Equation of state of the unitary Fermi gas

Igor Boettcher

Institute for Theoretical Physics, University of Heidelberg

with S. Diehl, J. M. Pawlowski, and C. Wetterich

Δ13, 11. 1. 2013
The many-body problem
The many-body problem

possibility of a statistical description

collective degrees of freedom
The many-body problem

1\textsuperscript{st} step: Find the right Hamiltonian $H$

2\textsuperscript{nd} step: Determine the partition function $Z$

$$Z(\mu, T) = \text{Tr} \left( e^{-\beta(H-\mu N)} \right)$$
The many-body problem

1\textsuperscript{st} step: Find the right Hamiltonian $H$

2\textsuperscript{nd} step: Determine the partition function $Z$

$$Z(\mu, T) = \text{Tr} \left( e^{-\beta (H - \mu N)} \right)$$

H is known for cold atoms and QCD!
The many-body problem

1\textsuperscript{st} step: Find the right Hamiltonian $H$

2\textsuperscript{nd} step: Determine the partition function $Z$

\[ Z(\mu, T) = \text{Tr} \left( e^{-\beta(H-\mu N)} \right) = \int \text{D}\phi e^{-S[\phi]} \]

path integral

Euclidean quantum field theory

H is known for cold atoms and QCD!
Shopping list

What are the generic features of quantum many-body systems?

What are reliable theoretical methods to describe such systems?

What observables reveal advancements and short-comings of theory?
Shopping list

What are the generic features of quantum many-body systems?

What are reliable theoretical methods to describe such systems?

What observables reveal advancements and short-comings of theory?

- cold atoms
- neutron stars
- high-Tc superconductors
- early universe
- nuclear matter
- heavy ion collisions
- quark gluon plasma
- neutron stars
Shopping list

Theory

- Phase diagram and Equation of state
  \[ P(\mu, T) = \frac{k_B T}{V} \log Z(\mu, T) \]
- Momentum distribution
- Transport coefficients
  \[ \eta(\mu, T) \]

Experiments with cold atoms

- Density images
- Collective mode frequencies and damping constants
- Expansion after release from trap
- Response functions

...
Shopping list

**Theory**

- Phase diagram and Equation of state
  \[ P(\mu, T) = \frac{k_B T}{V} \log Z(\mu, T) \]
- Momentum distribution
- Transport coefficients \[ \eta(\mu, T) \]

**Experiments with cold atoms**

- Density images
- Collective mode frequencies and damping constants
- Expansion after release from trap
- Response functions
The equation of state

Classical ideal gas: \[ P(n, T) = nk_B T \]

Virial expansion for interacting gas:

\[ P(n, T) = nk_B T (1 + B_2(T)n + \ldots) \]

Van-der-Waals equation of state:

\[ P(n, T) = \frac{nk_B T}{1 - bn} - an^2 \simeq nk_B T \left(1 + (b - \frac{a}{k_B T})n + \ldots\right) \]
Pressure \( P(\mu, T) \)

**Bose gas**

\[
\hat{T} = T a^2 m k_B / \hbar^2
\]

\[
\hat{\mu} = \mu a^2 m / \hbar^2
\]

Figure 2.1: Pressure \( \hat{P} = Pa^5 m / \hbar^2 \) as a function of \( \hat{T} \) and \( \hat{\mu} \)
Density \( n = (\partial P / \partial \mu)_T \)

Bose gas

\[ \hat{T} = Ta^2 m k_B / \hbar^2 \]
\[ \hat{\mu} = \mu a^2 m / \hbar^2 \]

Figure 2.3: Density \( \hat{n} = na^3 \) as a function of \( \hat{T} \) and \( \hat{\mu} \)
Isothermal compressibility \((\partial^2 P/\partial \mu^2)_T\)

Bose gas

\[
\hat{T} = Ta^2 mk_B/\hbar^2
\]
\[
\hat{\mu} = \mu a^2 m/\hbar^2
\]

Figure 2.5: \(\hat{P}^{\mu\mu} = P^{\mu\mu} a^2 \hbar^2 / m\) as a function of \(\hat{T}\) and \(\hat{\mu}\)
Isothermal compressibility $(\partial^2 P / \partial \mu^2)_T$

Bose gas

Position of critical line: phase diagram

Superfluid phase transition

$\hat{T} = T a^2 m k_B / \hbar^2$

$\hat{\mu} = \mu a^2 m / \hbar^2$
The BCS-BEC Crossover

Two cornerstones of quantum condensation:

- **BCS**
  - Cooper pairing of weakly attractive fermions

- **BEC**
  - Bose condensation of weakly repulsive bosons
The BCS-BEC Crossover

Two cornerstones of quantum condensation:

BCS

\[ \sigma = 4\pi a^2 \]

BEC

\[ (k_F a)^{-1} \]
The BCS-BEC Crossover

Two cornerstones of quantum condensation:

Unitary Fermi gas

\[ (k_F a)^{-1} = 0, \quad \sigma = \frac{4\pi}{p^2} \]

\[ \sigma = 4\pi a^2 \]
The BCS-BEC Crossover

3D BCS-BEC crossover
(results from Functional Renormalization Group)
The BCS-BEC Crossover

2D BCS-BEC crossover
(results from Functional Renormalization Group)

High Tc superconductors!?
The BCS-BEC Crossover

Key observables

Tan contact

\[ n_{\tilde{p}\sigma} \approx \frac{C}{p^4} \]

Equation of state

\[ P(\mu, T, a) \]

Dimer-dimer scattering length

\[ \frac{a_{dd}}{a} = 0.6 \]

Bertsch parameter

\[ \xi = \frac{\mu}{\varepsilon_F} \text{ at } a^{-1} = 0 \]

Critical temperature

\[ \frac{T_c}{T_F} \]
The BCS-BEC Crossover

Key observables

Tan contact
\[ n \bar{p}_\sigma \approx \frac{C}{p^4} \]

Equation of state
\[ P(\mu, T, a) \]

Dimer-dimer scattering length
\[ \frac{a_{dd}}{a} = 0.6 \]

Bertsch parameter
\[ \xi = \mu / \varepsilon_F \text{ at } a^{-1} = 0 \]

Critical temperature
\[ T_c / T_F \]
The BCS-BEC Crossover

One problem:

Tan contact

$n_{p\sigma}$

MOT Beams

Trapping beam

Objective

Offset coils

Gradient coils

CCD Camera

Equation of state

Dimer-dimer scattering length

Bertsch parameter

Critical temperature

One problem:
Thermodynamics from density profiles

$P(\mu, T) \rightarrow P(\mu - V_{\text{ext}}(\vec{x}), T)$

local density approximation

$P(\mu_z, T) = \frac{m\omega_r^2}{2\pi} \overline{n}(z)$

Ho, Zhou

S. Nascimbène et al.
Thermodynamics from density profiles

Unitary Fermi gas at MIT by Zwierlein group

Thermodynamics from density profiles


Tc/TF = 0.167(13)

Unitary Fermi gas at MIT by Zwierlein group
Bertsch parameter $\xi$: EoS at $T=0$

$$E(0) = \xi \frac{3}{5} N \varepsilon_F$$

Unitary Fermi gas at MIT by Zwierlein group
Thermodynamics from density profiles

Bertsch parameter $\xi$: EoS at $T=0$

$$E(0) = \frac{3}{5} N \varepsilon_F$$

$$F(0) = E(0)$$

Unitary Fermi gas at MIT by Zwierlein group
Thermodynamics from density profiles

Bertsch parameter $\xi$:
EoS at $T=0$

$$E(0) = \xi \frac{3}{5} N \varepsilon_F$$

$$F(T) \leq F(0) = E(0) \leq E(T)$$

Unitary Fermi gas at MIT by Zwierlein group
Thermodynamics from density profiles

Bertsch parameter $\xi$:
EoS at $T=0$

$$E(0) = \frac{3}{5} \xi N \varepsilon_F$$

$$F(T) \leq F(0) = E(0) \leq E(T)$$

Unitary Fermi gas at MIT by Zwierlein group
Bertsch parameter $\xi$: EoS at T=0

$E(0) - \xi^3 N_a$,

Precise characterization of $^6$Li Feshbach resonances using trap-sideband resolved RF spectroscopy of weakly bound molecules

G. Zürn,$^{1,2}$ T. Lompe,$^{1,2,3}$,* A. N. Wenz,$^{1,2}$ S. Jochim,$^{1,2,3}$ P. S. Julienne,$^4$ and J. M. Hutson$^5,^6$

$^1$Physikalisches Institut, Ruprecht-Karls-Universität Heidelberg, Germany
$^2$Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, 69117 Heidelberg, Germany
$^3$ExtreMe Matter Institute EMMI, GSI Helmholtzzentrum für Schwerionenforschung, Darmstadt, Germany
$^4$Joint Quantum Institute, NIST and the University of Maryland, Gaithersburg, Maryland 20899-8423, USA
$^5$Joint Quantum Centre (JQC) Durham/Newcastle, Department of Chemistry, Durham University, South Road, Durham, DH1 3LE, United Kingdom

(Dated: November 8, 2012)

Unitary Fermi gas at MIT by Zwierlein group
Equation of state from Functional RG

Experiment:

\[ \xi_{\text{exp}} = 0.370(5)(8) \]

\[ \left( \frac{T_c}{T_F} \right)_{\text{exp}} = 0.167(13) \]

Latest FRG:
(Floerchinger, Scherer, Wetterich)

\[ \xi_{\text{FRG}} = 0.51 \]

\[ \left( \frac{T_c}{T_F} \right)_{\text{FRG}} = 0.248 \]
Equation of state from Functional RG

Experiment:

\[ \xi_{\text{exp}} = 0.370(5)(8) \]

\[ (\frac{T_c}{T_F})_{\text{exp}} = 0.167(13) \]

Latest FRG:

(Floerchinger, Scherer, Wetterich)

\[ \xi_{\text{FRG}} = 0.51 \]

\[ (\frac{T_c}{T_F})_{\text{FRG}} = 0.248 \]
Equation of state from Functional RG

Experiment:

\[ \xi_{\text{exp}} = 0.370(5)(8) \]

Latest FRG:

\( (T_c/T_F) \)

Floerchinger, Scherer, Wetterich

Most likely error source:

Frequency-independent cutoffs

First estimate: \( \xi = 0.41 \) with 4D-fermion cutoff in the simplest truncation
Equation of state from Functional RG

Experiment:

$$\xi_{\text{exp}} = 0.370(5)(8)$$

Most likely error source:

Frequency-independent cutoffs

First estimate: $\xi = 0.41$ with 4D-fermion cutoff in the simplest truncation

→ contact me during the workshop if you are interested in that
Tan contact

\[ n_{\vec{p}\sigma} \approx \frac{C}{p^4} \]

Momentum distribution

\[ \left( \frac{\partial P}{\partial a^{-1}} \right)_{\mu, T} = \frac{C}{4\pi M} \]

Tan relation

\[ \Sigma_{\psi}(P) \approx \frac{4C}{-ip_0 + p^2 - \mu} - \delta\mu \]

Asymptotic fermion self-energy
Tan contact

FRG: IB, S. Diehl, J. M. Pawlowski, C. Wetterich
Tan contact

Thank you for your attention!

FRG: IB, S. Diehl, J. M. Pawlowski, C. Wetterich
Additional slides
Microscopic Model

Many-body Hamiltonian

\[ \hat{H} = \int d^3x \left( \sum_{\sigma=1,2} \hat{\psi}^\dagger_\sigma (-\nabla^2) \hat{\psi}_\sigma + \lambda_{\psi,\Lambda} \hat{\psi}^\dagger_1 \hat{\psi}^\dagger_2 \hat{\psi}_2 \hat{\psi}_1 \right) \]
Microscopic Model

Many-body Hamiltonian

\[ \hat{H} = \int d^3 x \left( \sum_{\sigma=1,2} \hat{\psi}_\sigma^\dagger (-\nabla^2) \hat{\psi}_\sigma + \lambda_{\psi,\Lambda} \hat{\psi}_1^\dagger \hat{\psi}_2^\dagger \hat{\psi}_2 \hat{\psi}_1 \right) \]

Microscopic action

\[ S[\varphi, \psi] = \int \mathcal{X} \left( \sum_{\sigma=1,2} \psi_\sigma^*(\partial_\tau - \nabla^2 - \mu) \psi_\sigma + m_{\varphi,\Lambda}^2 \varphi^* \varphi 
\]

\[ - h_{\varphi} (\varphi^* \psi_1 \psi_2 - \varphi \psi_1^* \psi_2^*) \right) \]
Macroscopic physics

How to compute the partition function?

\[ Z(\mu, T) = \int \mathcal{D}\varphi \mathcal{D}\psi e^{-S[\varphi, \psi]} \]  

Integration
Macroscopic physics

How to compute the partition function?

\[ Z_k(\mu, T) = \int D\varphi D\psi e^{-S[\varphi, \psi] + \Delta S_k} \]

scale dependent partition function

\[ \partial_k Z_k(\mu, T) = \ldots \]  Solve flow equation
Wetterich equation

\[ \Gamma[\Phi] = J \cdot \Phi - \log Z[J] \]

effective action

\[ \partial_k \Gamma_k = \frac{1}{2} \text{Str} \left( \frac{1}{\Gamma_k^{(2)} + R_k} \partial_k R_k \right) \]

\[ \Gamma_{k=\Lambda} = S \quad \text{fluctuations} \quad \Gamma_{k=0} = \Gamma \]

Microphysics \quad \text{fluctuations} \quad \text{Macrophysics}
Contact in the BCS-BEC Crossover
Momentum distribution

Ideal Fermi gas: Fermi-Dirac distribution

\[ n_{\psi, \bar{q}\sigma} = \frac{C}{q^4} \]
Momentum distribution

\[ n_{\vec{p}\sigma} \sim \frac{C}{p^4} \]

Tan contact \( C \)

Several exact relations, e.g.:

\[ \frac{1}{V} \frac{dE}{d(-1/a)} = \frac{C}{4\pi M} \]

\[ E = \frac{C}{4\pi Ma} + \sum_{\sigma=1,2} \int \frac{d^3 p}{(2\pi)^3} \frac{p^2}{2M} \left( n_{\vec{p}\sigma} - \frac{C}{p^4} \right) \]
Contact from the FRG

\[ n_{\bar{\sigma}\sigma} = - \int_{p_0} G_{\psi\sigma}(p_0, \bar{\rho}) \]

full macroscopic propagator
Contact from the FRG

Factorization of the RG flow for large $p$:

$$\partial_k G_{\psi,k}^{-1}(P) \simeq \frac{4}{-i\rho_0 + p^2 - \mu} \partial_k C_k$$
Contact from the FRG

Factorization of the RG flow for large $p$:

$$\partial_k G_{\psi,k}^{-1}(P) \approx \frac{4}{-i\rho_0 + p^2 - \mu} \partial_k C_k$$

Flowing contact

$$\partial_k C_k = \ldots$$
Contact from the FRG

Universal regime is enhanced for the Unitary Fermi gas

\[ \Sigma_\psi(P) \simeq \frac{4C}{-ip_0 + p^2 - \mu} - \delta \mu \]
Contact from the FRG

Universal regime is enhanced for the Unitary Fermi gas

\[ \Sigma_\psi(P) \approx \frac{4C}{-i\rho_0 + p^2 - \mu} - \delta \mu \]
Contact from the FRG

Temperature dependent contact of the Unitary Fermi gas

![Graph showing temperature dependent contact of the Unitary Fermi gas](image)
Contact from the FRG

Contact at $T=0$ in the BCS-BEC crossover
Contact from the FRG

Momentum distribution of the Unitary Fermi Gas at the critical temperature

without contact term

with contact term
Increase of density

Contribution from high energetic particles to the density

\[ n = 2 \int \frac{d^3 p}{(2\pi)^3} n\tilde{\rho}_\sigma \]

\[ \frac{\delta n^{(C)}}{n} = 27.5\% \text{ at } T_c \]

Substantial effect on \( \frac{T_c}{T_F} \propto \frac{T_c}{n^{2/3}} \)
Two-dimensional BCS-BEC Crossover
Two-dimensional BCS-BEC Crossover

Why two dimensions?

- Enhanced effects of quantum fluctuations → test and improve elaborate methods
- Understand pairing in two dimensions → high temperature superconductors

How?

Highly anisotropic traps!
What is different?

Scattering physics in two dimensions

\[ f_{2d}(q) \sim \frac{1}{\log(1/q^2 a_{2d}^2) + i\pi + \ldots} \]

\[ f_{3d}(q) \sim \frac{1}{-\frac{1}{a} + \frac{1}{2} r_e q^2 - iq + \ldots} \]

Scattering amplitude

Crossover parameter \( \log(k_F a_{2d}) \)

No scale invariance, but strong correlations for \( k_F \sim \frac{1}{a_{2d}} \)
Equation of state at $T=0$

\[ \frac{(\mu - \varepsilon_B/2)}{\varepsilon_F} = 0.874 \quad \text{for} \quad \log(k_F a_{2d}) = 0 \]
Equation of state at $T=0$

\[ \frac{(\mu - \varepsilon_B/2)}{\varepsilon_F} = 0.874 \quad \text{for} \quad \log(k_F a_{2d}) = 0 \]
$T_c/T_F = 0.172 \quad \text{for} \quad \log(k_F a_{2d}) = 0$
Superfluid phase transition

Thank you for your attention and enjoy lunch!

\[ T_c / T_F = 0.172 \quad \text{for} \quad \log(k_F a_{2d}) = 0 \]