The complex Langevin method: Successes and Difficulties

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Overview

1. Introduction
2. General discussion
3. Quadratic actions
4. Mathematical and Practical Problems
5. Some examples
6. Extension to manifolds?
7. Outlook
1. Introduction

Complex Langevin first (?) proposed:
Many studies in 1980’s and 1990’s, e.g.
In principle

Complex Langevin solves *sign problem*.

Sign problem arises in

- QCD at finite density
- Quantum Field Theory in Minkowski Space
- Relativistic Bose Gas
- …
Successes and Failures

In some simple cases good convergence to the right limit. Example: $U(1)$ LGT in $2D$ (Ambjørn et al 1986).

Practical Problems:

• Runaways (divergence)

• convergence to wrong limit.

Mathematical questions unresolved:

Quotes: ... *conspicuous absence of general spectral theorems* ... (Klauder&Petersen 1984)

... *a rather experimental character: for some situations the method works, while it fails for other choices of the action* ... (Haymaker&Wosiek 1988)
Resurrection


Continuation: Berges et al 2007, Berges & Sexty 2007

Finite density: Aarts & Stamatescu 2008
Complex relativistic Bose gas: Aarts 2009
– Numerically impressive results
– approach appears again promising
– but problems lingering.

Guralnik & Pehlevan 2008-2009: Effective potential to resolve ambiguities
2. General discussion

‘Flat’ case: defined on $\mathcal{M} = \mathbb{R}^n$, analytically continued to $\mathcal{M}_c \equiv \mathbb{C}^n$.

Complex Langevin:

$$dz = -\nabla S \, dt + dw$$

$dw$ increment of Wiener process on $\mathbb{R}^n$ (formally $dw = \eta(t) \, dt$, $\eta$ white noise).

This is real stochastic process:

$$dx = K_x \, dt + dw$$
$$dy = K_y \, dt,$$  \hspace{1cm} (1)
\[ K_x = -\text{Re}\nabla_x S(x + iy) \]
\[ K_y = -\text{Im}\nabla_x S(x + iy) \]  \hspace{1cm} (2)

\[ \implies \text{Real Fokker-Planck equation} \]

\[ \frac{\partial}{\partial t} P(x, y; t) = L_{FP} P(x, y; t) ; \quad P(x, y; 0) = \delta(x - x_0)\delta(y - y_0) , \]

\( P \) probability density in \( \mathbb{R}^{2n} \),

Real Fokker-Planck operator:

\[ L_{FP} \equiv \nabla_x [\nabla_x - K_x] - \nabla_y K_y \]
Complex Fokker-Planck Equation: Given $y_0$, define

$$\frac{\partial}{\partial t} \rho_{y_0}(x; t) = L^c_{y_0} \rho_{y_0}(x; t),$$

where $\rho_{y_0}(x; t)$ is complex density defined on $\mathbb{R}^n + iy_0$,

$$L^c_{y_0} \equiv \nabla_x [\nabla_x + (\nabla_x S(x + iy_0))].$$

Special case: $S(x)$ real for $x$ real:
Complex FPE $\rightarrow$ standard FPE
Real FPE lives still in $\mathbb{R}^{2n}$, but has stationary solution

$$P(x, y) \propto \exp[-S(x)] \delta(y).$$
**FP Hamiltonian**

$L_y^c$ operator on $\mathcal{H}_2 \equiv L^2(e^{Re S} dx)$.

Unitary map $U : L^2(dx) \rightarrow \mathcal{H}_2$:

$$U \psi = \exp(-\frac{1}{2} S) \psi ,$$

$$H_{FP} \equiv -U^{-1} L_y^c U = - (\nabla - \frac{1}{2}(\nabla S)) \left( \nabla + \frac{1}{2}(\nabla S) \right) ;$$

$S$ real: $H_{FP}$ manifestly positive.

**Fact:** spectrum and numerical range of $-H_{FP}$ and $L_y^c$ agree.
Goal and Questions

Goal: Produce expectation values of holomorphic observables $O$:

$$\langle O \rangle \equiv \frac{\int O(x+iy_0)e^{-S(x+iy_0)}d^n x}{\int e^{-S(x+iy)}d^n x} ;$$

independent of $y_0$ by Cauchy’s theorem.

Hope: obtainable as long time limit of

$$\langle O \rangle_{P,t} \equiv \frac{\int O(x+iy)P(x,y;t)d^n x d^n y}{\int P(x,y;t)d^n x d^n y} ;$$

and by ergodicity as

$$\lim_{t \to \infty} \frac{1}{t} \int O(z(t))dt .$$
**Question:** Relation to ‘\(\rho\)-expectations’

\[
\langle O \rangle_{\rho,t} \equiv \frac{\int O(x+iy_0)\rho(x;t)d^n x}{\int \rho_0(x;t)d^n x} \ ?
\]

**Transpose operator:**

\[
\left(L_{y_0}^c\right)^T \equiv \left[\nabla_x - (\nabla_x S(x + iy_0))\right] \nabla_x ,
\]

\[
L_{FP}^T \equiv \left[\nabla_x - \Re(\nabla_x S(x + iy))\right] \nabla_x - \Im(\nabla_x S(x + iy)) \nabla_y
\]

defined such that

\[
\partial_t \langle O \rangle_{\rho,t,y} = \langle \left(L_{y_0}^c\right)^T O \rangle_{\rho,t} \text{ and } \partial_t \langle O \rangle_{P,t} = \langle L_{FP}^T O \rangle_{P,t} .
\]
**Result**

Assume

- \( P(x, y; 0) = \delta(y - y_0)\rho(x; 0) \)
- for all \( y_0 \) \( L^c_{y_0} \) generates quasibounded holomorphic semigroup (i.e. \( \| e^{tL^c_{y_0}} \| \leq C_1 e^{C_2 t} \))
- \( L_{FP} \) generates quasibounded (strongly continuous) semigroup on \( L^2(\mathbb{R}^n) \) (i.e. \( \| e^{tL_{FP}} \| \leq C_1 e^{C_2 t} \))
- for all \( y_0 \) \( O(x + iy_0) \in L^2(\mathbb{R}^n, d^n x) \).

Then

\[
\langle O \rangle_{\rho,t} = \langle O \rangle_{P,t} \quad \forall y_0, \ t \geq 0
\]
Proof

1. Initial conditions agree.

2. Let $O(x + iy_0; t) \equiv \exp [t(L_{y_0}^c)^T] O(x + iy_0)$, the unique solution of DE

$$\partial_t O(x + iy_0; t) = (L_{y_0}^c)^T O(x + iy_0; t) \quad (t \geq 0);$$

$O(x + iy_0; t)$ still determines holomorphic $O(x + iy; t)$.

3. Consider $F(t, \tau) \equiv \int P(x, y; t - \tau) O(x + iy; \tau)$.

$$F(t, 0) = \langle O \rangle_{P,t}; \quad F(t, t) = \langle O \rangle_{\rho,t}$$

(second equation: use integration by parts)
Claim: $F(t, \tau)$ independent of $\tau$.

Reason:

$$\frac{\partial}{\partial \tau} F(t, \tau) = - \int (L_{FP} P(x, y; t - \tau) O(x + iy; \tau) d^m x d^m y$$
$$+ \int P(x, y; t - \tau) (L_{y_0}^c)^T O(x + iy; \tau) d^m x d^m y \quad (3)$$

Second term: can replace $(L_{y_0}^c)^T$ by $L_{FP}^T$ (Cauchy-Riemann equations).

Integration by parts $\Rightarrow \frac{\partial}{\partial \tau} F(t, \tau) = 0$. 
**Generalization**

Introduce $N_I, N_R > 0$, $N_R = N_I + 1$

Complex Langevin:

$$dz = -\nabla S dt + N_R dw_R + N_I dw_I$$

$w_R, w_I$ independent Wiener processes on $\mathbb{R}^{2n}$

Real FP operator:

$$L_{FP} \equiv N_R \nabla_x [\nabla x - K_x] + N_I \nabla_y [\nabla y - K_y]$$

Complex FP operator unchanged!

(Reason: Cauchy-Riemann equations)
Comments

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• Need: spectrum of $L_{y_0}^c$ in left half plane.

• $\text{spec}(L_{y_0}^c) \subset \text{spec}(L_{FP})$. Pseudospectrum?
**Equilibrium distribution**

Existence not proven!
Assume existence, $N_I > 0$. Stationary real FPE:

$$\left[ N_R \Delta x + N_I \Delta y - \vec{K} \cdot \vec{\nabla} - (\text{div } \vec{K}) \right] P(\vec{x}, \vec{y}) = 0$$

**Facts:**

- $P$ smooth
- $(\vec{x}_*, \vec{y}_*)$ stable fixed point $\implies P(\vec{x}_*, \vec{y}_*) \geq \langle P \rangle_\epsilon$, $\langle \cdot \rangle_\epsilon$ average over a circle of radius $\epsilon$ ($\epsilon$ small enough)
- $P(\vec{x}_*, \vec{y}_*)$ local maximum $\implies \text{div } \vec{K}(\vec{x}_*, \vec{y}_*) < 0$.

Analogous for local minima.
Reasons:

• Elliptic regularity

• Rescale $(\vec{x}, \vec{y}) = S(\vec{\xi}, \vec{\eta})$ to obtain

$$\left[ \Delta_\xi + \Delta_\eta - \vec{L} \cdot \vec{\nabla} - (\text{div } \vec{L}) \right] Q(\vec{\xi}, \vec{\eta}) = 0$$

Fixed point structure unchanged;
near $(\xi_*, \eta_*)$ \( \text{div } \vec{L} < 0 \implies Q\ \text{superharmonic};\)

• Near fixed point \( \vec{K} = A\vec{x} + O(\vec{x}^2) \implies \)

\[
\text{div } \vec{L}(\vec{x}_*, \vec{y}_*) = \text{div } \vec{K}(\vec{\xi}_*, \vec{\eta}_*)
\]
3. Quadratic Actions


Setting:

\[ S = -\frac{1}{2}(x, Ax), \quad x \in \mathbb{R}^n, \]

\[ A = A_r + iA_i \] complex symmetric matrix; \( A_r \) and \( A_i \) real symmetric matrices.

Assumptions:

• \(-A \textbf{ strictly dissipative:} \quad A_r = \frac{1}{2}(A + A^\dagger) > 0.\)

• \(A \textbf{ diagonalizable by a complex orthogonal matrix } O: \)
\[ A = O^T D O \text{ with } D = \text{diag}(\lambda_1, \ldots, \lambda_n). \textbf{ Generic!} \]
**Fact:** \( \text{Re} \lambda_1, \ldots \lambda_n > 0 \) because \(-A\) strictly dissipative.

Converse not true:

\[
A = \begin{pmatrix}
-1 & 2i \\
2i & 3
\end{pmatrix}
\]

has eigenvalues \( \lambda_1 = \lambda_2 = 1 \), but

\[
\frac{1}{2}(A + A^\dagger) = \begin{pmatrix}
-1 & 0 \\
0 & 3
\end{pmatrix}
\]

not positive definite, i.e. \(-A\) not dissipative.
1D example

\[ S = \frac{1}{2}ax^2, \quad a = a_r + ia_i, \quad a_r > 0 \]

\[ L_{FP} = \partial_x^2 + a_r(\partial_x x + \partial_y y) + a_i(-\partial_x y + \partial_y x). \]

\( L_{FP} \) not dissipative:

\[ \frac{1}{2}(L_{FP} + L_{FP}^\dagger) = \partial_x^2 + 2a_r. \]

But stationary solution:

\[ P(x, y; \infty) = c \exp \left[ -a_r x^2 - \frac{2a_r^2}{a_i} xy - \frac{a_r}{a_i^2}(2a_r^2 + a_i^2)y^2 \right]. \]

Integrable for \( a_r > 0 \).
Remark: Level lines of $P(x, y; \infty)$ are tilted ellipses:

$$P(x, y; \infty) = c \exp[-Q(x, y)]$$

with

$$Q(x, y) = \frac{a_r}{2} \left[ x + y(\alpha + \sqrt{1 + \alpha^2}) \right]^2 + \frac{a_r}{2} \frac{1 + \alpha^2 - \sqrt{1 + \alpha^2}}{1 + \alpha^2 + \sqrt{1 + \alpha^2}} \left[ x(\alpha + \sqrt{1 + \alpha^2}) - y \right]^2.$$

where $\alpha = a_r/a_i$. 
**Time-dependent solution**
(Haymaker&Peng 1989):

\[ X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad X_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad Z(t) = X - e^{-a_t} \begin{pmatrix} \cos a_t & \sin a_t \\ -\sin a_t & \cos a_t \end{pmatrix} X_0; \]

\[ P(x, y; t) = \exp \left[ -\frac{1}{2} Z(t)^T \Sigma^{-1}(t) Z(t) \right] \]

with \( \Sigma(t) = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \)
\[
\sigma_{11} = \frac{1}{ar} + \frac{ar}{2(a_r^2 + a_i^2)} + e^{-2ar t} \left[ \frac{-ar \cos(2a_i t) + a_i \sin(2a_i t)}{2(a_r^2 + a_i^2)} - \frac{1}{2ar} \right]
\]

\[
\sigma_{12} = -\frac{ar}{2(a_r^2 + a_i^2)} + e^{-2ar t} \left[ \frac{ar \sin(2a_i t) + a_i \cos(2a_i t)}{2(a_r^2 + a_i^2)} \right]
\]

\[
\sigma_{22} = \frac{1}{ar} - \frac{ar}{2(a_r^2 + a_i^2)} + e^{-2ar t} \left[ \frac{ar \cos(2a_i t) - a_i \sin(2a_i t)}{2(a_r^2 + a_i^2)} - \frac{1}{2ar} \right]
\]
Complex FP equation

\[ L^c_{y_0} = \partial_x^2 + a\partial_x (x + iy_0) ; \]

not dissipative if \( a_i \neq 0 \).

FP Hamiltonian:

\[ H_{FP} = -\partial_x^2 - \frac{1}{2}a + \frac{1}{4}a^2 (x + iy_0)^2 , \]

For \( y_0 = 0 \) and rescaled \( x \mapsto x\sqrt{2} \): standard harmonic oscillator

\[ H_{h.o.} = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} \omega^2 x^2 - \frac{\omega}{2} \]
Mehler formula

\[ \exp(-tH_{h.o.}(x, x_0)) \equiv Q_t(x, x_0), \]

with

\[ Q_t^\omega(x, x_0) = \sqrt{\frac{\omega}{\pi(1-e^{-2\omega t})}} \exp \left[ -\frac{\omega(x^2+x_0^2)}{2 \tanh(\omega t)} - \frac{\omega xx_0}{\sinh(\omega t)} \right]. \]

Using unitary map \( U \):

\[ \exp(tL_0^c)(x, x_0) = e^{-ax^2/4}Q_t^\omega \left( \frac{x}{\sqrt{2}}, \frac{x_0}{\sqrt{2}} \right) e^{ax_0^2/4}. \]

Reintroduce \( y_0 \):

\[ \exp(tL_{y_0}^c)(x, x_0) = \exp(tL_0^c)(x + iy_0, x_0 + iy_0). \]
Higher dimensions

\[ L_{FP} = \Delta x + \nabla_x \cdot A_r x + \nabla_y \cdot A_r y - \nabla_x \cdot A_i y + \nabla_y \cdot A_i x , \]

\[ L_{FP}^{\dagger} = \Delta x - (A_r x) \cdot \nabla_x - (A_r y) \cdot \nabla_y + \nabla_x \cdot A_i y - \nabla_y \cdot A_i x . \]

\[ \frac{1}{2} (L_{FP} + L_{FP}^{\dagger}) = \Delta x + 2 \text{tr } A , \]

so \( L_{FP} \) is again not dissipative.
Solution by Mehler kernel

First $A_i = 0$: $\exists O$ (orthogonal)

$$A = O^T D$$

with $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$.

Put $Ox = x'$, $Ox_0 = x'_0$:

$$\exp(-tH_{FP})(x, x_0) = \prod_{i=1}^{n} Q_{t}^{\lambda_{i}} \left( \frac{(Ox)_i}{\sqrt{2}}, \frac{(Ox_0)_i}{\sqrt{2}} \right).$$

$$e^{L_{y_0}t}(x, x_0) = \exp\left(-\frac{S(x+iy_0)}{2}\right) \prod_{i=1}^{n} Q_{t}^{\lambda_{i}} \left( \frac{(Ox)_i}{\sqrt{2}}, \frac{(Ox_0)_i}{\sqrt{2}} \right) \exp\left[\frac{S(x_0+iy_0)}{2}\right].$$
Remarks:

• By analytic continuation this remains valid for complex $A$. 
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• Relaxation to equilibrium if $\text{Re} \lambda_i > 0$, $i = 1, \ldots, n$. 
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• Relaxation to equilibrium if $\text{Re} \lambda_i > 0$, $i = 1, \ldots, n$.

• Moral reason: all classical trajectories attracted to origin.
4. Problems

Mathematical and practical difficulties:

• *Existence* of the semigroup generated by $L_{FP}$. Not known: $L_{FP}$ never manifestly dissipative. Hope: with new scalar product $L_{FP}$ dissipative.
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Mathematical and practical difficulties:

• **Existence** of the semigroup generated by $L_{FP}$. Not known: $L_{FP}$ never manifestly dissipative. **Hope:** with new scalar product $L_{FP}$ dissipative.

• **Runaways:** In typical cases deterministic motion can go to $\infty$ in finite time. **Reason:** Drift $\nabla S$ grows in some directions. **1D:**

\[
\dot{z} = -S' \implies t - t_0 = -\int \frac{dz}{S'}
\]

(integration on curve with $dz$ real multiple of $S'$).
• Pseudospectrum (see below)
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• Convergence to wrong limit
  Noticed by Klauder & Petersen 1985, Ambjørn et al 1986:
  “Quantum mechanical desasters of the first degree”:

  \[ S = -\beta \cos \theta - i\theta \]

  works for large \( \beta \), fails for small \( \beta \).

  “Non-abelian desasters of the third degree”:

  \[ S = -\beta \text{tr } U - \log \text{tr } U, \quad U \in SU(2), SU(3), \]

  works for large \( \beta \), fails for small \( \beta \).
– Haymaker&Wosiek 1987:  

\[ S = -\beta \cos \theta - \log \cos \theta \]

Simulates restricted range \([-\pi/2, \pi/2]\).  
Reason: zero of \(\cos \theta\).

  (1) 1D, \(S\) polynomial, \(e^{-S} \in S\)  
  (2) \(\int_{\mathbb{R}} e^{-S(x)} \, dx \neq 0\)  
  (3) \(\forall k \in \mathbb{R} \quad \lim_{t \to \infty} \langle e^{ikz} \rangle_{P,t} \) exists and is \(\in S(\mathbb{R})\).  

Not really practical.
5. Examples

Example 1 (Aarts & Stamatescu 2008)

\[ S = -\beta \cos x - \kappa \cos(x - i\mu) \]

Complex Langevin equation

\[ dx = K_x dt + dw, \quad dy = K_y dt \]

with

\[ K_x = -\sin x \left[ \beta \cosh y + \kappa \cosh(y - \mu) \right] \]
\[ K_y = -\cos x \left[ \beta \sinh y + \kappa \sinh(y - \mu) \right] \] (2)
From (Aarts & Stamatescu 2008): Drift pattern
Real FP operator:

\[ L_{FP} = \partial_x [\partial_x - K_x] - \partial_y K_y \]

Complex FP operator:

\[ L^c_{y_0} = \partial_x [\partial_x + \beta \sin(x + iy_0) + \kappa \sin(x + iy_0 - i\mu)] \]

Drift \( K_x, K_y \) parallel to gradient of

\[ G(x, y) = \exp \left[ -\frac{\cos x}{\beta \cosh y + \kappa \cosh(y - \mu)} \right]. \]
$G$ is candidate Lyapunov function:

$$\frac{d}{dt} G(x(t), y(t)) = (K_x \partial_x + K_y \partial_y) G(x, y) =$$

$$- \left[ \sin^2 x + \cos^2 x \left( \frac{\beta \sinh y + \kappa \sinh(y-\mu)}{\beta \cosh y + \kappa \cosh(y-\mu)} \right)^2 \right] G \leq 0 ,$$

Vanishes only on fixed points \((0, y_*)\), \((\pi, y_*)\);

\(\Rightarrow\) all points with \(x \neq \pi\) attracted to \((O, y_*)\).
$G$ also candidate stochastic Lyapunov function:

$$L^T_{FP} G < 0$$

for $|y|$ large enough.

Need (Khasminskii 1980):

$$L^T_{FP} G \to -\infty \text{ for } |y| \to \infty.$$  

Open problem.

**Practically** large excursions cause problems even if stationary $P(x, y)$ exists.
Simulation

\[ \beta = 100. \]

\[ \kappa = 0.0 \]

\[ N_I = 1.0 \]
Problem

Convergence to wrong limit for $N_I > 0$: What is going on?

$\beta = 1.$

$\kappa = 0.5$

$\mu = 1.0$
Possible reason: Non-Gaussian large fluctuations?

\[ \beta = 1. \]
\[ \kappa = 0.5 \]
\[ \mu = 1.0 \]
\[ N_I = 1.0 \]
Example 2 (Guralnik&Pehlevan 2009)

\[ S = -\beta (iz + \frac{i}{3}z^3) \]

Attractive fixed point: \( z = i \)
Repulsive fixed point: \( z = -i \)
Classical orbits: **Circles**

\[ z(t) = \frac{z_0 + it\tanh t}{1 - iz_0 \tanh t} \]

Möbius transformation \( w \equiv \tanh t \mapsto z(t) \),
\( z(0) = z_o, \quad z(\infty) = i \)
Simulation

\[ N_I = 1.0, \beta = 1.0 \]
**Exact results**

\[ \beta = 1.0 : \quad Z(j) = \int dx \exp[ix + \frac{i}{3}x^3 + jx] \]

Schwinger-Dyson eq.:

\[ -iZ'' - iZ = jZ \]

leads to

\[ \langle z \rangle = -i \frac{Ai'(1)}{Ai(1)} \approx 1.17632i \]

\[ \langle z^2 \rangle = -1.0, \quad \langle z^3 \rangle = i - \langle z \rangle \approx -0.17632i \]
Problem

Again convergence to wrong limit for $N_I > 0$.

Reason: Non-Gaussian large fluctuations?

$\beta = 1.0, \quad N_I = 1.0$
Problem

But convergence to right limit for $N_I \ll 1$.

In spite of: Small non-Gaussian fluctuations

\[ \beta = 1.0, \quad N_I = 0.0 \]
Pseudospectrum

**Typically:**
Spectrum of $L_{FP}^{c}$ and $-H_{FP}$ in left half plane, but not dissipative: $\text{Re} (\psi, H_{FP} \psi) < 0$ for some $\psi$

**Price** to pay: Pseudospectrum

**Definition:**  $\text{spec}_\epsilon(A) \equiv \{ z \in \mathbb{C} | \| (A - z)^{-1} \| > \epsilon^{-1} \}$

Signifies instability:

$$\text{spec}_\epsilon(A) = \bigcup_B \{ \text{spec}(A + B), \| B \| < \epsilon \}$$

Tiny perturbation can eliminate “pseudo”
Example 3 (Davies & Kuijlaars, 2004): Spectral projections $P_n$ of complex harmonic oscillator grow:

$$\|P_n\| \geq a C^{2n+1}, \quad C > 1;$$

poor convergence of eigenfunction expansions:

$$e^{-Ht}\psi = \sum_n e^{-\omega(n+1/2)t} P_n \psi$$

- Eigenfunctions do not form Riesz basis
- $e^{-Ht}$ not bounded semigroup
- $\exists$ pseudospectrum far from spectrum!

(Davies 1999)
Riesz basis $(\phi_n)_{n=1}^\infty$: 

$\exists$ bounded operator $S$ with $S^{-1}$ bounded such that 

$$S\phi_n = e_n \quad n = 1, \ldots, \infty,$$

where $(e_n)_{n=1}^\infty$ orthonormal basis.]
6. Extension to manifolds

Gausterer&Thaler 1998, Aarts&Stamatescu 2008:
Compact connected Lie groups.

Examples:
- $U(1)$ complexified to $U(1) \times \mathbb{R}$
- $SU(N)$ complexified to $SL(N, \mathbb{C})$

More generally:
- $\mathcal{M}$ Riemannian manifold $\Rightarrow \exists$ Wiener process $\Rightarrow$
  noise in real directions well defined
- Real manifold $\mathcal{M}$ has to have complexification $\mathcal{M}_C$.

Formal arguments carry over; problems remain.
7. Outlook

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- Validation necessary: check with analytic or otherwise known result.
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• Method shows some promise
• Practical usefulness has to be checked
• Validation necessary: check with analytic or otherwise known result.
• Hope for the best, be prepared for the worst