J. Phys. G: Nucl. Part. Phys. **26** (2000) 377–385. Printed in the UK PII: S0954-3899(00)10180-X

Unitary flow of the bosonized large-N Lipkin model

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Received 7 December 1999

Abstract. The flow equations describing the continuous unitary transformation which brings the Hamiltonian closer to diagonality are derived and solved exactly for the Lipkin model in the Holstein–Primakoff boson representation and for a large particle number N. The transformed Hamiltonian is diagonal in order $1/N^3$, extending known linear transformations to next-higher orders in the inverse particle number. This approach quite naturally allows one to preserve the tridiagonal structure of the original Lipkin Hamiltonian in the course of the transformation. Exact analytical results for the coupling functions and explicit expressions for the ground state energy and for the energy gap to the first excited state in order $1/N^2$ are presented and are compared with the accurate numerical values.

1. Introduction

The two-level shell model introduced by Lipkin *et al* [1–3] in nuclear physics is a standard test-Hamiltonian used to assess the validity of approximative methods [4–7], since it has a finite Hamiltonian matrix. This model can be expressed by bilinear products of fermionic creation and annihilation operators which correspond to a quasi-spin representation. It is thus also accessible by common bosonization techniques [8] suitable for spins. The approximate diagonalization of such a bosonized model may then be performed by a Bogoliubov transformation, which eliminates the bilinear nondiagonal terms. Higher-order nondiagonal terms corresponding to interactions of the bosons which do not conserve the number of particles cannot be removed by a standard Bogoliubov rotation and conventionally must be treated perturbatively [9].

The method of continuous unitary transformations [10, 11] allows one to circumvent this difficulty inherent to the single-step Bogoliubov transformation. Instead of applying the diagonalizing transformation in one step, it may also be performed in infinitesimally small steps, thus bringing the Hamiltonian continuously closer to diagonality. In this way, it is not necessary to know explicitly the total transformation from the beginning, one only needs to specify its generator.

In this paper, a continuous unitary transformation is applied to the bosonized Lipkin model with a large number of particles. This corresponds to a large quasi-spin, and allows an expansion around the semiclassical limit of the resulting spin Hamiltonian. In contrast to previous work using continuous transformations [12], in this way the bosonized version of the model offers a systematic classification of the contributions to the effective Hamiltonian. Utilizing the Holstein–Primakoff representation of spin operators, the interactive part of the bosonic Hamiltonian can also be diagonalized, i.e. also terms which are of quartic and higher

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order in the bosons. This extends known transformations to the next-higher orders of an expansion in the inverse number of particles. The central aim of the paper is to show how the flow equation method works for a standard test model, allowing the explicit construction of a diagonalizing unitary transformation *beyond* the harmonic approximation, which is usually the limit for more complicated models and especially for conventional transformation methods.

2. The flow equations

The Lipkin model describes a nuclear system consisting of two n-fold degenerate single-particle shells separated by an energy J'. Each fermion state is specified by two quantum numbers (σ, m), where $\sigma = \pm 1$ designates the lower and upper levels and $m = 1 \ldots n$ enumerates the states within each shell. A two-particle interaction V introduces scattering of pairs between the two shells. The fermionic Hamiltonian thus reads

$$\mathcal{H} = -\frac{J'}{2} \sum_{m,\sigma} \sigma c^\dagger_{m,\sigma} c_{m,\sigma} + \frac{V}{2} \sum_{m,m',\sigma} c^\dagger_{m,\sigma} c^\dagger_{m',\sigma} c_{m',-\sigma} c_{m,-\sigma}. \tag{1}$$

Utilizing the quasi-spin representation

$$\hat{S}_z = \tfrac{1}{2} \sum_{m,\sigma} \sigma c^\dagger_{m,\sigma} c_{m,\sigma}, \qquad \hat{S}_+ = \sum_m c^\dagger_{m,+1} c_{m,-1}, \qquad \hat{S}_- = \sum_m c^\dagger_{m,-1} c_{m,+1}, \tag{2}$$

one obtains the spin-S Hamiltonian

$$\mathcal{H} = -JS\hat{S}_z + \frac{V}{2}(\hat{S}^2_+ + \hat{S}^2_-), \tag{3}$$

where the total spin S is related to the number of particles in both shells, $N = 2S$. In (3) the shell separation energy $J' = JS$ has been rescaled by a factor S to make the $1/S$-expansion of the Hamiltonian well defined. The classical ground state of this quantum model then depends on the parameter $\epsilon = 2V/J$. While for $\epsilon < 1$ the classical ground state is unique with $\langle \hat{S}_z \rangle = S$, for $\epsilon > 1$ there are two degenerate classical vacua with $\langle \hat{S}_z \rangle = S/\epsilon$ and $\langle \hat{S}_y \rangle = \pm S(1 - \epsilon^{-2})^{1/2}$, involving the possibility of tunnelling in the quantum Hamiltonian. In the following we therefore restrict ourselves to the case $\epsilon < 1$.

To study the quantum spin model (3) near the classical limit where $S \gg 1$, we utilize the Holstein–Primakoff bosonic representation [13] of spin operators,

$$\hat{S}_z = S - a^\dagger a, \qquad \hat{S}_+ = (2S - a^\dagger a)^{1/2} a, \qquad \hat{S}_- = a^\dagger (2S - a^\dagger a)^{1/2}, \tag{4}$$

where the $a, a^\dagger$ are Bose operators. Inserting these expressions in (3) and expanding the square roots results in an expansion of the Hamiltonian in powers of the inverse spin,

$$\mathcal{H}(\ell) = -\Omega E_0(\ell) + \Omega \sum_{k=1}^{\infty} \frac{:H_k(\ell):}{S^k}, \tag{5}$$

where $\Omega = JS^2$ is the classical ground state energy scale. Due to the continuous transformation which is applied to the Hamiltonian, $\mathcal{H}(\ell) = U(\ell)\mathcal{H}(0)U^\dagger(\ell)$, the various couplings appearing in the H_k depend on the flow parameter $\ell \geqslant 0$. The interaction contributions $:H_k(\ell):$ are written in normal-ordered form, where the normal ordering $:\ldots:$ is initially performed with respect to the classical vacuum $|0\rangle$, with $a|0\rangle = 0$. Since in the course of the transformation the classical vacuum evolves to the approximate quantum ground state, generally normal ordering is with respect to the actual transformed ground state. We use the convention that the interaction terms of the Hamiltonian (5) are assigned to the lowest-order H_k in which they appear. As a consequence, the couplings of the interactions will in general also have contributions which

are of higher order in $1/S$ and which originate from the transformation and normal ordering of higher-order interaction terms.

The transformation flow of the Hamiltonian is described by the differential equation [10]

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\mathcal{H}(\ell) = [\eta(\ell), \mathcal{H}(\ell)], \tag{6}$$

where the anti-Hermitian flow generator $\eta(\ell)$ is also expanded in powers of $1/S$,

$$\eta(\ell) = \frac{\mathrm{d}U(\ell)}{\mathrm{d}\ell}U^{\dagger}(\ell) = \sum_{k=0}^{\infty}\frac{\eta_k(\ell)}{S^k}. \tag{7}$$

Instead of applying the canonical choice of the generator [10] which apart from degeneracies ensures the final diagonalization of the Hamiltonian, the $\eta(\ell)$ employed here is more suitable for the tridiagonal structure of the Lipkin Hamiltonian. Generally, at kth order of the $1/S$-expansion, the Hamiltonian can be written in the form

$$: H_k(\ell) := H_k^{(+)}(\ell) + H_k^{(-)}(\ell) + H_k^{(0)}(\ell), \tag{8}$$

where the different contributions raise or lower the number of the Holstein–Primakoff bosons or leave it unchanged, respectively. We then take

$$\eta_k(\ell) = \tfrac{1}{2}(H_{k+1}^{(+)}(\ell) - H_{k+1}^{(-)}(\ell)). \tag{9}$$

This choice of the generator preserves tridiagonality of the original Lipkin model and avoids populating initially empty off-diagonals of the Hamiltonian in the course of the transformation [14, 15].

2.1. The one-particle excitations

To recover the results of common linearization schemes, only terms up to order $\mathcal{O}(1/S)$ are retained in the $1/S$-expansion of the Hamiltonian (5),

$$H_1(\ell) = f(\ell)a^{\dagger}a + \frac{\epsilon}{2}g(\ell)(a^{\dagger}a^{\dagger} + aa), \tag{10}$$

where one initially has $f(0) = g(0) = 1$. This leading correction beyond the classical contribution is bilinear in the boson operators and may be diagonalized directly in a single step by a conventional Bogoliubov rotation. However, since we are finally aiming at a diagonalization at order $\mathcal{O}(1/S^3)$ which requires some results of the continuous transformation in the leading order, we will present here a continuous version of the Bogoliubov transformation. In order $\mathcal{O}(1/S)$ the 'single-step' and the continuous versions yield the same final results for the transformed bosonic Hamiltonian.

In the framework of the continuous transformation the terms in (10) which do not conserve the number of spin excitations are eliminated by choosing

$$\eta_0(\ell) = \frac{\epsilon}{4}g(\ell)(a^{\dagger}a^{\dagger} - aa), \tag{11}$$

in accordance with (9) and resulting in the leading-order flow equation

$$\frac{\mathrm{d}}{\mathrm{d}\ell}H_1(\ell) = [\eta_0(\ell), H_1(\ell)]. \tag{12}$$

The differential equations which describe the transformation of the coupling functions of (10) are then given by

$$\frac{\mathrm{d}}{\mathrm{d}\ell}f(\ell) = -\epsilon^2 g^2(\ell), \qquad \frac{\mathrm{d}}{\mathrm{d}\ell}g(\ell) = -f(\ell)g(\ell). \tag{13}$$

The quantum corrections to the ground state energy give rise to the flow of $E_0(\ell)$,

$$\frac{\mathrm{d}}{\mathrm{d}\ell}E_0(\ell) = \frac{1}{2S}\epsilon^2 g^2(\ell). \tag{14}$$

Introducing the bare excitation energy $\omega = (1-\epsilon^2)^{1/2}$, one identifies $f^2(\ell) - \epsilon^2 g^2(\ell) = \omega^2$ as a hyperbolic invariant of the transformation flow. The solutions of (13) are thus readily obtained:

$$f(\ell) = \omega \coth(\omega\ell + \ell_0), \qquad g(\ell) = \frac{\omega}{\epsilon \sinh(\omega\ell + \ell_0)}, \tag{15}$$

with the initial values correctly reproduced by

$$\ell_0 = \frac{1}{2}\ln\frac{1+\omega}{1-\omega}. \tag{16}$$

Since $\ell_0 \geqslant 0$, the functions $f(\ell)$ and $g(\ell)$ are continuous and monotonically decreasing for $\ell \geqslant 0$. Asymptotically, since $\omega > 0$, for $\ell \to \infty$ they converge exponentially to the values $f(\infty) = \omega$ and $g(\infty) = 0$. Even in the degenerate case $\omega = 0$ which is excluded here, one would find an algebraic decay $f(\ell) = g(\ell) = 1/(\ell+1)$.

2.2. The two-particle interactions

Expanding the Hamiltonian (5) to next higher order in $1/S$ yields additional two-particle interactions of the bosonic excitations,

$$H_2(\ell) = \epsilon h(\ell) a^\dagger a^\dagger a a - \frac{\epsilon}{4}\zeta(\ell)(a^\dagger a^\dagger a^\dagger a + a^\dagger a a a). \tag{17}$$

The terms with coupling $h(\ell)$ describe a two-particle interaction of the Holstein–Primakoff bosons, while $\zeta(\ell)$ is related to the creation or annihilation of bosonic excitations. Only the contributions given by the coupling $\zeta(\ell)$ are initially present in the bosonized Lipkin Hamiltonian, and the coupling functions of the interaction terms (17) obey the initial conditions $h(0) = 0$ and $\zeta(0) = 1$.

At this point, the usual bosonization techniques do not involve any further transformation of the interactive part of the Hamiltonian which is beyond the linear Bogoliubov transformation considered in the preceding section. As a consequence, in $H_2(\ell)$ nondiagonal terms are not removed. Correspondingly, the flow of $H_2(\ell)$, as completely described by the generator $\eta_0(\ell)$ only, does not eventually lead to a particle-conserving expression. However, the interaction terms which connect subspaces with a different number of spin excitations can be transformed away by an extension of the continuous generator. Choosing

$$\eta_1(\ell) = -\frac{\epsilon}{8}\zeta(\ell)(a^\dagger a^\dagger a^\dagger a - a^\dagger a a a), \tag{18}$$

at second order in $1/S$ the flow equation for the Hamiltonian reads

$$\frac{\mathrm{d}}{\mathrm{d}\ell}H_2(\ell) = [\eta_0(\ell), H_2(\ell)] + [\eta_1(\ell), H_1(\ell)]. \tag{19}$$

The transformation (19) does not generate new types of nondiagonal interaction terms in the Hamiltonian, in contrast to the usual linear Bogoliubov transformation which at order $\mathcal{O}(1/S^2)$ leads to new interactions of the form $(aaaa + a^\dagger a^\dagger a^\dagger a^\dagger)$. This is due to a cancellation of the corresponding contributions in (19), which does not occur without the term related to $\eta_1(\ell)$. This cancellation is a peculiar feature of the specific choice for the generator; additionally in related work [14, 15] it was observed that an analogous type of transformation is able to avoid the population of initially empty off-diagonals.

The flow equation (19) then yields

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\begin{pmatrix}\zeta(\ell)\\ h(\ell)\end{pmatrix}=\begin{pmatrix}-f(\ell) & 4\epsilon g(\ell)\\ \frac{3}{2}\epsilon g(\ell) & 0\end{pmatrix}\begin{pmatrix}\zeta(\ell)\\ h(\ell)\end{pmatrix}. \tag{20}$$

The solutions for this system of coupled linear differential equations are given by

$$\zeta(\ell)=g(\ell)\left(\left(1+\frac{3\epsilon^2}{\omega^2}(1+\ell)\right)\left(1+\frac{3\epsilon^2}{2\omega^2}g^2(\ell)\right)-\frac{9\epsilon^2}{2\omega^4}f(\ell)\right) \tag{21}$$

and

$$h(\ell)=\frac{3\epsilon}{4\omega^2}\left(1+\frac{3\epsilon^2}{\omega^2}g^2(\ell)-f(\ell)g^2(\ell)\left(1+\frac{3\epsilon^2}{\omega^2}(1+\ell)\right)\right), \tag{22}$$

where the final elimination of the terms related to $\zeta(\ell)\sim g(\ell)$ for large ℓ is apparent.

2.3. The three-particle interactions

Although the initial Hamiltonian (5) does not contain contributions of order $\mathcal{O}(1/S^3)$, such terms are generated by the applied transformation. However, these terms preserve the tridiagonality of the original Lipkin model. One thus has

$$H_3(\ell)=-\frac{\epsilon^2}{2}k(\ell)a^\dagger a^\dagger a^\dagger aaa+\frac{\epsilon^2}{2}\xi(\ell)(a^\dagger a^\dagger a^\dagger a^\dagger aa+a^\dagger a^\dagger aaaa), \tag{23}$$

with initial values $k(0)=\xi(0)=0$. To remove the newly generated nondiagonal terms, we extend the generator by

$$\eta_2(\ell)=\frac{\epsilon^2}{4}\xi(\ell)(a^\dagger a^\dagger a^\dagger a^\dagger aa-a^\dagger a^\dagger aaaa). \tag{24}$$

This third-order transformation results in the flow equation for the Hamiltonian:

$$\frac{\mathrm{d}}{\mathrm{d}\ell}H_3(\ell)=[\eta_0(\ell),H_3(\ell)]+[\eta_1(\ell),H_2(\ell)]+[\eta_2(\ell),H_1(\ell)]. \tag{25}$$

The coupling functions $k(\ell)$ and $\xi(\ell)$ are then determined by the inhomogeneous differential equations

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\begin{pmatrix}\xi(\ell)\\ k(\ell)\end{pmatrix}=\begin{pmatrix}-f(\ell) & \frac{3}{2}\epsilon g(\ell)\\ 8\epsilon g(\ell) & 0\end{pmatrix}\begin{pmatrix}\xi(\ell)\\ k(\ell)\end{pmatrix}+\begin{pmatrix}\zeta(\ell)h(\ell)\\ \zeta^2(\ell)\end{pmatrix}. \tag{26}$$

Without quoting the lengthy solutions for this system of coupled equations, we note that in the large-ℓ limit the leading term $\frac{\mathrm{d}}{\mathrm{d}\ell}\xi(\ell)\sim -f(\ell)\xi(\ell)$ ensures the exponential decay of the nondiagonal couplings.

3. The low-energy spectrum

To extract the physical properties of the quantum ground state and the one-particle excitation spectrum from the flow equations presented in the preceding section, one first notes that expanding (13) in powers of $1/S$ results in the following equations for the $\mathcal{O}(1/S)$ contributions $f^{(1)}(\ell)$ and $g^{(1)}(\ell)$:

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\begin{pmatrix}f^{(1)}(\ell)\\ g^{(1)}(\ell)\end{pmatrix}=-\begin{pmatrix}0 & 2\epsilon^2 g(\ell)\\ g(\ell) & f(\ell)\end{pmatrix}\begin{pmatrix}f^{(1)}(\ell)\\ g^{(1)}(\ell)\end{pmatrix}+\begin{pmatrix}b_1(\ell)\\ b_2(\ell)\end{pmatrix}, \tag{27}$$

with initial conditions $f^{(1)}(0)=0$ and $g^{(1)}(0)=-\frac{1}{4}$. The inhomogeneous contribution to this system of differential equations is given by

$$b_1(\ell)=\tfrac{3}{2}\epsilon^2 g(\ell)\zeta(\ell),\qquad b_2(\ell)=-\epsilon g(\ell)h(\ell). \tag{28}$$

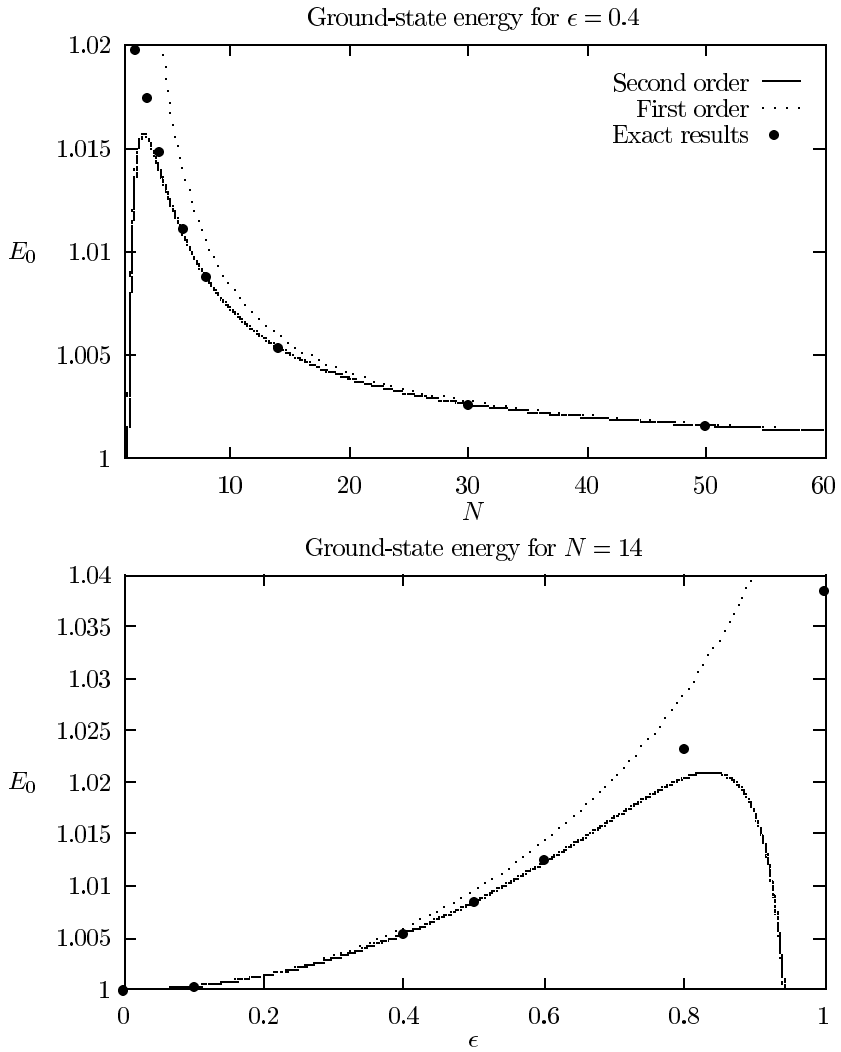

Figure 1. The ground state energy E_0 (cf (42)) for $\epsilon = 0.4$ and for $N = 14$.

The integral basis for the homogeneous part of (27) consists of the two independent solutions

$$\begin{pmatrix} \epsilon^2 g^2(\ell) \\ f(\ell)g(\ell) \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} f(\ell) - \ell\epsilon^2 g^2(\ell) \\ g(\ell) - \ell f(\ell)g(\ell) \end{pmatrix}. \tag{29}$$

Utilizing these expressions (29) and including the inhomogeneities of (27), one derives the explicit solutions for the functions $f^{(1)}(\ell)$ and $g^{(1)}(\ell)$ which are lengthy and are not reproduced here. However, one may directly consider the asymptotic values

$$g^{(1)}(\infty) = 0 \qquad \text{and} \qquad f^{(1)}(\infty) = \frac{\epsilon^2}{4\omega} + \frac{1}{\omega}\int_0^\infty \mathrm{d}\ell(f(\ell)b_1(\ell) - \epsilon^2 g(\ell)b_2(\ell)). \tag{30}$$

Inserting here (28) and using (20), yields the result

$$f^{(1)}(\infty) = \frac{(1-\omega^2)(3-\omega)}{2\omega^2}, \tag{31}$$

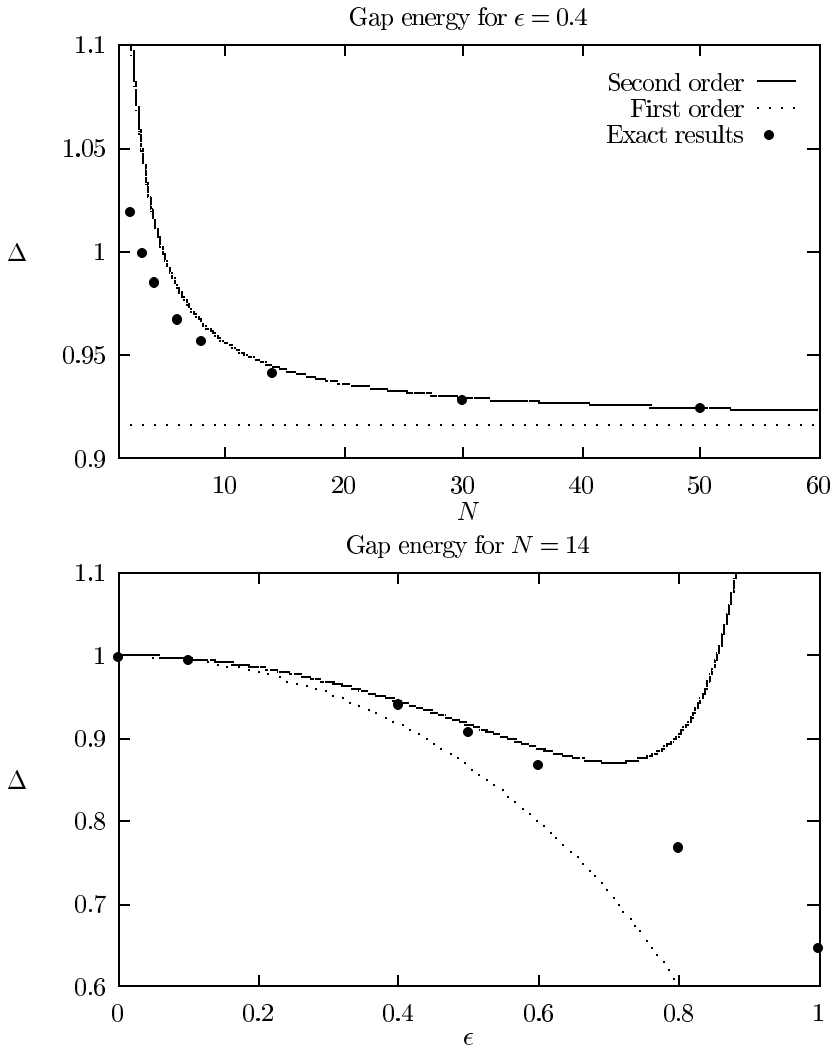

Figure 2. The gap energy Δ (cf (43)) for $\epsilon = 0.4$ and for $N = 14$.

which directly enters into the calculations for the ground state energy and the energy gap to the first excited state.

Extending the expansion to $\mathcal{O}(1/S^2)$, one finds for the corresponding second-order corrections

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\begin{pmatrix} f^{(2)}(\ell) \\ g^{(2)}(\ell) \end{pmatrix} = -\begin{pmatrix} 0 & 2\epsilon^2 g(\ell) \\ g(\ell) & f(\ell) \end{pmatrix}\begin{pmatrix} f^{(2)}(\ell) \\ g^{(2)}(\ell) \end{pmatrix} + \begin{pmatrix} b_3(\ell) \\ b_4(\ell) \end{pmatrix}, \tag{32}$$

with initial conditions $f^{(2)}(0) = 0$ and $g^{(2)}(0) = -\frac{1}{32}$, and the inhomogenities

$$b_3(\ell) = -\tfrac{3}{8}\epsilon^2\zeta^2(\ell) - \epsilon^2[g^{(1)}(\ell)]^2, \qquad b_4(\ell) = -f^{(1)}(\ell)g^{(1)}(\ell). \tag{33}$$

One again finds $g^{(2)}(\infty) = 0$, in accordance with the diagonalizing property of the continuous transformation.

3.1. Ground state energy

Classifying also the contributions to the ground state energy according to their power in the inverse spin,

$$E_0(\ell) = \sum_{k=0}^{\infty} \frac{E_0^{(k)}(\ell)}{S^k}, \tag{34}$$

from the result (14) of linear spin-wave theory one finds the flow equation for the leading-order quantum correction,

$$\frac{\mathrm{d}}{\mathrm{d}\ell} E_0^{(1)}(\ell) = \frac{\epsilon^2}{2} g^2(\ell), \tag{35}$$

with the solution

$$E_0^{(1)}(\ell) = \tfrac{1}{2}(1 - f(\ell)). \tag{36}$$

Also the second-order correction of E_0 is entirely encapsulated in (14), since there are no further contributions to the flow equation for the ground state energy arising from the transformation and subsequent normal ordering of the interaction terms. Including $g^{(1)}(\ell)$ in (35), one thus obtains

$$\frac{\mathrm{d}}{\mathrm{d}\ell} E_0^{(2)}(\ell) = \epsilon^2 g(\ell) g^{(1)}(\ell). \tag{37}$$

From (20) and (27) it then follows that

$$E_0^{(2)}(\ell) = -\frac{1}{2} f^{(1)}(\ell) + \frac{\epsilon}{2} h(\ell) \tag{38}$$

and

$$E_0^{(2)}(\infty) = \frac{(1-\omega^2)(2\omega - 3)}{8\omega^2}, \tag{39}$$

where we have made use of (22) and (31). Yet another order higher in $1/S$, one obtains the flow of the third-order contribution to the ground state energy:

$$\frac{\mathrm{d}}{\mathrm{d}\ell} E_0^{(3)}(\ell) = \epsilon^2 g(\ell) g^{(2)}(\ell) + \tfrac{1}{2}\epsilon^2 [g^{(1)}(\ell)]^2, \tag{40}$$

with the implicit solution

$$E_0^{(3)}(\ell) = -\frac{1}{2} f^{(2)}(\ell) - \frac{3\epsilon^2}{16} \int_0^{\ell} \mathrm{d}\ell' \zeta^2(\ell'), \tag{41}$$

which follows from (32). However, at this point we restrict ourselves to consider the $\mathcal{O}(1/S^2)$ contributions to the energy, for which we have found explicit analytical solutions. Therefore, we find our final second-order result for the ground state energy:

$$E_0(\infty) = 1 + \frac{1}{N}(1-\omega) - \frac{1}{N^2} \frac{(1-\omega^2)(3-2\omega)}{2\omega^2}, \tag{42}$$

which is shown in figure 1 in comparison with the accurate numerical results, yielding quite good agreement for ϵ not too large ($\epsilon \lesssim 0.7$) and also for rather small particle number ($N \gtrsim 4$). The singularity $E_0 \sim 1/\omega^2$ indicates the breakdown of the bosonic expansion as $\epsilon \to 1$, where the ground state assumed in representation (4) is no longer appropriate.

3.2. Energy gap to first excited state

From the diagonalized one-particle Hamiltonian (10) one can directly read off the excitation spectrum, so that with (31) the $\mathcal{O}(1/S)$ energy gap Δ between the ground state and the first excited state is given by

$$\Delta = f(\infty) + \frac{1}{S} f^{(1)}(\infty) = \omega + \frac{1}{N} \frac{(1-\omega^2)(3-\omega)}{\omega^2}. \tag{43}$$

This result is displayed in figure 2, where a surprisingly satisfactory agreement with the exact values is observed for $N \gtrsim 10$ and $\epsilon \lesssim 0.6$. Again, this expression exhibits singular behaviour $\Delta \sim 1/\omega^2$ for $\epsilon \to 1$.

4. Summary

In this work a new continuous unitary transformation has been applied to the bosonized Lipkin model, utilizing the Holstein–Primakoff spin representation near the limit of a large number of particles. With this continuous transformation, the Hamiltonian can be explicitly diagonalized in order $1/N^3$, which incorporates interaction terms and extends known linear transformations to higher orders. Exact analytical solutions for the flow equations are derived to order $1/N^2$, yielding results for the ground state energy and the energy of the first excited state which agree surprisingly well with the accurate values.

Acknowledgments

The author wishes to thank F Wegner and H J Pirner for useful discussions.

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