1. Presence exercise for the lecture Statistical Physics

for Thursday/Friday October 21/22, 2010
For active participation you get 2 points !

Exercise P1.1: *Reminder: Density matrix / statistical operator*
If the system is in the normalized state $|\psi\rangle$ the expectation value of the observable $\hat{A}$ is given by

$$\langle \hat{A} \rangle \;=\; \langle \psi | \hat{A} | \psi \rangle$$

With the density matrix

$$\hat{\rho} := |\psi\rangle\langle\psi|$$

of the pure state $|\psi\rangle$ show that

(a) $\langle \hat{A} \rangle = tr(\hat{\rho}\hat{A})$

(b) $tr(\hat{\rho}) = 1$

(c) $\hat{\rho}^2 = \hat{\rho}$

(d) $\hat{\rho}^\dagger = \hat{\rho}$

The trace $tr(\hat{X})$ of an operator $\hat{X}$ is defined by

$$tr(\hat{X}) \;=\; \sum_n \langle n | \hat{X} | n \rangle$$

with an arbitrary complete orthonormal basis $\{|n\rangle\}$.

(e) Show that $tr(\hat{X})$ is independent of the choice of the basis.

Now consider a so called mixed state, that is an ensemble of different states $|\psi_i\rangle$ with $\langle\psi_i|\psi_i\rangle = 1$. Let p_i be the probability to find the state $|\psi_i\rangle$ in the ensemble with

$$\sum_i p_i \;=\; 1 \;.$$

The expectation value of an observable $\hat{A}$ is then given by

$$\langle \hat{A} \rangle \;=\; \sum_i p_i \langle \psi_i | \hat{A} | \psi_i \rangle$$

The density matrix for this ensemble ist defined by (=density matrix of a so called mixed state)

$$\hat{\rho} \;:=\; \sum_i p_i |\psi_i\rangle\langle\psi_i| \;.$$

Show that

(e) $\langle \hat{A} \rangle \; = \; tr\left(\hat{\rho}\hat{A}\right)$

(f) $tr(\hat{\rho}) \; = \; 1$

(g) $\hat{\rho}^{\dagger} \; = \; \hat{\rho}$

(h) $\hat{\rho}^2 \; \neq \; \rho$ and $tr(\hat{\rho}^2) \; < 1$ if $p_i \; \neq 0$ for more than one i.

Exercise P1.2: *Density matrix / statistical Operator: Stern-Gerlach-experiment*

A monoenergetic electron beam with low intensity (interaction of the particles with one another can be neglected), which has a spin state given by the density matrix

$$\rho \; = \; |\uparrow> r_+ <\uparrow| \; + \; |\downarrow> \frac{3}{4} <\downarrow|$$

enters the field of a Stern-Gerlach-magnet, where in its interior a mask absorbs particles which are in the state $|\downarrow>$.

a) Calculate r_+.

b) Which is the density matrix for the electrons, which have passed through the magnet ?

c) Is this then a pure state ?

d) How large is the probability that an electron passes through the magnet ?

Exercise P1.3: Probalistic observables
Consider a general two-state system with the 2x2 density matrix (see lecture of October 14)

$$\hat{\rho} \; = \; \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12}^* & (1-\rho_{11}) \end{pmatrix} \qquad \text{with} \qquad \rho_{11}^* = \rho_{11} \,\text{and}\, 0 \leq \rho_{11} \leq 1$$

Furthermore consider an observable presented by the general matrix

$$\hat{A} \; = \; \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \; = \; \hat{A}^{\dagger}$$

Calculate the quantum mechanical expectation value $\langle \hat{A} \rangle = tr\left(\hat{\rho}\hat{A}\right)$ and show that one gets the same result for $\langle \hat{A} \rangle$ with

$$\langle \hat{A} \rangle \; = \; p_1 \, \bar{A}_1 \; + \; p_2 \, \bar{A}_2 \qquad \text{with} \qquad p_1 \; = \; \rho_{11} \qquad \text{and} \qquad p_2 \; = \; (1-\rho_{11})$$

$$\bar{A}_1 \; = \; A_{11} \; + \; \rho_{12} A_{12}^* \; + \; \rho_{12}^* A_{12}$$
$$\bar{A}_2 \; = \; A_{22} \; + \; \rho_{12} A_{12}^* \; + \; \rho_{12}^* A_{12}$$