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Effective theory for thermal lattice QCD with heavy quarks

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General outline

- Lattice QCD: Sign problem at finite chemical potential
- Define effective theory by integrating out spatial degrees of freedom
- Effective theory can be simulated very fast by different algorithms
- No solution to the sign problem, but a huge reduction of its severity
- Disadvantage: Expansion starts from the unphysical strong coupling and infinite quark mass region

Starting point: QCD with Wilsons Action

- Partition function

$$Z = \int [dU_0][dU_i] e^S \qquad S = S_g + S_q$$

- Gauge part:

$$S_g = \frac{\beta}{3} \sum_p \mathrm{Re}\,\mathrm{Tr}\, U_p$$

- Quark part after Grassmann integration (per flavor and omitting spin and color indices):

$$e^{S_q} = \det \left[\delta_{xy} - \kappa \sum_{\pm\nu} (1+\gamma_\nu) U_\nu(x)\, \delta_{x,y-\hat{\nu}} \right]$$

- Finite T: Compact temporal extent with (a)pbcs
- Chemical potential: Additional factor $e^{\pm\mu}$ for temporal links

The effective action

- Integrate out spatial link variables

$$Z = \int [dU_0][dU_i]e^S \equiv \int [dU_0]e^{S_{\rm eff}}$$

- Crucial point: $S_{\rm eff}$ depends only on Polyakov loops
$\rightarrow$ (3+1)d theory can be reduced to effective 3d theory
- Dofs: Complex numbers instead of group elements
- Disadvantages:
  - Need in principal infinite number of effective interaction terms and effective couplings
  - Couplings only known to some order in strong coupling and hopping parameter expansion
- Nevertheless: Leading interaction terms and orders in β and κ can be calculated without too much effort

Leading order effective theory

Quark part

- Neglect spatial plaquettes and spatial quark hops
  $\rightarrow$ The spatial integrations can be calculated exactly
- The quark part has no spatial link dependence at all

$$
\begin{aligned}
e^{S_q} &= \det\left[\delta_{xy} - \kappa e^{\mu}(1+\gamma_0)U_0(x)\,\delta_{x,y-\hat{0}}\right] \\
&* \det\left[\delta_{xy} - \kappa e^{-\mu}(1-\gamma_0)U_0^{\dagger}(x)\,\delta_{x,y+\hat{0}}\right] \\
&= \prod_{\vec{x}} \det\left[1 + h_1\, W(\vec{x})\right]^2 \left[1 + \bar{h}_1\, W^{\dagger}(\vec{x})\right]^2
\end{aligned}
$$

- Effective coupling: $h_1(\mu) = (2\kappa e^{\mu})^{N_\tau} = \bar{h}_1(-\mu)$
- Polyakov loop:

$$
\mathrm{Tr}\, W(\vec{x}) = \mathrm{Tr} \prod_{\tau=1}^{N_\tau} U_0(\tau,\vec{x}) = L(\vec{x})
$$

Leading order effective theory

Gauge part

- Character expansion

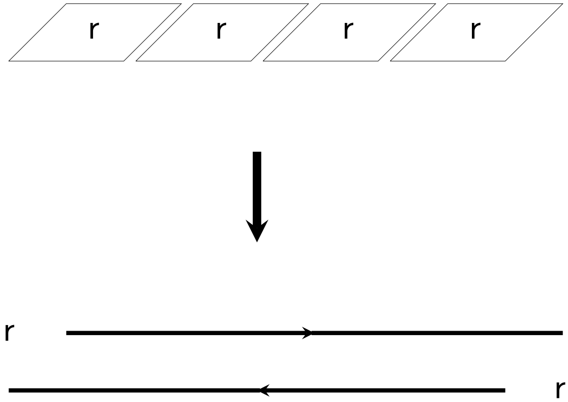
$$e^{S_g} = \prod_{tp} \left[1 + \sum_r d_r a_r(\beta) \chi_r(U_p) \right] \qquad \chi_r(U) = \mathrm{Tr} D^r(U)$$

- Spatial links: At most two plaquettes in nontrivial representations r and s
- Then: Use character orthogonality at each spatial link

$$\int dU\, D^r_{ij}(U) D^s_{kl}(U^\dagger) = \frac{\delta^{rs}}{d_r} \delta_{il} \delta_{jk}$$

- Surviving terms: Chains of N_τ plaquettes in the same rep. [Polonyi, Szlachanyi (1982)]

Leading order effective theory



Leading order effective theory

- The gauge part then reads

$$\int [dU_i] e^{S_g} = \prod_{\vec{x},i} \left[1 + \sum_r a_r^{N_\tau} \chi_r\Big(W_{\vec{x}}\Big) \chi_r\Big(W^\dagger_{\vec{x}+\vec{e}_i}\Big) \right]$$

- Higher representations are suppressed: E.g. in SU(2)

$$r = \frac{1}{2}, 1, \frac{3}{2}, \ldots \qquad a_r(\beta) \sim \beta^{2r} + \ldots$$

- Defining $u = a_f(\beta)$ and $\lambda_1 = u^{N_\tau}$:

$$\int [dU_i] e^{S_g} = \prod_{\vec{x},i} \left[1 + \sum_r \lambda_1 \Big(L_{\vec{x}} L^*_{\vec{x}+\vec{e}_i} + L^*_{\vec{x}} L_{\vec{x}+\vec{e}_i} \Big) \right]$$

Leading order effective theory: Remarks

$$Z = \int [dW] \prod_i \det \left[1 + h_1 W_i\right]^2 \left[1 + \bar{h}_1 W_i^\dagger\right]^2 \prod_{<ij>} \left[1 + 2\lambda_1 \mathrm{Re}\, L_i L_j^*\right]$$

- Simulation yields critical h_1^c and λ_1^c $\quad \rightarrow \quad$ β^c and κ^c
- The well-known $SU(N)$ spin model is the first order approximation to this [DeGrand, DeTar (1983), Green, Karsch (1984), Aarts, James [2011], Delgado, Gattringer (2012)]
- Pure gauge: Next-to-nearest neighbor interactions are due to the inclusion of spatial plaquettes
- Spatial plaquettes and quark hops contribute higher orders to the leading couplings and introduce new interaction terms

Full effective theory

- Including corrections effective theory may be written as:

$$Z = \int [dW] \prod_n \left[1 + \lambda_n \Delta_n^s\right] \prod_m \left[1 + h_m \Delta_m^a\right] \left[1 + \bar{h}_m \Delta_m^{a,\dagger}\right]$$

- $Z(N)$-symmetric terms Δ^s and asymmetric terms Δ^a
- Generic leading orders of the effective couplings:

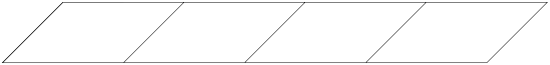
$$\begin{aligned} \lambda_n(u, \kappa, N_\tau) &\sim u^{r_n N_\tau}\left[1 + \dots\right] + (2\kappa)^{s_n N_\tau}\left[1 + \dots\right] \\ h_m(u, \kappa, \mu, N_\tau) &\sim (2\kappa e^{\mu})^{t_m N_\tau}\left[1 + \dots\right] = \bar{h}_m(u, \kappa, -\mu, N_\tau) \\ r_n, s_n, t_m &\in N \end{aligned}$$

- Corrections in brackets depend on (u, κ, N_τ) only: μ-dependence of h_m completely determined by t_m

Gauge corrections

Corrections to the leading coupling λ_1

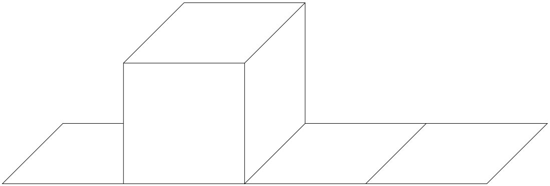
- Proper treatment: Cluster expansion, see e.g. textbook of Montvay/Münster
- Essence: Starting with leading order graph and attach an increasing number of plaquettes
- Example $\sim u^{N_\tau}$



Gauge corrections

Corrections to the leading coupling λ_1

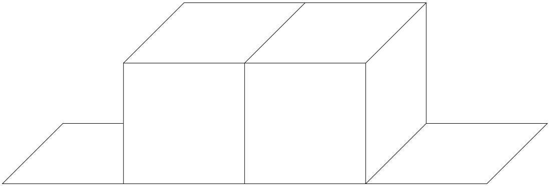
- Proper treatment: Cluster expansion, see e.g. textbook of Montvay/Münster
- Essence: Starting with leading order graph and attach an increasing number of plaquettes
- Example $\sim N_\tau u^{N_\tau+4}$



Gauge corrections

Corrections to the leading coupling λ_1

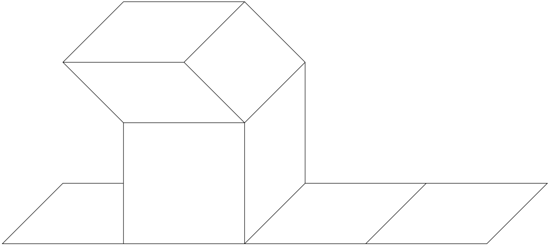
- Proper treatment: Cluster expansion, see e.g. textbook of Montvay/Münster
- Essence: Starting with leading order graph and attach an increasing number of plaquettes
- Example $\sim N_\tau u^{N_\tau+6}$



Gauge corrections

Corrections to the leading coupling λ_1

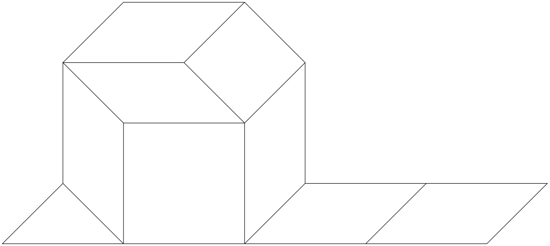
- Proper treatment: Cluster expansion, see e.g. textbook of Montvay/Münster
- Essence: Starting with leading order graph and attach an increasing number of plaquettes
- Example $\sim N_\tau u^{N_\tau+8}$



Gauge corrections

Corrections to the leading coupling λ_1

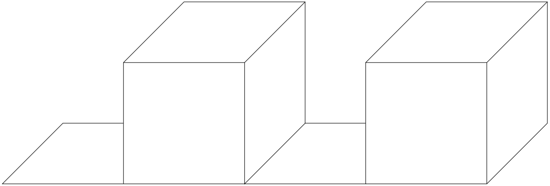
- Proper treatment: Cluster expansion, see e.g. textbook of Montvay/Münster
- Essence: Starting with leading order graph and attach an increasing number of plaquettes
- Example $\sim N_\tau u^{N_\tau + 10}$



Gauge corrections

Corrections to the leading coupling λ_1

- Proper treatment: Cluster expansion, see e.g. textbook of Montvay/Münster
- Essence: Starting with leading order graph and attach an increasing number of plaquettes
- Example $\sim \frac{1}{2} N_\tau^2 u^{N_\tau+8}$



Gauge corrections

Corrections to the leading coupling λ_1

- Repetitions of these decorations exponentiate

$$\lambda_1(u, N_\tau) = u^{N_\tau} \exp\left[N_\tau\left(P_{N_\tau}(u)\right)\right]$$

- For large enough N_τ after truncating in u:

$$P_{N_\tau}(u) = P_{N_\tau^*(u)} \equiv P(u) \qquad\qquad \forall \quad N_\tau > N_\tau^*$$

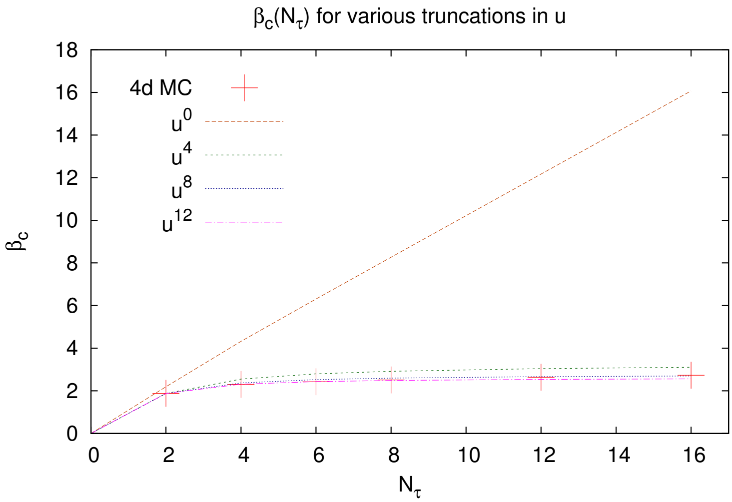
- E.g. $SU(2)$ up to $\mathcal{O}(u^{12})$: $N_\tau^* = 6$ and

$$P(u) = 4u^4 - 4u^6 + \frac{140}{3}u^8 - \frac{37664}{405}u^{10} + \frac{863524}{1215}u^{12}$$

- $\lambda_1(u, N_\tau < 6)$ also known to this order
- Details in [Langelage, Lottini, Philipsen (2010)]

Deconfinement transition

Evolution of $\beta^c(N_\tau)$ for different truncations and $SU(2)$



Deconfinement transition

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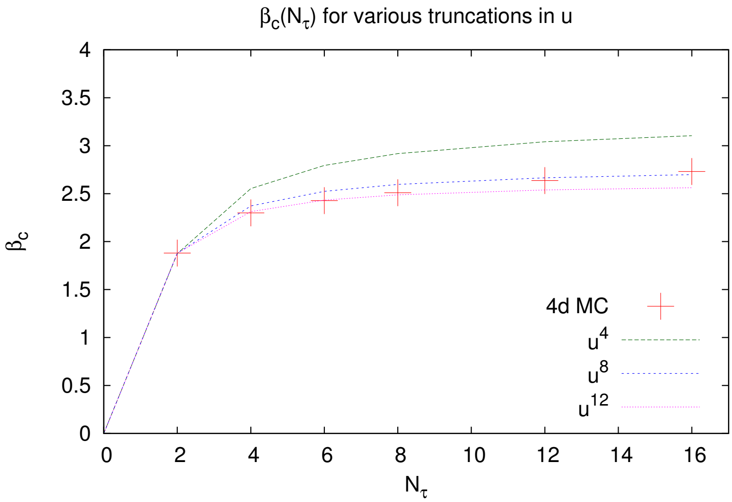


Comparison with full simulations

$SU(2)$

N_τ	3d Eff. Th.	4d YM
2	2.1929(13)	2.1768(30)
4	2.3102(08)	2.2991(02)
6	2.4297(05)	2.4265(30)
8	2.4836(03)	2.5104(02)
12	2.5341(02)	2.6355(10)
16	2.5582(02)	2.7310(20)

4d Monte Carlo results taken from [Fingberg et al. (1992), Bogolubsky et al. (2004) and Velytsky (2007)]

Comparison with full simulations

$SU(3)$

N_τ	3d Eff. Th	4d YM
2	5.1839(2)	5.10(5)
4	6.09871(7)	5.6925(2)
6	6.32625(4)	5.8941(5)
8	6.43045(3)	6.001(25)
12	6.52875(2)	6.268(12)
16	6.57588(1)	6.45(5)

4d Monte Carlo results taken from [Fingberg et al. (1992)]

Fermionic corrections

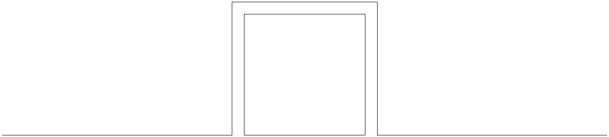
- Terms from the hopping expansions fall into two classes: Winding number i) $n = 0$ or ii) $n > 0$
- i) The leading contribution of these graphs for large quark masses comes from a κ^4-plaquette (4 quark hops). We absorb this effect in a shift of β

$$u(\beta) \quad \rightarrow \quad u\left(\beta + 48 N_f \kappa^4\right)$$

- ii) These terms contribute higher orders to the effective coupling h_1 or give rise to new interaction terms. In the latter case they have $n > 1$ and wind at at least two different spatial sites
- Details in [Fromm, Langelage, Lottini, Philipsen (2011)]

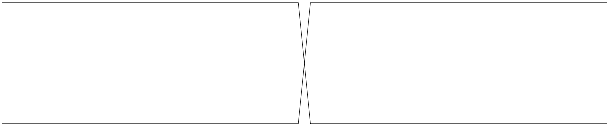
Fermionic corrections: Examples

- Corrections to the leading coupling: $\mathcal{O}(\kappa^{N_\tau+2}u)$



$\longrightarrow$ Deconfinement transition

- New interaction terms: $\mathcal{O}(\kappa^{2N_\tau+2})$



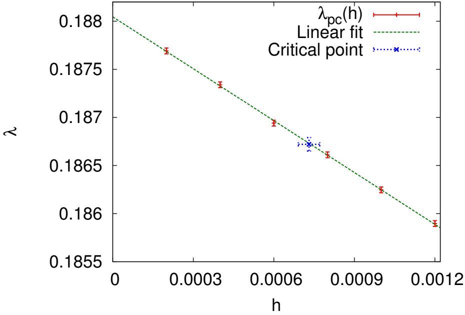
$\longrightarrow$ Cold, dense matter

Deconfinement transition: $\mu = 0$

- Use the leading order effective theory and $\bar{h}_1 = h_1$

$$Z = \int [dW] \prod_i \det \left[1 + h_1\, W_i\right]^2 \left[1 + h_1\, W_i^\dagger\right]^2 \prod_{<ij>} \left[1 + 2\lambda_1 \mathrm{Re}\, L_i L_j^*\right]$$

- With increasing h_1, the transition turns from first order to crossover at a second order endpoint
- Corrections of higher interaction terms negligible



Deconfinement transition: $\mu = 0$

Comparison with other approaches

- Comparison with 4d simulations
- Conversion to quark masses via $\kappa = \frac{1}{2} e^{-aM_q}$

N_f	M_c/T	$\kappa_c(N_\tau = 4)$	$\kappa_c(4)$, Ref. [1]	$\kappa_c(4)$, Ref. [2]
1	7.22(5)	0.0822(11)	0.0783(4)	~ 0.08
2	7.91(5)	0.0691(9)	0.0658(3)	–
3	8.32(5)	0.0625(9)	0.0595(3)	–

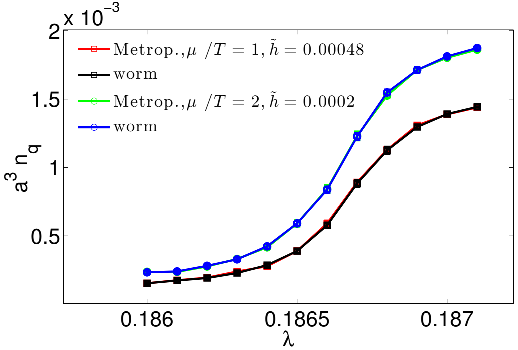
Table : Location of the critical point for $\mu = 0$ and $N_\tau = 4$. The first two columns report our results, the last two compare with existing literature ([1] Saito et al. (2011), [2] Alexandrou et al (1998)).

Deconfinement transition: $\mu \neq 0$

$$Z = \int [dW] \prod_i \left[1 + h_1 L_i\right]^2 \left[1 + \overline{h}_1 L_i^*\right]^2 \prod_{<ij>} \left[1 + 2\lambda_1 \mathrm{Re}\, L_i L_j^*\right]$$

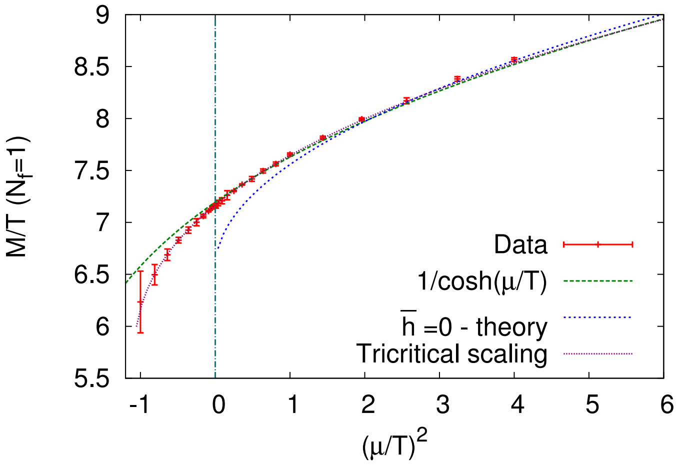
- Metropolis algorithm: Mild sign problem
- Worm algorithm: No sign problem

Comparison of the two algorithms: Quark number density for $\frac{\mu}{T} = 1; 2$



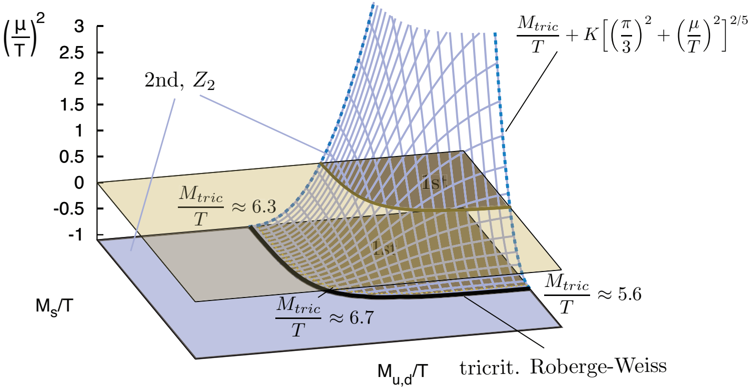
Deconfinement transistion: $\mu \neq 0$

Critical $\frac{M}{T}$ for all chemical potentials



Deconfinement transition: $\mu \neq 0$

3d columbia plot



Cold and dense matter

- $T \simeq 0$ is at finite a realized by large N_τ

$$\lambda_1(\beta = 5.7, N_\tau = 115) \sim 10^{-27}$$

- $\Rightarrow$ Effective gauge part can be neglected
- Not to be confused with strong coupling limit:
  λ_1 is small, not β
- Effective theory then reads:

$$Z = \int [dW] \prod_i \det \left[1 + h_1 W_i\right]^2 \left[1 + \bar{h}_1 W_i^\dagger\right]^2$$

- No interactions, single-site problem: Can be solved analytically
- [Fromm, Langelage, Lottini, Neuman, Philipsen (2012)]

Cold and dense matter

Static limit: $N_f = 1$

- Analytic solution $\left(M = h_1\overline{h}_1,\ B = h_1^3\right)$:

$$
\begin{aligned}
Z_1 &= \left[1 + 4M + 10M^2 + 20M^3 + 10M^4 + 4M^5 + M^6\right] \\
&+ \left[4 + 6M + 6M^2 + 4M^3\right]\left[B + \overline{B}\right] + B^2 + \overline{B}^2
\end{aligned}
$$

- Free gas of mesons and baryons $\Rightarrow$ Hadron Resonance Gas
- Holds also for $N_f > 1$ with much more terms
- Dense system: Neglect negative μ contributions

$$
Z_1 = 1 + 4B + B^2 = 1 + 4h_1^3 + h_1^6
$$

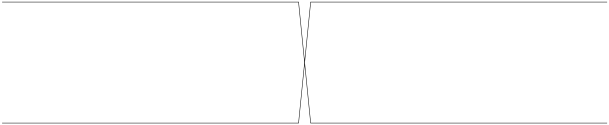
- Baryon density for $T = 0$:

$$
\lim_{T\to 0} a^3 n_B = \begin{cases} 0, & \mu_B < m_B \\ 2N_c, & \mu_B > m_B \end{cases}
$$

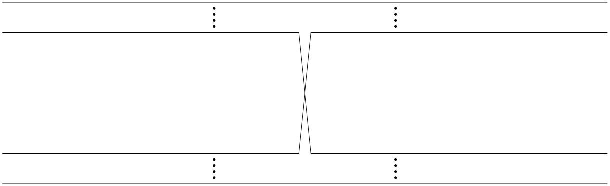
Cold and dense matter

Interactions

- Leading interaction term:



- This graph alone spoils baryon saturation at large densities: Need to resum all winding numbers



Cold and dense matter

Interactions

- After resummation the new interaction term reads:

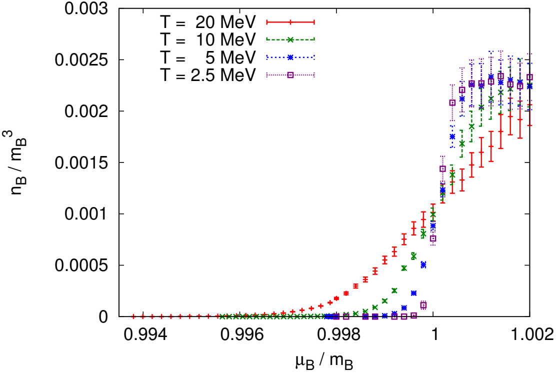
$$\begin{aligned}\Delta_2^a &= \mathrm{Tr}\frac{cW_i}{1+cW_i}\,\mathrm{Tr}\frac{cW_j}{1+cW_j}\\ h_2 &= -\frac{\kappa^2}{N_c}\left[1+2\frac{u-u^{N_\tau}}{1-u}\right]\end{aligned}$$

- Here: $c = (2\kappa e^{\mu})^{N_\tau}$ equals the leading order of h_1
- Term goes to a constant for $\mu \to \infty$, i.e. no additional contribution to $n_B = -\partial_\mu f$
- Drawback: No worm formulation and Metropolis inefficient for large volumes
- $\to$ Use Complex Langevin

Cold and dense matter

Results:

- Transition to nuclear matter:



- Not yet clear, if this happens at $T = 0$ or $T > 0$ (as in nature)
- $\rightarrow$ Include higher order corrections

Conclusions

- Constructed effective theory with much milder sign problem
- In good agreement with full simulations, where comparison is possible (heavy quarks)
- Gauge part seems to be under control
- Fermionic sector needs further developments: Higher order expansions or even nonperturbatively determined couplings
- Main advantage: Dependence of the couplings on chemical potential is trivial, determination at $\mu = 0$ suffices

Outlook

- For fermions κ^4 corrections have been computed and are currently simulated with CLE
- Cold, dense region: Combination $\kappa^2 N_\tau$ seems to be the proper expansion parameter $\rightarrow$ Try to resum all powers $(\kappa^2 N_\tau)^n$
- Measure correlation functions (of Polyakov loops) and compare with full simulations
- Extract effective couplings nonperturbatively
- Apply method to other theories (QC_2D, $Z(2)$, ...)