Sommerfeld approximation

(1) normalization condition

$$\int_{-\infty}^{\infty} d\varepsilon\, \Omega_1(\varepsilon)\left\{\frac{1}{e^{\beta(\varepsilon-\mu)}+1} - \Theta(\varepsilon_F-\varepsilon)\right\} = 0$$

expand: $\Omega_1(\varepsilon) = \Omega_1(\varepsilon_F) + \Omega_1'(\varepsilon_F)(\varepsilon-\varepsilon_F)$

$$\mu = \varepsilon_F + \Delta\mu$$

variable $x = \beta(\varepsilon-\varepsilon_F)$

$$\int_{-\infty}^{\infty} dx \left(\Omega_1(\varepsilon_F) + \frac{\Omega_1'(\varepsilon_F)x}{\beta}\right)\left(\frac{1}{e^x e^{-\beta\Delta\mu}+1} - \Theta(-x)\right) = 0$$

$$\int_{-\infty}^{\infty} dx \left(\Omega_1(\varepsilon_F) + \underbrace{\frac{\Omega_1'(\varepsilon_F)x}{\beta}}_{\hookrightarrow \pi^2/6}\right)\left(f(x) + \beta\Delta\mu \underbrace{\frac{e^x}{(e^x+1)^2}}_{\hookrightarrow 1}\right) = 0$$

$$f(x) = \frac{1}{e^x+1} - \Theta(-x) = -f(-x)$$

$$\int_{-\infty}^{\infty} dx\, f(x) = 0, \quad \int_{-\infty}^{\infty} dx\, x\, f(x) = \frac{\pi^2}{6}$$

$$\int_{-\infty}^{\infty} dx\, \frac{e^x}{(e^x+1)^2} = 1$$

$$\beta \Delta\mu\, \Omega_1(\varepsilon_F) = -\frac{\pi^2}{6}\frac{1}{\beta}\, \Omega_1'(\varepsilon_F)$$

$$\boxed{\mu = \varepsilon_F - \frac{\pi^2}{6}\frac{\Omega_1'(\varepsilon_F)}{\Omega_1(\varepsilon_F)}(kT)^2}$$

(2) energy difference

$$\frac{E-E_0}{N} = \int_{-\infty}^{\infty} d\varepsilon\, \varepsilon\, \Omega_1(\varepsilon)\left\{\frac{1}{e^{\beta(\varepsilon-\varepsilon_F-\Delta\mu)}+1} - \Theta(\varepsilon_F-\varepsilon)\right\}$$

$$= \int_{-\infty}^{\infty} dx\, \frac{1}{\beta}\underbrace{\left(\varepsilon_F + \frac{x}{\beta}\right)}_{\varepsilon}\left(\Omega_1(\varepsilon_F) + \Omega_1'(\varepsilon_F)\frac{x}{\beta}\right)\left(f(x) + \beta\Delta\mu\frac{e^x}{(e^x+1)^2}\right)$$

$$= \varepsilon_F \Delta\mu\, \Omega_1(\varepsilon_F) + \frac{\pi^2}{6}\frac{1}{\beta^2}\left(\Omega_1(\varepsilon_F) + \Omega_1'(\varepsilon_F)\varepsilon_F\right)$$

insert $\Delta\mu = -\frac{\pi^2}{6}\frac{\Omega_1'}{\Omega_1}(kT)^2$

$$\boxed{E = E_0 + \frac{\pi^2}{6}\Omega_1(\varepsilon_F)(kT)^2 N}$$