10. Exercise sheet for the lecture Statistical Physics

Hand in your solutions on Thursday, 13.01.2011 or Friday, 14.01.2011 in the exercise group.

Exercise 10.1 *Diluted gas at high temperature - velocity distribution* **(9 points)**
In the lecture, you got to know the velocity distribution

$$\hat{p}(\vec{v})\, d^3v \;=\; \left(\frac{M}{2\pi kT}\right)^{3/2} \exp\left\{-\frac{M\vec{v}^{\,2}}{2kT}\right\} d^3v$$

of the atoms in an ideal gas.

a) Use the velocity distribution to determine the most likely, the mean and the mean-square velocity of the particles in an ideal gas. (3 points)

b) Consider now a container with an ideal gas at temperature T. The container has a hole at one side (see illustration).

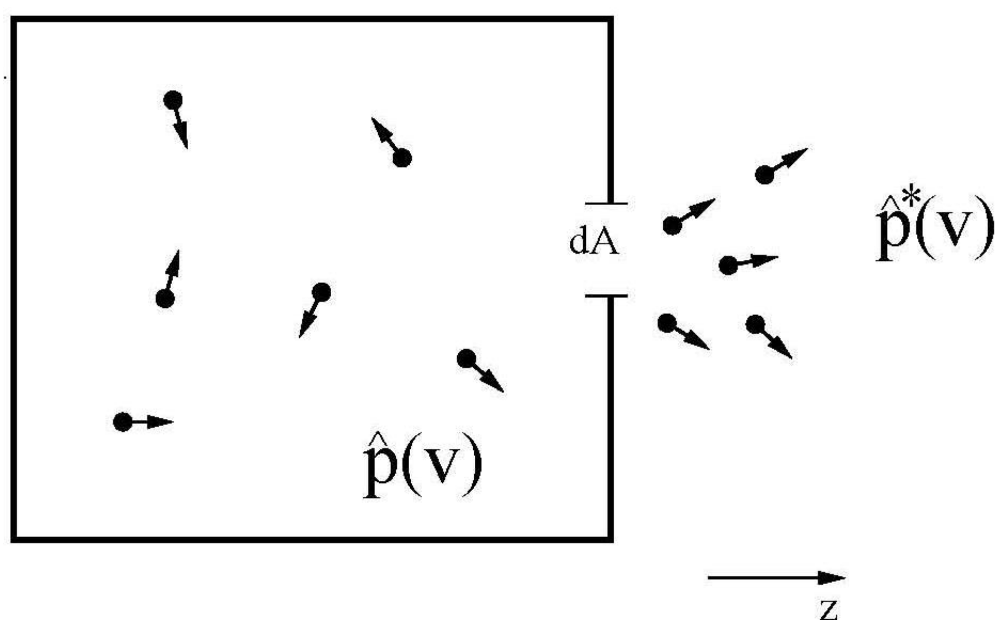
1) Calculate the velocity distribution $\hat{p}^*(\vec{v})$ of the particles, which leave the hole. Assume that the equilibrium in the interior of the container is not disturbed by the escaping particles. (2 points)

Note: The velocity distribution $\hat{p}^*(\vec{v})$ of the escaping particles is proportional to $v_z\hat{p}^*(\vec{v})$ (with v_z being the velocity component of the particles in $z-$ direction), i.e. $\hat{p}^*(\vec{v}) \;=\; c\, v_z\hat{p}(\vec{v})$. The constant c can be determined from the normalization.

2) Calculate the mean velocity of the escaping particles in $z-$ direction (see illustration). (1 point)

3) Calculate the mean kinetic energy of the escaping particles.
(1 point)

4) Calculate the rate $R \;=\; \frac{d^2N}{dtdA}$ of the particles, which leave the container per unit of time and unit of hole area. (2 points)



see reverse

Exercise 10.2:*(Ideal Bose gas, Bose-Einstein condensation)* **(6 points)**
Consider an ideal Bose gas with total number of particles N. The one-particle energy levels are normalized such that $E_0 = 0$, $E_0 \leq E_1 \leq E_2 \leq ....$

a) Argue that based on the mean occupation number
$$\bar{n}_\nu \;=\; \frac{1}{e^{\beta(E_\nu - \mu)} - 1}$$
of the νth one-particle state, it must hold $\mu \leq 0$ for the chemical potential μ. (2 points)

b) Assume that $\mu < 0$. What do you get then in the limit $T \to 0$ for $\bar{n}_\nu$ (ν arbitrary) ? Which problem arises in this case for N ? (2 points)

c) Assume now that for $T \to 0$ it is always $0 < -\beta\mu \ll 1$. What do you get in this case for the mean occupation number $\bar{n}_0$ of the energetically lowest one-particle level and how does this solve the problem which occurred in exercise 10.2 b) ? (2 points)

Exercise 10.3: *(Ideal Bose gas)* **(10 points)**
The grand canoncial potential $J(T, V, \mu)$ of an ideal Bose gas is given by
$$\begin{aligned}\beta J(T,V,\mu) &= -\ln(Z_{gc}^{(B)}) = \sum_\nu \ln\left(1 - e^{-\beta(E_\nu - \mu)}\right) \\ &= g_I \frac{V}{(2\pi)^3} 4\pi \int_0^\infty dk\, k^2 \ln\left(1 - z e^{-\beta\epsilon(\vec{k})}\right) + g_I \ln(1-z)\end{aligned}$$
with $z = e^{\beta\mu}$.
The last term $g_I \ln(1-z)$ is the contribution of the ground state for which we assume $\epsilon(\vec{0}) = 0$.

a) Why is it necessary to split off the ground state contribution when the sum $\sum_\nu$ is replaced by the integral? (2 points)

b) Consider non-relativistic bosons with $\epsilon(\vec{k}) = \frac{\hbar^2 k^2}{2m}$ and show, that
$$\beta J(T,V,\mu) \;=\; -\frac{g_I}{\lambda^3} V\, g_{5/2}(z) \;+\; g_I \ln(1-z)$$
with
$$g_\alpha(z) \;=\; \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{x^{\alpha-1}}{z^{-1}e^x - 1}\, dx \qquad (\alpha \in \mathbb{R}, 0 \leq z \leq 1),$$
the thermal De Broglie wavelength $\lambda = \sqrt{\frac{2\pi\beta\hbar^2}{m}}$ and the Gamma function $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$.

c) Show that (2 points)
$$g_\alpha(z) \;=\; \sum_{n=1}^\infty \frac{z^n}{n^\alpha}$$

d) Show that the mean particle density n is given by (2 points)
$$n = \frac{\langle \hat{N} \rangle}{V} \;=\; \frac{g_I}{\lambda^3} g_{3/2}(z) \;+\; \frac{g_I}{V} \frac{z}{1-z}$$

e) Show that the mean energy of the system U is given by (4 points)
$$U = \frac{3}{2} \frac{V}{\beta} \frac{g_I}{\lambda^3} g_{5/2}(z)$$

MERRY CHRISTMAS AND HAPPY NEW YEAR!