0 | \, T \left(\Psi_\alpha(x) \bar{\Psi}_\beta(y) \right) | 0 \rangle \quad (11.13)$$

where α and β are spinor indices. It is often notated as $S_{\text{F}}(x, y)$, and fulfills the equation

$$\boxed{((i\partial\!\!\!/ - m)_x)_{\alpha\beta} \left(S_{\text{F}}(x, y) \right)_{\beta\gamma} = i \delta_{\alpha\gamma} \delta^4(x - y)} \quad (11.14)$$

We can check that $S_F(x-y) = (i\partial\!\!\!/ + m)_x D_F(x-y)$ is the solution with Feynman boundary conditions using

$$(i\partial\!\!\!/ - m)\,(i\partial\!\!\!/ + m) = -\partial^2 - m^2 = -(\partial^2 + m^2) \tag{11.15}$$

The Fourier transform turns out to be

$$S_F(p) = \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon} \tag{11.16}$$

Of course, this propagator has direction, and the corresponding propagator line has an arrow, since Ψ is a complex field.

Note

In the time ordering of the LSZ reduction formula for Dirac fermions commutations *cause minus signs*. Similarly, in the Wick formula, contraction can only be done after commuting through the field in between and thereby picking up minus signs. There are $u, \bar{u}, v, \bar{v}$ in the LSZ-formula corresponing to ingoing and outgoing particles and ingoing and outgoing antiparticles. The vacuum graphs again cancel. Inserting the vacuum expectation value into the LSZ formula the outer propagators are cancelled, which is also referred to as "truncated", and we are left with a $\sqrt{Z}$ factor. This will be important for loop calculations later on.

Later, we will see that in the dressed propagator for fermions, the "self-energy" insertions also have the Dirac structure:

$$\frac{i(p\!\!\!/ + m_0)}{p^2 - m^2 + i\epsilon} + S_{\mathrm{F}}(p, m_0)\Pi S_{\mathrm{F}}(p, m_0) + \cdots = \frac{iZ_2(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon} + \text{rest}$$

We then obtain the Feynman rules for S-matrix elements (in momentum space):

For Yukawa theory

(i) Dirac propagator (eq. (11.16), instead of the scalar propagator before)

(ii) $-if$ or $-if\gamma_5$, for scalar and pseudoscalar vertices, respectively

(iii) statistical factors

(iv) vertex integration

(v) u^s for incoming particles, $\bar{u}^s$ for outgoing particles, v^s for outgoing antiparticles, $\bar{v}^s$ for incoming antiparticles

(vi) $\sqrt{Z}$ for outer scalar, $\sqrt{Z_2}$ for outer fermion fields

(vii) -1 for closed (inner) fermion lines (to be explained later)

Note: fermion propagator lines end in outer particles or are closed, but do not cross each other.

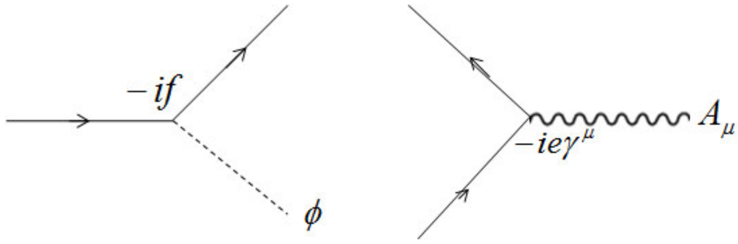


Figure 11.1: Yukawa scalar and QED vertices

For quantum electrodynamics

The rules will just be given here, in the Lorentz gauge ($\partial^\mu A_\mu = 0$). The derivation will come later, in the context of the fermionic path integral.

The interaction Hamiltonian is

$$\mathbf{H}_{\text{int}} = -eQ\bar{\Psi}(x)\gamma^\mu\Psi(x)A_\mu(x) \tag{11.17}$$

(Q is the charge number, e.g. -1 for electrons). The rules are:

(i) photon propagator: $\frac{-ig_{\mu\nu}}{p^2+i\epsilon}$ (the '$-m$'-term in the denominator disappears, since $m = 0$ for photons); Dirac propagator as before

(ii) $ie\gamma^\mu Q$ for vertices

(iii) statistical factors

(iv) vertex integration

(v) u^s, $\bar{u}^s$, v^s, $\bar{v}^s$ as before

(vi) $\sqrt{Z_3}$ for outer photons

(vii) -1 for fermion loops

Note

Spin averaging over incoming particles and spin *summation* over outgoing particles are often required when calculating cross sections. This gives, for example,

$$\frac{1}{2}\sum_{s,s'} \bar{u}^s(p_2)\gamma_5 u^{s'}(q_2)(\bar{u}^s(p_2)\gamma_5 u^{s'}(q_2))^* =$$
$$\frac{1}{2}\sum_{s,s'} \bar{u}^s(p_2)\gamma_5 u^{s'}(q_2)\bar{u}^{s'}(q_2)\gamma_5 u^s(p_2) =$$
$$\frac{1}{2}\sum_{s} \mathrm{tr}\left(u^s(p_2)u^s(p_2)\gamma_5\frac{\not{q}_2+m}{2m}\gamma_5\right) =$$
$$\frac{1}{2}\mathrm{tr}\left(\frac{\not{p}_2+m}{2m}\gamma_5\frac{\not{q}_2+m}{2m}\gamma_5\right) =$$
$$\frac{1}{2}\mathrm{tr}((\not{p}_2+m)(-\not{q}_2+m)) = 2(-p_2\cdot q_2+m^2)$$

(the factor of $\frac{1}{2}$ comes from spin averaging).

11.4 Simple example in Yukawa theory

Let us consider "nucleon scattering" with pion-exchange as force mediator as an example (between quotes because, of course, we know nowadays that nucleons are composite objects). The interaction Hamiltonian density is given by

$$\mathcal{H}_{\mathrm{int}} = f\bar{\Psi}\gamma_5\Psi\Phi \tag{11.18}$$

N-N scattering

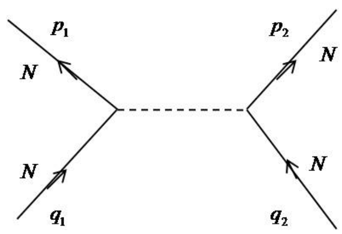


Figure 11.2: t-channel exchange

This is *t-channel exchange*: $t = (p_1-q_1)^2$. The expression corresponding to the diagram (see figure) is the following:

$$(-if)^2\bar{u}(p_1)\,\gamma_5 u(q_1)\bar{u}(p_2)\gamma_5 u(q_2)\frac{i}{(p_1-q_1)^2-m^2+i\epsilon}\times$$
$$(Z_2)^2(2\pi)^4\delta^4(p_1+p_2-q_1-q_2) =$$
$$i\mathbf{M}(2\pi)^4\delta^4(p_1+p_2-q_1-q_2) \tag{11.19}$$

There is a second diagram for N-N scattering, where the two outgoing lines are exchanged. According to the above recipe, this will give a minus sign; this can also be derived from a careful evaluation of the LSZ-formula.

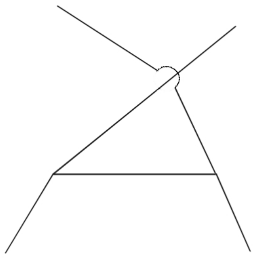


Figure 11.3: u-channel exchange

The Z-factors are not of great interest here; they will become important in the discussion of renormalization. The propagator $1/(p_1-q_1)^2-m^2$ in the t-channel gives, after Fourier transformation of the evaluated $i\mathbf{M}$ (the amplitude), the Yukawa potential, which is well known in nuclear physics.

N-N̄ scattering

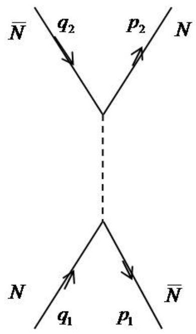


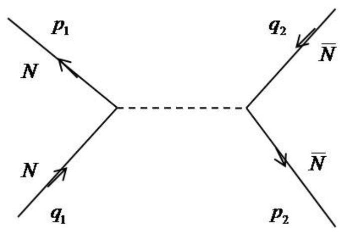
Figure 11.4: s-channel exchange

This is s-channel exchange: $s=(q_1+p_2)^2$. The diagram is the same, except for the exchange of p_1 and q_2. The corresponding expression is

$$(-if)^2\bar{v}(-p_1)\,\gamma_5 u(q_1)\bar{u}(p_2)\gamma_5 v(-q_2)\frac{i}{(p_1-q_1)^2+m^2+i\epsilon}\times$$
$$(Z_2)^2(2\pi)^4\delta^4(p_1+p_2-q_1-q_2) \qquad (11.20)$$

Another diagram that represents this type of scattering is shown below, and has the following expression:

$$(-if)^2\bar{u}(p_1)\gamma_5 u(q_1)\bar{v}(-p_2)\gamma_5 v(-q_2)\frac{i}{(p_1-q_1)^2+m^2+i\epsilon}\times\ldots \tag{11.21}$$



Notes

- In Peskin & Schröder, in the spirit of time-ordered perturbation theory, outer one-particle states are directly contracted with vertex-fields:

$$\langle p_1 q_2|\,\bar{\Psi}\gamma_5\Psi\bar{\Psi}\gamma_5\Psi\,|q_1 p_2\rangle$$

  The required permutations give rise to extra minus signs, which include signs related to the fermion statistics of the outer states. Altogether, this gives the $\bar{u}, u, \bar{v}$ and v as above.

- A closed fermion loop gives a minus sign, as mentioned above:

$$\begin{aligned}&\langle 0|\,\mathrm{T}(\bar{\Psi}(x_1)\Psi(x_1)\bar{\Psi}(x_2)\Psi(x_2))\,|0\rangle \to \\ &\qquad -\,\Psi(x_1)\bar{\Psi}(x_2)\Psi(x_2)\bar{\Psi}(x_1)\end{aligned}$$

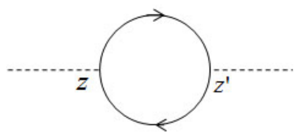


Figure 11.5: Closed fermion loop

Feynman approach vs. time-ordered perturbation theory

In the Feynman approach the propagators are off-shell and one has 4-momentum conservation at the vertices.

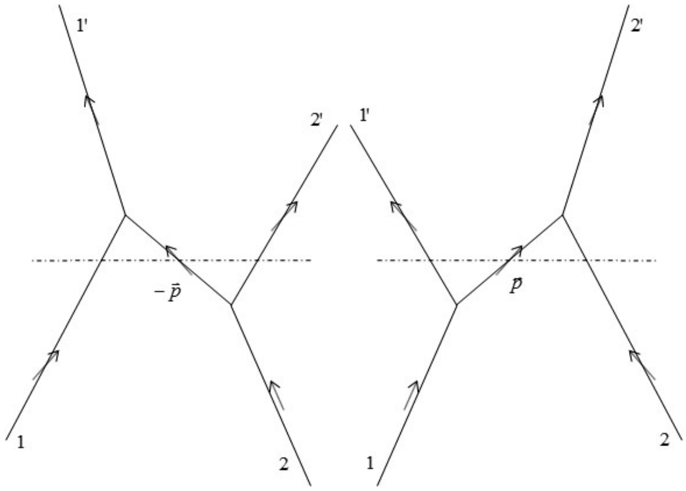


Figure 11.6: Cases (α) (left) and (β) (right)

In *time-ordered perturbation theory*, on the other hand, one has the well-known energy denominators, like in QM, and, which is new in field theory, 3-momentum conservation at the vertices. Of course the incoming and outgoing particles have the same total energy, but in between, energy is not conserved, as a consequence of the energy-time uncertainty relation. The intermediate particles are *on shell* in this approach.

For tree-level diagrams, the relation between the two approaches can be seen by decomposing the propagator:

$$\frac{1}{p^2 - m^2 + i\epsilon} = \frac{1}{p_0^2 - \vec{p}^2 - m^2 + i\epsilon}$$

In the Feynman case, one rewrites this as

$$\left(\frac{1}{p_0 - \sqrt{\vec{p}^2 + m^2} + i\epsilon} - \frac{1}{p_0 + \sqrt{\vec{p}^2 + m^2} - i\epsilon} \right) \frac{1}{2p_{0,+}}$$

In the time-ordered perturbation theory, one has the following denominators:

$$E_1' + \sqrt{\vec{p}^2 + m^2} + E_2 - E_1 - E_2 = -p_0 + \sqrt{\vec{p}^2 + m^2} \qquad (\alpha)$$
$$E_1 + \sqrt{\vec{p}^2 + m^2} + E_2' - E_1' - E_2' = p_0 + \sqrt{\vec{p}^2 + m^2} \qquad (\beta)$$

These are the same as the ones from the Feynman case.

For loop integrals, the Feynman approach has

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(p-k)^2 - m^2 + i\epsilon} \frac{1}{k^2 - m^2 + i\epsilon}$$

Time-ordered perturbation theory has

$$\int d^3k \frac{1}{\text{energy denominators}}$$

Here, the connection between the two approaches can be seen by doing the dk^0-integration using Cauchy's theorem (the residue theorem), closing the integration contour in one of the two halfplanes. One gets poles in k_0, yielding the required energy denominator in the residue.

For example, in Φ^3 scalar theory, for the graph in figure 11.6, one has

$$(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p-k)^2 - m^2 + i\epsilon} =$$

$$(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^0 - \sqrt{\vec{k}^2 + m^2} + i\epsilon} - \frac{1}{k^0 + \sqrt{\vec{k}^2 + m^2} - i\epsilon} \right) \frac{1}{2k^{0,+}} \times$$

$$\left(\frac{1}{(p^0 - k^0) - \sqrt{(\vec{p} - \vec{k})^2 + m^2} + i\epsilon} - \frac{1}{(p^0 - k^0) + \sqrt{(\vec{p} - \vec{k})^2 + m^2} - i\epsilon} \right) \times$$

$$\frac{1}{2(p^0 - k^0)^+}$$

(This is a rough calculation, just to see the principle; we will not worry about infinities here.) Closing the integration contour in the lower half plane and applying the residue theorem gives

$$(i\lambda)^2 \int \frac{d^3k}{(2\pi)^4} (-2\pi i) \frac{1}{2k^{0,+}} \frac{1}{2(p^0 - k^0)^+} \times$$

$$\left(\frac{1}{p^0 - \sqrt{\vec{k} + m^2} - \sqrt{(\vec{p} - \vec{k})^2 + m^2}} - \frac{1}{p^0 + \sqrt{\vec{k} + m^2} + \sqrt{(\vec{p} - \vec{k})^2 + m^2}} \right.$$

$$\left. - \frac{1}{p^0 - \sqrt{\vec{k} + m^2} + \sqrt{(\vec{p} - \vec{k})^2 + m^2}} + \frac{1}{p^0 - \sqrt{\vec{k} + m^2} + \sqrt{(\vec{p} - \vec{k})^2 + m^2}} \right)$$