12. Exercise sheet for the lecture Statistical Physics

Hand in your solutions on Thursday, 27.01.2011 or Friday, 28.01.2011 in the exercise group.

This is the last sheet!

This sheet is an EXTRA sheet: The points you could get with this sheet are extra points (allowing you to earn some extra points). That means that in the end with this points you could have more than 100 % ! Or: With this points it is maybe your last chance to reach the 60% to get the admission for the test!

Exercise 12.1 *The Ising model*

We consider again the one dimensional Ising model. This time we take N spins and denote these with s_i ($i = 1 \dots N$). Each spin can be in two states: $s_i = +1$ or -1 (up or down). The magnetic moment of each spin is μ_B, the Bohr magneton. The spins only interact with their direct neighbours with a coupling strength J. In the presence of an external magnetic field B the Hamiltonian is

$$H = -\sum_{\langle ij \rangle} J_{ij} s_i s_j - B\mu_B \sum_{i=1}^{N} s_i,$$

where $\langle ij \rangle$ means that the sum goes over any pair of spins and $J_{ij} = J$ for nearest neighbours and zero otherwise.

a) We take $B = 0$ and $J > 0$. What is/are the groundstates and what is the groundstate energy? (For $J > 0$, the Ising model is a model for ferromagnetism). Same question for $J < 0$ (for $J < 0$ it is a model for paramagnetism). (2 points)

From now on we take $J > 0$ and $B \neq 0$. The canonical partition function is

$$Z_{can} = \sum_{\{s_i\}} \exp\left(-\beta H\right),$$

where the sum runs over both values of s_i for all i.

b) The magnetization is $M = \mu_B \sum_{i=1}^{N} s_i$. Show how the average magnetization $\langle M \rangle$ can be obtained from $\ln Z$. (2 point)

c) The susceptibility $\chi = \langle \Delta M^2 \rangle / kT|_{B=0}$, with $\Delta M = M - \langle M \rangle$, is a measure for fluctuations of the magnetization in the absence of a magnetic field. Show that χ can be found from

$$\chi = \frac{\partial \langle M \rangle}{\partial B}\Big|_{B=0}.$$

(2 points)

d) The one-dimensional Ising model can be solved completely. The Hamiltonian reads

$$H = -J \sum_{i=1}^{N} s_i s_{i+1} - B\mu_B \sum_{i=1}^{N} s_i,$$

with $s_i = \pm 1$. The N spins lie on a ring (periodic boundary conditions) such that $s_{N+1} = s_1$.

(i) Show that the partition function Z_{can} can be written as

$$Z_{can} = \sum_{s_1=\pm} \dots \sum_{s_N=\pm} K(s_1, s_2)K(s_2, s_3) \dots K(s_N, s_1),$$

with $K(s_i, s_{i+1}) = \exp\left\{\beta[\frac{1}{2}\mu_B B(s_i + s_{i+1}) + J s_i s_{i+1}]\right\}$

(3 points)

(ii) Show that Z can be written as $Z = \mathrm{Tr}\left(K^N\right)$ with

$$K = \begin{pmatrix} e^{j+b} & e^{-j} \\ e^{-j} & e^{j-b} \end{pmatrix}, \quad b = \beta\mu_B B, \quad j = \beta J.$$

(3 points)

(iii) Why can the matrix K be diagonalized? Are the eigenvalues λ_+ and λ_- real, imaginary, or complex? Calculate the eigenvalues. (2 point)

(iv) Show that $Z = \lambda_+^N + \lambda_-^N$ and that in the thermodynamic limit ($N \to \infty$) only λ_+ ($\lambda_+ > \lambda_-$) contributes.

(2 points)

We consider the thermodynamic limit from now on.

(v) Calculate the magnetization $\langle M \rangle$. Can there be spontaneous magnetization in one dimension? Make a plot of $\langle M \rangle$ versus B for several values of the temperature. Calculate the susceptibility χ. (2 points)